TOPOLOGY

M.A. Mathematics (Previous)

PAPER-III

Directorate of Distance Education

Maharshi Dayanand University

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Contents

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Paper-III **M.Marks: 100**
Time: 3 Hrs.

Note: *Question paper will consist of three sections. Section-I consisting of one question with ten parts of 2 marks each covering whole of the syllabus shall be compulsory. From Section-II, 10 questions to be set selecting two questions from each unit. The candidate will be required to attempt any seven questions each of five marks. Section-III, five questions to be set, one from each unit. The candidate will be required to attempt any three questions each of fifteen marks.*

Unit-I

Definition and examples of topological spaces, closed sets and closure, dense subsets. Neighbourhoods interior Exterior and boundary operations, Accumulation points and Derived sets. Bases and subbase. Subspaces and relative topology. Alternative method of defining a topology in terms of Kuratowski closure operator and neighbourhood systems. Continuous functions and homoemorphisms.

Connected spaces. Connectedness on the real time. Components, Locally connected spaces.

Unit-II

Compactness, continuous functions and compact sets. Basic properties of compactness and finite intersection property. Sequentially and countably compact sets, Local compactness and one point compactification.

Separation axioms T_0, T_1 and T_2 spaces, Their characterization and basic properties, Convergence on T_0 space.First and second countable spaces, Lindelof's Theorems, Separable spaces and separability.

Unit-III

Regular and normal spaces, Urysohn's Lemma and Tietze Extension Theorem, T_3 and T_4 spaces, Complete regularity and complete normality, T_3/½ and T_5 spaces.

Embedding and Metrization. Embedding Lemma and Tychonoff embedding Urysohn's Metrization Theorem.

Unit-IV

Product topological spaces, Projection mapppings, Tychonoff product topology in terms of standard subbases and its characterization, Separation axioms and product spaces, Connectedness, locally connectedness and Compactness of product spaces. Product space as first axiom space.

Nets and filters. Topology and convergence of nets. Hausdorffness and nets. Compactness and nets. Filters and their convergence. Canonical way of converting nets to filters and vice-versa. ultra filters and compactness. Stone-Cech compactification.

Unit-V

Homotopy of paths, Fundamental group, Covering spaces, The fundamental group of the circle and fundamental theorem of algebra.

Covering of a space, local finiteness, paracompact spaces, Michaell theorem on characterization of paracompactness in regular space, Paracompactness as normal, Nagata-Smirnov Metrization theorem.

TOPOLOGY

The word topology is derived from two Greek words, topos meaning surface and logs meaning discourse or study. Topology thus literally means the study of surfaces or the science of position. The subject of topology can now be defined as the study of all topological properties of topological spaces. A topological property is a property which if possessed by a topological space X, is also possessed by every homeomorphic image of X. If very roughly, we think of a topological space as a general type of geometric configuration, say, a diagram drawn on a sheet of rubber, then a homemorphism may be thought of as any deformation of this diagram (by stretching bending etc.) which does not tear the sheet. A circle can be deformed in this way into an ellipse, a triangle, or a square but not into a figure eight, a horse shoe or a single point. Thus a topological property would then be any property of the diagram which is invariant under (or unchanged by) such a deformation. Distances, angles and the like are not topological properties because they can be altered by suitable non-tearing deformations. Due to these reasons, topology is often described to non-mathematicians as “rubber sheet geometry”.

Maurice Frechet (1878-1973) was the first to extend topological considerations beyond Euclidean spaces. He introduced metric spaces in 1906 in a context that permitted one to consider abstract objects and not just real numbers or n-tupls of real numbers. Topology emerged as a coherent discipline in 1914 when Felix Hausdorff (1868-1942) published his classic treatise Grundzuge der Mengenlehre. Hausdorff defined a topological space in terms of neighbourhoods of member sof a set.

1

TOPOLOGICAL SPACES

We begin with the study of topology with a brief motivational introduction to metric spaces. The ideas of metric and metric spaces are abstractions of the concept of distance in Euclidean space. These abstractions are fundamental and useful in all branches of mathematics.

Definition. A metric on a set X is a function $d : X \times X \to R$ that satisfies the following conditions.

(a) $d(x, y) \geq 0$ for all $x, y \in X$
(b) $d(x, y) = 0$ if and only if $x = y$
(c) $d(x, y) = d(y, x)$ for all $x, y \in X$.
(d) $d(x, z) \leq d(x, y) + d(y, z)$ for all $x, y, z \in X$.

If d is a metric on a set X, ordered pair (X, d) is called a metric space and if $x, y \in X$, Then d(x, y) is the distance from x to y.

Metric Space

Note that a **metric space** is simply a set together with a distance function on the set.

The function $d : R \times R \to R$ defined by $d(x, y) = |x-y|$ satisfies the four conditions of the definition and hence this function is a metric on R. It is called the usual metric on R. Also the function $d : R^2 \times R^2 \to R$ defined by $d\{(x_1\ x_2), (y_1, y_2)\} = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$ is called the usual metric on R^2.

Definition. A subset U of a metric space (X, D) is open if for each $x \in U$, there is an open ball $B_d(x, \in)$ such that $B_d(x, \in) \subseteq U$.

The open subsets of a metric space (X, d) have the following properties.

(a) X and ϕ are open sets
(b) The union of any collection of open sets is open.
(c) The intersection of any finite collection of open sets is open.

Metrizable

A metrizable space is a topological space X with the property that there exists at least one metric on the set X whose class of generated open sets is precisely the given topology i.e. it is a topological space whose topology is generated by some metric. But metric space is a set with a metric on it. The following example shows that **there are topological spaces that are not metrizable.**

Example. Let X be a set with at least two members and T be the trivial topology on X, then (X, T) is not metrizable.

Definition and examples of Topological spaces

Definition. A topological space is a pair (X, T) consisting of a set X and a family T of subsets of X satisfying the following axioms :

(0_1) $\phi, X \in T$
(0_2) The intersection of any finite number of sets in T is in T.

(0_3) Any union (countable or not) of sets in T is in T.

The set X will be called a space, its elements points of the space and the subsets of X belonging to the family T, sets open in the space. The collection T is called a topology for X.

Axioms (0_1)$\to$(0_3) of the family of open sets can be formulated in the following manner

(0_1) The empty set and the whole space are open

(0_2) The intersection of two open sets is open

(0_3) The union of arbitrary many open sets is an open set.

Examples of Topological Spaces

1. Example. Let X {a, b, c, d, e}. Consider the following classes of subsets of X .

$S_1 = \{\phi, X, \{a\}, \{c, d\} \{a, c, d\}, \{b\ c, d\ e\}\}$

$S_2 = \{\phi, X, \{a\}, \{c, d\}, \{a, c, d\}, \{b, c, d\}\}$

$S_3 = \{\phi, X, \{a\}, \{c, d\}, \{a, c, d\}, \{a, b, d, e\}\}$

We observe that S, is a topology on X since it satisfies the three axioms.

But S_2 is not a topology on X since the union $\{a, c, d\} \cup \{b, c, d\} = \{a, b, c, d\}$ of two members of S_2 is not in S_2 and so S_2 does not satisfy the third axiom.

Similarly it can be seen that S_3 is not a topology on X since the intersection

$$\{a, c, d\} \cap \{a, b, d, e\} = \{a, d\}$$

of two sets in S_3 does not belong to S_3 and so S_3 does not satisfy the second axiom

2. **Example.** A metric space is a special kind of topological space. The open sets being defined as usual the axioms (0_1) – (0_3) hold since

(0_1) ϕ and X in a metric space (X, d) are open.

(0_2) The intersection of any finite number of open sets in (X, d) is open.

(0_3) Any union (countable or not) of open sets in (X, d) is open.

This topology defined on metric space is called **usual topology** on a metric space.

3. **Example.** Let D denote the class of all subsets of X. Then D satisfies all axioms for a topology on X. This topology is called the **Discrete topology** and (X, D) is called a Discrete topological space or simply a Discrete Space.

4. Example. Let X be a nonempty set. The family $I = \{\phi, X\}$ consisting of ϕ and X is itself a topology on X and is called the **Indiscrete topology** or simply an **Indiscrete space.** It is the **coarsest topology**.

Remark. When X is a singleton, then the two topologies discrete and indiscrete coincide.

5. **Example.** Let X be any infinite set and T be the family consisting of ϕ and complements of finite subsets of X. Show that T is a topology on X.

Analysis. Let X be an infinite set and T be the family consisting of ϕ and subsets of X whose complements in X are finite.

To prove that T is a topology on X.

(1) $\phi \in T$ and since ϕ is finite $\Rightarrow \phi^C = X \in T$

(2) Let $G_1, G_2 \in T$ $\Rightarrow G_1 \cap G_2 \in T$

If $G_1 \cap G_2 = \phi$ then $G_1 \cap G_2 \in T$

If $G_1 \cap G_2 \neq \phi$, then $G_1 \neq \phi$, $G_2 \neq \phi$

and $G_1, G_2 \in T$ $\Rightarrow X-G_1$ and $X-G_2$ are finite

$\Rightarrow (X-G_1) \cap (X-G_2)$ is finite

$\Rightarrow X - (G_1 \cap G_2)$ is finite

$\Rightarrow G_1 \cap G_2 \in T$

Thus in either case $G_1, G_2 \in T \Rightarrow G_1 \cap G_2 \in T$

(3) Let $G_\alpha \in T \quad \Rightarrow X - G_\alpha$ is finite

$\Rightarrow \underset{\alpha}{\cap} (X - G_\alpha)$ is finite

$\Rightarrow \underset{\alpha}{\cap} G_\alpha^C$ is finite

$\Rightarrow X - (\underset{\alpha}{\cap} G_\alpha^C) \in T$

$\Rightarrow \underset{\alpha}{\cup} (G_\alpha^C)^C \in T$

$\Rightarrow \underset{\alpha}{\cup} G_\alpha \in T$

Hence all the axioms for a topology are satisfied.

$\Rightarrow$ T is a topology on X.

Remark. This topology is called **cofinite topology**.

6 Example. Let X be a set and T = {$\cup \in$ P(X), $\cup = \phi$ or X $-$ $\cup$ is countable}. Then T is a topology on X and is called the countable complement topology on X.

Remark. If X is a finite set, then the finite complement topology, the countable complement topology and the discrete topology are the same.

7. Example. Let T = {B $\in$ **P**(R); B is an interval of the form [a, b)}

Then T satisfies all the conditions for a topology.

This topology is called lower limit topology on R.

8. Example. Let T {B $\in$ **P**(R); B is an interval of the form (a, b]}

Then T satisfies all the conditions for a topology and this topology is called upper limit topology on R.

9. Example. Let X be a linearly ordered set. Then order topology for X is obtained by choosing as a subbase for all sets which are either of the form {x, x > a} or of the form {x; x < a} for some a$\in$X.

10. Example. Let X = {1,2,3} . List some of topologies on X. Are there any collection of subsets of X that are not topologies on X.

Solution. Of course we have the trivial topology and the discrete topology. We list some other topologies,

$T_1 = \{\phi, X, \{1\}\}$

$T_2 = \{\phi, X, \{1, 2\}\}$

$T_3 = \{\phi, X, \{1\}, \{2\}, \{1, 2\}\}$

$T_4 = \{\{1\}, \{2, 3\}, \phi, X\}$

$T_5 = \{\phi, X, \{2\}, \{1, 2\}, \{2, 3\}\}$

$T_6 = \{\phi, X, \{1\}, \{1, 2,\}\}$

$T_7 = \{\phi, X, \{1\}, \{2\}, \{1, 2\}, \{2, 3\}\}$

There are collections of subsets of X = {1, 2, 3} that are not topologies on X.

e.g. $\{\phi, X, \{1, 2\}, \{2, 3\}\}$

and $\{\phi, X, \{1\}, \{2\}\}$.

Theorem. 1. The intersection $T_1 \cap T_2$ of any two topologies T_1 and T_2 on X is also a topology on X.

Proof. Since T_1 and T_2 are topologies on X, therefore $\phi, X \in T$ and $\phi, X \in T_2$

$\Rightarrow \quad \phi, X \in T_1 \cap T_2$

That is $T_1 \cap T_2$ satisfies first axiom for a topology

Also if $G, H \in T_1 \cap T_2$, Then

$G, H \in T_1$ and $G, H \in T_2$

Since T_1 and T_2 are topologies

$\Rightarrow \quad G \cap H \in T_1$ and $G \cap H \in T_2$

$\Rightarrow \quad G \cap H \in T_1 \cap T_2$

i.e. $T_1 \cap T_2$ satisfies the second axiom for a topology.

Further, let $G_\alpha \in T_1 \cap T_2$ for every $\alpha \in S$, where S is an arbitrary set

Then $G_\alpha \in T_1$ and $G_\alpha \in T_2$ for every $\alpha \in S$. but T_1 and T_2 are topologies

$\Rightarrow \quad \cup_\alpha G_\alpha \in T_1$ and $\cup_\alpha G_\alpha \in T_2 \Rightarrow \cup_\alpha G_\alpha \in T_1 \cap T_2$.

i.e. $T_1 \cap T_2$ satisfies third axiom for a topology.

Hence the result follows.

Remark. The union $T_1 \cup T_2$ of two topologies on a set X need not be a topology on X for example, let X = {a, b, c}, then

$T_1 = \{\phi, X, \{a\}, \{b\} \{a, b\}\}$

$T_2 = \{\phi, X, \{a\}, \{c\}, \{a, c\}\}$

are two topologies on X but

$T_1 \cup T_2 = \{\phi, X, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}\}$

is not a topology on X since union of {b} and {c} is not in $T_1 \cup T_2$

Definition. Let (X, T_1) and (X, T_2) be topological spaces with the same set X. Then T_1 is said to be finer than T_2 if $T_1 \supset T_2$. The topology T_2 is then said to be coarser than T_1.

Clearly the discrete topology is the finest topology and the indiscrete topology is the coarset topology defined on a set.

Accumulation Points and Derived Sets.

Definition. Let (X, T) be a topological space and $E \subseteq X$. A point x is said to be a limit point or accumulation point of the set E if for every open set G containing x, we have

$$E \cap G - \{x\} \neq \phi$$

The set of all limit points of a set E is called the derived set of E and is denoted by d(E).

Example 1. Let (X, T) be a discrete topological space. Then each point is an open set and

$$E \cap \{x\} - \{x\} = \phi$$

and therefore the derived set of every set E is empty

Example 2. Let (X, T) be an indiscrete topological space then the only open set containing a point x is X itself. Consider

$$E \cap X - \{x\} = E - \{x\}$$

If E consists of two points, then $E - \{x\} \neq \phi$ and therefore the derived set of any set containing at least two points is the entire space X. If E is empty, then the derived set of E is empty since

$$\phi \cap X - \{x\} = \phi$$

If E consists of exactly one point, then the derived set of E is the complement of that point.

Example 3. Let X = {a, b, c} and $T = \{\phi, \{a\}, \{b\}, \{a, b, X\}$

Then we note that a is contained in {a}, {a, b} and X and

$$\{a\} \cap \{a\} - \{a\} = \phi$$
$$\{a\} \cap \{a, b\} - \{a\} = \phi$$
$$\{a\} \cap X - \{a\} = \phi$$

This implies that a is not a limit point of {a} . Similarly b is not a limit point of {a}.
Since X is the only open set containing {c}
Hence $\{a\} \cap X - \{c\} \neq \phi$
This implies that c is a limit point of {a}

Remark. The definition of a limit point is equivalent to "A point x is said to be a limit point of E if every open set containing x contains a point of E different from x".

Theorem 2. If A, B and E are subsets of the topological space (X, T), then the derived set has following properties.

(i) $d(\phi) = \phi$
(ii) If $A \subseteq B$, then $d(A) \subseteq d(B)$
(iii) If $x \in d(E)$, then $x \in d\{E - \{x\}\}$
(iv) $d(A \cup B) = d(A) \cup d(B)$

Proof. (i) holds since $\phi \cap G - \{x\} = \phi$ for any $x \in X$ and $G \in T$.

(ii) Since $A \subseteq B$ $\Rightarrow A \cap G - \{x\} \subseteq B \cap G - \{x\}$
$\Rightarrow$ $d(A) \subseteq d(B)$

This proves (ii)

(iii) To prove (iii), we note that

$$[E - \{x\}] \cap G - \{x\} = [E \cap \{x\}^C] \cap G \cap \{x\}^C$$
$$= E \cap G \cap [x\}^C$$
$$= E \cap G - \{x\}$$

Therefore if $x \in d(E)$, then

$$X \in d\,[E - \{x\}]$$

(iv) Since $A \subseteq A \cup B$ and $B \subseteq A \cup B$
$\Rightarrow$ $d(A) \subseteq d\,(A \cup B)$ and $d(B) \subseteq d(A \cup B)$
$d(A) \cup d(B) \subseteq d(A \cup B)$

Conversely suppose that $x \notin d(A) \cup d(B)$ and so $x \notin d(A)$ and $x \notin d(B)$. Therefore by definition, there must exist G_A and G_B containing x such that

$$G_A \cap A - \{x\} = \phi$$
$$G_B \cap B - \{x\} = \phi$$

Let $G = G_A \cap B_B$. By axiom of topology, this is an open set since $x \in G_A, G_B$
$\Rightarrow$ $x \in G$.
But $G \cap A - \{x\} = G \cap B - \{x\} = \phi$
and so $G \cap \{A \cup B\} - \{x\} = \phi$
$\Rightarrow$ $x \notin d(A \cup B)$
$\Rightarrow$ $d(A \cup B) \subseteq d(A) \cup d(B)$

Closed Sets and Closure

The concept of a topological space has been introduced in terms of the axioms for the open sets. Similarly closed sets are used as the fundamental notion of topology.

Definition. Let (X, T) be a topological space. A set $F \subseteq X$ is said to be closed if it contains all of its limit points. Thus F is closed iff $d(F) \subseteq F$.

Theorem 3. If $x \notin F$, where F is a closed subset of a topological space (X, T), then there exists an open set G such that $x \in G \subseteq F^C$

Proof. Suppose no such open set exists.
Then $x \in G \in T$ would imply that $G \cap F \neq \phi$.

Since $x \notin F$, $G \cap F - \{x\} \neq \phi$.

This implies that x is a limit point of F that is $x \in d(F)$. F however is a closed set and so $d(F) \subseteq F$, so that x must belong to F. but this is contradiction to $x \notin F$. This contradiction shows that such an open set must exist.

Cor. 1. If F is a closed set, then F is open.

Proof. If $x \in F^C$, then $x \notin F$ where F is a closed set. By the above theorem, there exists an open set G_x such that

$$x \in G_x \subseteq F^C. \text{ But then}$$
$$F^C = \cup \{x \,;\, x \in F^C\} \subseteq \cup \{G_x \,;\, x \in F^C\} \subseteq F^C$$

Thus $$F^C = \cup \{G_x \,;\, x \in F^C\}$$
which is the union of open sets and hence an open set.

Cor. 2. If F^C is an open set, then F is closed.

Proof. Suppose x is a limit point of F and let $x \notin F$. Then $x \in F^C$, and
$$F \cap F^C - \{x\} = \phi$$
which implies that x is not a limit point of F. Hence the assumption that $x \notin F$ is wrong. Therefore, every limit point of F is in F and so F is closed.

Remark. 1. From Cor 1 and Cor 2, it follows that "A set is a closed subset of a topological space if and only if its complement is an open subset of the plane."

Remark. 2. From the De-Morgan's Laws and from the three axioms of a topological space, the following properties of the closed set follows

- (c_1) The empty set and the whole space are closed
- (c_2) The union of two closed sets is closed.
- (c_3) The intersection of arbitrary many closed sets is a closed set.

As an example, we give a proof for property (c_2) and (c_3).
(c_2) Suppose F and G are two closed sets then F^C and G^C are open since union of two open sets is open

$$F^C \cup G^C \text{ is open.}$$
$\Rightarrow$ $(F \cap G)^C$ is open [by De-Morgan's Law]
$\Rightarrow$ $F \cap G$ is closed
$\Rightarrow$ Intersection of two closed sets is closed.

(c_3) Let a family $\{F_s\}_{s\in S}$ of closed sets be given. By definition the set $U_s = X - F_s$ is open for every $s \in S$.

Since $\bigcap_{s\in S} F_s = \bigcap_{s\in S} (X - U_s) = X - \bigcup_{s\in S} U_s$.

Since the set $\bigcup_{s\in S} U_s$ is open, $\bigcap_{s\in S} F_s$ is a closed set.

Theorem. 4. If $d(F) \subseteq A \subseteq F$ and F is a closed set, then A is closed.

Proof. Since $A \subseteq F \quad \Rightarrow d(A) \subseteq d(F)$

But F is closed, hence

$$d(A) \subseteq d(F) \subseteq A \subseteq F \text{ i.e. } d(A) \subseteq A$$

which implies that A is closed.

Cor. The derived set of a closed set is closed.

Proof. Let F be a closed set we have to prove that d(F) is closed. Now as F is closed, $d(F) \subseteq F$

$$\Rightarrow d(d(F)) \subseteq d(F) \subseteq F.$$ Thus by the above theorem, d(F) is closed.

Definition. Let (X, T) be a topological space and $E \subseteq X$. Then closure of E denoted by C(E) is defined by C(E) = {$\cap$ all closed sets containing E}

Since by (c_1), the family of closed sets containing E is non-void and by (c_3), the intersection of all elements of this family is closed. Hence closure of a set is closed and it is the smallest closed set containing E.

Theorem. 5. For any set E in a topological space,

$$C(E) = E \cup d(E)$$

Proof. Suppose $x \notin E \cup d(E)$, so that $x \notin E$ and $x \notin d(E)$, there exists an open set G_x containing x such that $E \cap G_x - \{x\} = \phi$

Since $x \notin E$, this actually means that $E \cap G_x = \phi$ so $G_x \subseteq E^C$. Since G_x is open set disjoint from E, no point of G_x can be a limit point of E, that is $G_x \subseteq (d(E))^C$,

Thus $\qquad [E \cup d(E)]^C = \cup \{G_x; x \notin E \cup d(E)\}$

which is an open set since arbitrary union of open sets is open. Therefore $E \cup d(E)$ is closed, which obviously contains E. Hence C(E) being the smallest closed set containing E, we have $C(E) \subseteq E \cup d(E)$.

Conversely suppose that $x \in E \cup d(E)$ and suppose that F is any closed set containing E. If $x \in d(E)$, then $x \in d(F)$ and so $x \in F$ [since $d(E) \subseteq d(F) \subseteq F$]. But if $x \in E$, then again we have $x \in F$ since $E \subseteq F$, Thus x belongs to any closed set containing E and hence to the intersection of all such sets, which is the closure of E. Thus

$$E \cup d(E) \subseteq C(E).$$

Hence $C(E) = E \cup d(E)$

Remark. For arbitrary subsets A and B of the space X if $A \subset B$, then $C(A) \subset C(B)$. Indeed if $A \subseteq B$, then the family of closed sets containing A is contained in the family of closed sets containing B, then $C(A) \subseteq C(B)$.

Theorem. 6. E is closed if and only of E = C(E)

Proof. We suppose first that E is closed

Then $d(E) \subseteq E$. Since $C(E) = E \cup d(E)$, therefore it follows that if E is closed, then

$$C(E) = E \cup d(E) = E$$

Conversely if E = C(E), then C(E) being the intersection of all closed sets containing E is closed. Hence E is closed

Kurtowski Closure Axioms

Definition. An operator C of $\rho(X)$ into itself which satisfies the following four properties (known as Kuratowski closure axioms) is called a closure operator on the set X.

Theorem. 7. In the topological space (X, T), the closure operator has the following properties.

(K_1) $C(\phi) = \phi$

(K_2) $E \subseteq C(E)$

(K_3) $C(C(E)) = C(E)$ and

(K_4) $C(A \cup B) = C(A) \cup C(B)$

Proof. (K_1) Because the void set is closed and also we know that a set A is closed if and only if A = C(A), therefore it follows that $C(\phi) = \phi$.

(K_2) It follows from the definition as C(E) is the smallest closed set containing E

(K_3) Since C (E) is the smallest closed set containing E, we have C(C(E)) = C(E) by the result that a set is closed if and only if it is equal to its closure

(K_4) Since $A \subset A \cup B$ and $B \subset A \cup B$, therefore

$$C(A) \subset C(A \cup B) \text{ and } C(B) \subset C(A \cup B) \text{ and so}$$

$$C(A) \cup C(B) \subset C(A \cup B) \qquad (1)$$

By (K_2), we have

$$A \subset C(A) \text{ and } B \subset C(B)$$

Therefore $A \cup B \subset C(A) \cup C(B)$

Since C(A) and C(B) are closed sets and so $C(A) \cup C(B)$ is closed. By the definition of closure, we have

$$C(A \cup B) \subset C\,[C(A) \cup C(B)] \qquad (2)$$

From (1) and (2), we have

$$C(A \cup B) = C(A) \cup C(B).$$

Remark. $C(A \cap B)$ may not be equal to $C(A) \cap C(B)$. e.g. if A = (0, 1), B = (1, 2), then

$$C(A) = [0, 1],\ C(B) = [1, 2]$$

Therefore,

$$C(A) \cap C(B) = \{1\} \text{ where } A \cap B = \phi$$

But $C(\phi) = \phi$. Therefore $C(A \cap B) = \phi$ and thus $C(A \cap B) \neq C(A) \cap C(B)$

Remark. The closure operator completely determines a topology for a set A is closed iff A = C(A). In other words the closed sets are simply the sets which are fixed under the closure operator. We shall prove it in the form of the following.

Defining Topology in Terms of Kuratowski Closure Operator

Theorem. 8. Let C* be a closure operator defined on a set X. Let F be the family of all subsets F of X for which C*(F) = F and let T be a family of all complements of members of F . Then T is a topology for X and if C is the closure operator defined by the topology T. Then C*(E) = C(E) for all subsets $E \subseteq X$.

Proof. Suppose $G_\lambda \in T$ for all λ. We must show that $\bigcup_\lambda G_\lambda \in T$ i.e. $(\bigcup_\lambda G_\lambda)^C \in F$.

Thus we must show that

$$C^*[(\bigcup_\lambda G_\lambda)^C] = (\bigcup_\lambda G_\lambda)^C$$

By (K_2) $\qquad (\bigcup_\lambda G_\lambda)^C \subseteq C^*[(\bigcup_\lambda G_\lambda)^C]$

so we need only to prove that

$$C^*[(\bigcup_\lambda G_\lambda)^C] \subseteq (\bigcup_\lambda G_\lambda)^C$$

By De-Morgan’s Law, this reduces to

$$C^*[\bigcap_\lambda (G_\lambda)^C] \subseteq \bigcap_\lambda (G_\lambda)^C$$

Since $(\bigcap_{\lambda} (G_\lambda)^C) \subseteq (G_\lambda)^C)$ for each particular λ.

$$C^*[\bigcap_{\lambda} (G_\lambda)^C] \subseteq C^*[(G_\lambda)^C] \text{ for each } \lambda \text{ and}$$

so $$C^*[\bigcap_{\lambda} (G_\lambda)^C] \subseteq \bigcap_{\lambda} C^*[(G_\lambda)^C] \qquad (1)$$

But $G_\lambda \in T \quad \Rightarrow (G_\lambda)^C \in F$, so

$$C^*[(G_\lambda)^C] = (G_\lambda)^C$$

Thus we have from (1)

$$C^*[\bigcap_{\lambda} (G_\lambda)^C] \subseteq \bigcap_{\lambda} (G_\lambda)^C$$

Hence if $G_\lambda \in T$, then $\bigcup_{\lambda} G_\lambda \in T$

To check that ϕ, $X \in T$, we observe that by Kuratowaksi closure axiom (K_2)

$$X \subseteq C^*(X) \subseteq X \quad \Rightarrow C^*(X) = X \quad \Rightarrow X \in F$$

Hence $X^C = \phi \in T$.

Also by (Kurtowaski closure axiom K_1), we have $C^*(\phi) = \phi \Rightarrow \phi \in F$

$\Rightarrow \qquad \phi^C = X \in T$.

Finally suppose that G_1 $G_2 \in T$. Then by hypothesis

$$C^*(G_1)^C = G_1^C \text{ and } C^*(G_2)^C = G_2^C.$$

We may now calculate that

$$C^*[(G_1 \cap G_2)^C] = C^*[G_1^C \cup G_2^C]$$
$$= C(G_1^C) \cup C^*(G_2^C)$$
$$= G_1^C \cup G_2^C = (G_1 \cap G_2)^C$$

$\Rightarrow \qquad (G_1 \cap G_2)^C \in F \quad \Rightarrow G_1 \cap G_2 \in T$.

Hence all the axioms for a topology are satisfied and hence T is a topology. We now prove that $C^* = C$.

We have shown above that T is a topology for X. Thus members of T are open sets therefore the closed sets are just the members of the family *F*.

By (K_3), $C^*[C^*(E)] = C^*(E)$

which implies that $C^*(E) \in F$. Now by (K_2) $E \subseteq C^*(E)$. Thus $C^*(E)$ is a closed set containing E and hence

$$C^*(E) \supseteq C(E) \qquad (1)$$

as $\hat{C}(E)$ is the smallest closed set containing E. On the other hand by (K_2)

$$E \subseteq C(E) \in F \text{ so } C^*(E) \subseteq C^*(C(E)) = C(E) \qquad (2)$$

Thus by (1) and (2),

$$C^*(E) = C(E)$$

for any subset $E \subseteq X$

Dense Subsets

Definition. Let A be a subset of the topological space (X, T). Then A is said to be dense in X if

$$\overline{A} = X.$$

Trivially the entire set X is always dense in itself. Q is dense in R since

$$\overline{Q} = R.$$

Let T be finite complement topology on R. Then every infinite subset is dense in R.

Theorem. 9. A subset A of topological space (X, T) is dense in X iff for every nonempty open subset B of X, $A \cap B \neq \phi$.

Proof. Suppose A is dense in X and B is a non empty open set in X. If $A \cap B = \phi$, then $A \subset X - B$ implies that $\bar{A} \subset X - B$ since $X - B$ is closed. But then $X - B \underset{\neq}{\subset} X$ contradicting that $\bar{A} = X$ [since $\bar{A} \subset X - B \underset{\neq}{\subset} X$]

Conversely assume that A meets every non-empty open subset of X. Thus the only closed set containing A is X and consequently $\bar{A} = X$. Hence A is dense in X.

Theorem. 10. In a topological space (X, T)

(i) Any set C, containing a dense set D, is a dense set.

(ii) If A is a dense set, and B is dense on A, then B is also a dense set.

Proof. (1) Since $D \subset C \quad \Rightarrow \bar{D} \subset \bar{C}$

But $\bar{D} = X$ hence $X \subset \bar{C}$ also $\bar{C} \subset X$ so that $\bar{C} = X$.

Thus C is dense in (X, T)

(iii) Since A is dense, $\bar{A} = X$

Also B is dense on A

$\Rightarrow \quad A \subset \bar{B} \quad\quad \Rightarrow \bar{A} \subset \bar{\bar{B}} = \bar{B}$ (By closure property)

$\Rightarrow \quad \bar{A} \subset \bar{B}$

$\Rightarrow \quad X = \bar{A} \subset \bar{B}$

Thus B is dense in (X, T).

1.5. Neighbourhood

Definition. Let (X, T) be a topological space and let x be a point of X. Then a subset N of X is said to be a T-nbd of x if there exists a T-open set G such that $x \in G \subseteq N$. That is a nbd of a point is any set which contains an open set containing the point.

Remark.(i) It is evident from the definition that a T-open set is a T-nbd of each of its points but a T-nbd of a point need not be T-open. Also every open set containing x is a nbd of x we shall call such a nbd an open nbd of x

(ii) Clearly E is a nbd of x if and only if $x \in i(E)$

Example. 1. Let X = {1, 2, 3, 4, 5}

and T = {ϕ, X {1}, {1, 2}, {1, 2, 3} {1, 2, 4}, {1, 2, 3, 4}}

is a topology on X.

For such a topological space all subsets containing {1} will be nbds of 1.

Example. 2. Let (X, D) be a discrete topological space and $x \in X$. In such a case all possible subsets of X will be nbd of x because here each subset of X is open.

(3) If (X, I) is an indiscrete topological space, then X is the nbd of each of its points.

Theorem 11. Let (X, T) be a topological space and $A \subset X$. Then A is open if and only if it contains nbd of each of its points.

Proof. Let A be an open set and $x \in A$ be an arbitrary element. Since A is an open set such that $x \in A \subseteq A$, therefore A is a nbd of x. Thus A contains a nbd (namely A) of each of its points.

Conversely, let A contains a nbd of each of its points. So there exists a nbd N_a for each $a \in A$ such that $N_a \subset A$. Therefore by definition of nbd, there exists an open set S_a such that

$$a \in S_a \subseteq N_a.$$

Define $\quad S = \cup\{S_a \,;\, a \in A\}$

We shall show that S = A and then clearly A will be open. If $x \in A$, then by the definition of S_x $x \in S_x$

and so $$x \in \cup \{S_a ; a \in A\}$$

$$\Rightarrow \quad A \subseteq \cup \{S_a ; a \in A\}$$

$$\Rightarrow \quad A \subseteq S. \qquad (1)$$

Also if $y \in S$, then $y \in S_a$ for some $a \in A$.

But $$S_a \subset N_a \subset A$$

Therefore $y \in A$ which implies that

$$S = \cup \{S_a ; a \in A\} \subset A \qquad (2)$$

(1) and (2) yield S = A

Thus A is open.

Properties of Neighbourhood

Theorem. 12. Prove that

(N_1) Every point of X is contained in at least one neighbourhood and is contained in each of its nbds.

(N_2) The intersection of any two nbds of a point is a nbd of that point.

(N_3) Any set which contains a nbd of a point is itself a nbd of that point.

(N_4) If N is a nbd of a point x, then there exists a nbd N* of x such that N is a nbd of each point of N*.

Proof. (N_1) Since X is T-open, it is a T-nbd of every point. Hence there exists at least one T-nbd (namely X) for each $x \in X$.

(N_2) Let N and M be two neighbourhoods of $x \in X$. Then by definition, there exists open set G_1 and G_2 in X such that

$$x \in G_1 \subset N, \quad x \in G_2 \subset M$$

which imply that

$$x \in G_1 \cap G_2 \subset N \cap M, \text{ where } G_1 \cap G_2 \text{ is open}$$

Hence $N \cap M$ is a nbd of x.

(N_3) Let N be any nbd of x in X and M be a set in X which contains N. By the definition of nbd, there exists an open set G of x such that

$$x \in G \subseteq N$$

But $N \subset M$, it follows that

$$x \in G \subseteq N \subset M$$

$$\Rightarrow \quad x \in G \subset M.$$

Hence M is a nbd of x in X.

(N_4) Since N is a nbd of x. Therefore there exists an open set N* such that x $N^* \in N$. Since N* is open, it is a nbd of each of its points. But $N^* \subset N$, thus by (N_3) N is a nbd of each of points of N*.

Alternating Method of defining a topology in terms of neighbourhood system

Theorem. 13. Let there be associated with each point x of a set X, a collection N_x^* of subsets called nbds subject to the above four properties. Let T be the family of all subsets of X which are nbds of each of their points i.e. $G \in T$ iff $x \in G$ implies that $G \in N_x^*$. Then T is a topology for X and if N_x is the collection of all nbds of x defined by the topology T, then $N_x^* = N_x$ for every $x \in X$.

Proof. First we prove that T is a topology for X

(0_1) of course $\phi \in T$ since it is a nbd of each of its points.

We also note that for any $x \in X$, x is contained in at atleast one nbd by (N_1) and this nbd is contained in X, so X is a nbd of x by (N_3). Thus X is a nbd of each of its points and so $X \in T$.

(0_2) Suppose G_1 and G_2 belong to T. If $x \in G_1 \cap G_2$, then $x \in G_1$ and $x \in G_2$. By the definition of T, G_1 and G_2 are both nbds of each of their points. So $G_1 \in N_x^*$, $G_2 \in N_x^*$. Now by the property (N_2) of nbd $G_1 \cap G_2 \in N_x^*$ so $G_1 \cap G_2$ is a nbd of each of its points.

(0_3) Suppose $G_\lambda \in T$ for all λ. If $x \in \bigcup_\lambda G_\lambda$, then $x \in G_\lambda$ for some λ. Now by the definition of T, G_λ is a nbd of each of its points, so $G_\lambda \in N_x^*$, By (N_3) property of nbd, since $G_\lambda \subseteq \bigcup_\lambda G_\lambda$ we have $\bigcup_\lambda G_\lambda \in N_x^*$. Thus $\bigcup_\lambda G_\lambda$ is a nbd of each of its points and hence belongs to T.
Thus T is a topology.

IInd Part. To prove $N_x^* = N_x$

If $N \in N_x$, then $\exists$ an open set G such that $x \in G \subseteq N$, from the definition of T, $x \in G$ implies that $G \in N_x^*$ and so $N \in N_x^*$ by (N_3) property of nbd.

Thus $$N_x \subseteq N_x^* \qquad (1)$$

Now suppose $N \in N_x^*$. Let us define the set G to be all points which have N as a nbd. Clearly x is one of these points and so $x \in G$. While by property (N_1) of nbd, every point with N as a nbd is in N. So $G \subseteq N$. We shall show that $G \in T$ that is G is in N_y^* for every $y \in G$. Let $y \in G$, so that $N \in N_y^*$. By (N_4), there exists a set N^* s that $N^* \in N_y^*$ such and if $z \in N^*$, then $N \in N_z^*$ and the definition of G shows that $N \in N_z^*$ implies that $z \in G$, hence $N^* \subseteq G$ by (N_3), $G \in N_y^*$

$\Rightarrow$ $G \in T$

$\Rightarrow$ N is a nbd of x i.e. $N \in N_x$

$\Rightarrow$ $N_x^* \subseteq N_x$ (2)

[since $N \in N_x^*$] .

From (1) and (2)
$$N_x = N_x^*$$

1.6. Interior, Exterior and Boundary Operators

Definition. The interior of a set E is the largest open set contained in E or equivalently the union of all open sets contained in E called the interior of the set E. The interior of the set E is denoted by i (E). The points of i(E) of E are known as interior points of E. A set E is open if and only if E = i(E)

Theorem. 14. For any set E in a topological space (X, T),
$$i(E) = [C(E^C)]^C$$

Proof. Let $x \in i(E)$, then i(E) is itself an open set containing x which is disjoint from E and so $x \notin d(E^C)$. But $x \in E^C$, $x \notin d(E^C)$. This implies that $x \notin E^C \cup d(E^C)$

$\Rightarrow$ $x \notin C(E^C)$

$\Rightarrow$ $x \in [C(E^C)]^C$

$\Rightarrow$ $i(E) \subseteq [C(E^C)]^C$

Conversely suppose that $x \in [C(E^C)]^C$

$\Rightarrow$ $x \notin C(E^C)$

$\Rightarrow \quad x \notin [E^C \cup d(E^C)]$
$\Rightarrow \quad x \notin E^C$ and $x \notin d(E^C)$

Thus $x \in E$ and x is not a limit point of E^C.

Thus there exists an open set G containing x such that $E^C \cap G - \{x\} = \phi$
Since $x \notin E^C$, we have $E^C \cap G = \phi$ and so $G \subseteq E$.

Thus $x \in G \subseteq E$ for some open set G and so x belongs to the union of all open sets contained in E, which is i(E). Thus

$$[C(E^C)]^C \subseteq i(E)$$

Hence it follows that

$$i(E) = [C(E^C)]^C$$

Definition. Interior operator is a mapping which maps ρ(X) into itself satisfying

(I_1) $i(X) = X$
(I_2) $i(E) \subseteq E$
(I_3) $i(i(E)) = i(E)$
(I_4) $i(A \cup B) \supseteq i(A) \cup i(B)$
and $i(A \cap B) = i(A) \cap i(B)$

These properties are called interior Axioms we now prove these axioms.

Theorem. 15. Prove that (I_1), (I_2), (I_3), (I_4) holds.

Proof. (I_1) we know that

$$i(E) = [C(E^C)]^C$$

Therefore, $i(X) = [C(X^C)]^C = [C(\phi)]^C$
$= \phi^C = X$
$\Rightarrow \quad i(X) = X.$

(I_2) It is evident from the definition as i(E) is the largest open set contained in E.

(I_3) Since i(E) is open and we know that a set is open if and only if it is equal to its interior. Therefore i(E) being open, we have

$$i[i(E)] = i(E)$$

(I_4) Since $A \subset A \cup B$ and $B \subset A \cup B$

Therefore, $i(A) \subset i(A \cup B)$ and $i(B) \subset i(A \cup B)$

Which imply that

$$i(A) \cup i(B) \subset i(A \cup B)$$

To prove the second part, we note that

$$\begin{aligned} I(A \cap B) &= [C(A \cap B)^C]^C \\ &= [C(A^C \cup B^C)]^C \\ &= [C(A^C) \cup C(B^C)]^C \\ &= [C(A^C)]^C \cap [C(B^C)]^C \\ &= i(A) \cap i(B). \end{aligned}$$

Definition. The exterior of a set e is the set of all points interior to the complement of E and is denoted by e(E)

Thus $e(E) = i(E^C)$ (1)

If we replace E by E^C in (1), we get

$$e(E^C) = i(E)$$

Definition. Exterior operator C is a mapping which maps **P**(X) into itself satisfying the following exterior axioms.

(E_1) $e(\phi) = X$

(E_2) $e(E) \subseteq E^C$

(E_3) $e(E) = e[(e(E))^C]$

(E_4) $e(A \cup B) = e(A) \cap e(B)$

Theorem. 16. Prove that (1), (E_1), (E_2), (E_3), (E_4) holds.

Proof of the exterior axioms

(E_1) since $e(\phi) = i(\phi^C)$

$\Rightarrow \quad e(\phi) = i(X) = X$

$\Rightarrow \quad e(\phi) = X.$

(E_2) $e(E) = i(E^C) \subseteq E^C$ by the definition of interior.

(E_3) we note that

$$\begin{aligned} e[(e(E))^C] &= e[(i(E^C))^C] \\ &= i[(i(E^C))^C]^C \\ &= i[i(E^C)] \\ &= i(E^C) = e(E) \end{aligned}$$

(E_4)
$$\begin{aligned} e(A \cup B) &= i[(A \cup B)^C] = i[A^C \cap B^C] \\ &= i(A^C) \cap i(B^C) \\ &= e(A) \cap e(B) \end{aligned}$$

$\Rightarrow \quad e(A \cup B) = e(A) \cap e(B)$

Definition. The boundary of a subset A of a topological space X is the set of all points interior to neither A nor A^C and is denoted by b(A). Thus from the definition,

$$b(A) = [i(A) \cup i(A^C)]^C \qquad (1)$$

Now, $x \in b(A) \Rightarrow x \notin [i(A) \cup i(A^C)]$

$\Rightarrow x \notin i(A)$ and $x \notin i(A^C)$

Replacing A by A^C in (1), we get

$$b(A^C) = [i(A^C) \cup i(A)]^C \qquad (2)$$

From (1) and (2), it follows that

$$b(A) = b(A^C)$$

Using De-Morgan's law (1) yields

$$\begin{aligned} b(A) &= [i(A^C)]^C \cap [i(A)]^C \\ &= [(C(A))^C \cap [(C(A^C))^C]^C \end{aligned}$$

Since $i(A) = [C(A^C)]^C$

$$= C(A) \cap C(A^C)$$

Thus $b(A) = C(A) \cap C(A^C)$

Thus boundary of a set A is defined as follows

$$b(A) = C(A) \cap C(A^C)$$

Being the intersection of two closed sets, b(A) is closed.

Theorem 17. The boundary operator has the following properties

(1) $i(A) = A - b(A)$

(2) $C(A) = A \cup b(A)$

(3) $b(A \cup B) \subset b(A) \cup b(B)$

(4) $b(A \cap B) \subset b(A) \cup b(B)$

(5) $b(A) = b(A^C)$

(6) If A is an open set, then $b(A) = C(A) - A$

(7) $b(A) = \phi$ if and only if A is open and closed

Proof. (i) $A - b(A) = A - [C(A) \cap C(A^C)]$

$= [A - C(A)] \cup [A - C(A^C)]$

$= \phi \cup [A - C(A^C)]$ [since $A \subset C(A)$]

$= A \cap [C(A^C)]^C$

$= A \cap i(A)$ [since $i(A) = [C(A^C)]^C$]

$= i(A)$ since $i(A) \subset A$

Hence $i(A) = A - b(A)$

(2) R.H.S. $= A \cup b(A) = A \cup [C(A) \cap C(A^C)]$

$= [A \cup C(A)] \cap [A \cup C(A^C)]$

$= C(A) \cap [A \cup \{A^C \cup d(A^C)\}]$

since $C(A^C) = A^C \cup d(A^C)$

$= C(A) \cap [A \cup A^C \cup d(A^C)]$

$= C(A) \cap X$

$= C(A) =$ L. H. S.

(3) $b(A \cup B) = C(A \cup B) \cap C[(A \cup B)^C]$

$= [C(A) \cup C(B)] \cap [C(A^C \cap B^C)]$

$\subseteq [C(A) \cup C(B)] \cap [C(A^C) \cap C(B^C)]$

Since $C(A \cap B) \subset C(A) \cap C(B)$

$= [C(A) \cap (A^C) \cap C(B^C)]$

$\cup [C(B) \cap C(A^C) \cap C(B^C)]$

$= [b(A) \cap C(B^C)] \cup [b(B) \cap C(A^C)]$

$\subset b(A) \cup b(B)$

(4) $b(A \cap B) = C(A \cap B)^C \cap C(A \cap B)$

$= C(A^C \cup B^C) \cap C(A \cap B)$

$C [C(A^C) \cup C(B^C)] \cap [C(A) \cap C(B)]$

$= [C(A^C) \cap C(A) \cap C(B)] \cup [C(B^C) \cap C(A) \cap C(B)]$

$= [b(A) \cap C(B)] \cup [b(B) \cap C(A)]$

$\subset b(A) \cup b(B)$

Hence $b(A \cap B) \subset b(A) \cup b(B)$.

(5) $b(A) = C(A) \cap C(A^C)$

$\Rightarrow$ $b(A) = C(A^C) \cap C(A) = C(A) \cap C(A^C)$

$b(A) = b(A^C)$

(6) $b(A) = C(A) \cap C(A^C)$

$= C(A) \cap A^C$ since A is open $\Rightarrow A^C$ is closed $\Rightarrow C(A^C) = A^C$

$= C(A) - A$

Hence $b(A) = C(A) - A$

(7) $b(A) = [C(A) \cap C(A^C)]$

Since A is closed $\Rightarrow C(A) = A$

A is also open $\Rightarrow A^C$ is closed $\Rightarrow C(A^C) = A^C$.

Thus $b(A) = A \cap A^C = \phi$

Base and Subbase for a Topology

A topology on a set can be complicated collection of subsets of a set and it can be difficult to describe the entire collection. In most cases one describes a subcollection that generates the topology. One such collection is called a basis and another is called a subbasis.

Definition. A sub family B of T is called a base for the topology T on X iff for each point x of the space and each nbd U of x, There is a member V of B such that $x \in V \subset U$

For example, in a metric space every open set can be expressed as a union of open balls and consequently the family of all open balls is a base for the topology induced by the metric.

The following is a simple characterization of basis and is frequently used as a definition.
Definition. A subfamily **B** of a topology T is a base for T if and only if each member of T is the union of members of B.

To prove that this second definition is equivalent to first one, suppose that B is a base for the topology T and that $U \in T$.

Let V be the union of all members of B which are subsets of U and suppose that $x \in U$. Then there is W in B such that $x \in W \subset U$ and consequently $x \in V$ and since V is surely a subset of U, V = U. So the first definition ⇒ second.

Conversely let B be a subfamily of T and each member of T is the union of members of B. If $U \in T$, then U is the union of members of subfamily B and for each x in U, there is a V in B such that $x \in V \subset U$. Consequently **B** is a base for T.

Example. (1) The collection **B** of all open intervals is a basis for the usual topology on R.
(2) The collection B of all open disks is a basis for the usual topology on the plane
(3) If X is a set, then
$B = \{\{x\} [x \in X\}$ is a basis for the discrete topology on X.
(4) Let (X, d) be a metric space, then the family
$B = \{B(x, \in) ; x \in X \text{ and } \in > 0\}$ is a basis for the topology generated by d.

Remark. The following example shows that there is a subset of ρ(X) that is not a basis for a topology on X. For example let X = {1, 2, 3} and B = {1, 2}, {2, 3}, X}.
Then B is not a basis for a topology on X.

Analysis. Suppose *B* is a basis for a topology T on X. Then by definition of basis $B \subseteq T$. Hence $\{1, 2\}, \{2, 3\} \in T$ and so $\{1, 2\} \cap \{2, 3\} = \{2\} \in T$. But $\{2\} \neq \phi$ and there is no subcollection B' of *B* such that $\{2\} = \cup \{B ; B \in B'\}$. Hence *B* is not a basis for T.

Remark. The following example provides a necessary and sufficient condition for a subset of P(X) to be a basis for a topology on X.

Theorem. 18. A family **B** of sets is a base for some topology for the set $X = U\{B ; B \in \mathbf{B}\}$ if and only if for every $B_1, B_2 \in B$ and $x \in B_1 \cap B_2$, there exists a $B \in \mathbf{B}'$ such that $x \in B \subseteq B_1 \cap B_2$, that is the intersection of any two members of *B* is a union of members of B.

Proof. Suppose that *B* is a base for a topology T, $B_1, B_2 \in \mathbf{B}$. Then by definition every member of T is the union of members of **B**.

Since **B** is a subfamily of T, the members of base **B** are open sets and the intersection of two open sets is an open set. Thus $B_1 \cap B_2$ is an open set and therefore is a member of T. Since every member of T is a union of members of **B**, it follows that $B_1 \cap B_2$ is a union of members of **B**.

Conversely, let **B** be a family of sets satisfying the condition of the theorem and let T be the family of all union of members of **B**. We shall show that T is a topology for X with base **B**. We check that all the axioms for a topology are satisfied.

(0_1) The whole set X was defined to be the union of all members of **B** and so is a member of T. Also the empty set is the union of the empty collection of members of **B** implies that $\phi \in T$.

(0_2) Since each member of T is a union of members of **B**, the union of any number of members of T is a union of members of **B** and so belongs to T.

(0_3) Suppose that $G_1, G_2 \in T$. If $x \in G_1 \cap G_2$, then $x \in G_1$ and $x \in G_2$ by the definition of T, G_1 and G_2 are union of members of **B** and so there exists sets B_1 and B_2 belonging to **B** such that $x \in B_1 \subseteq G_1$ and $x \in B_2 \subseteq G_2$

Now $x \in B_1 \cap B_2$ and so by hypothesis, there exists a $B \in$ **B** such that $x \in B \subseteq B_1 \cap B_2$.
Since $B_1 \subseteq G_1$, $B_2 \subseteq G_2$ $\Rightarrow B_1 \cap B_2 \subseteq G_1 \cap G_2$

We have thus shown that every point of $G_1 \cap G_2$ is contained in a member of **B** which is itself contained in $G_1 \cap G_2$. Thus $G_1 \cap G_2$ is the union of members of **B** and so belongs to T.

Hence T is a topology for the set X with base *B.*

Remark. Note that if X = {1, 2, 3} and **B** = {{2}, {1, 2}, {2, 3}}, then **B** satisfies the conditions for a base. Therefore it is a base for the topology T = {ϕ, {2}, {1, 2}, {2, 3}, X} on X.

Remark. If **B** is a basis for a topology T on a set X, then T is the topology generated by **B**.
Definition. Let $\mathbf{B}_1$ and $\mathbf{B}_2$ be two basis for the topologies T_1 and T_2 on a set X. Then $\mathbf{B}_1$ and $\mathbf{B}_2$ are equivalent provided that $T_1 = T_2$.

The collection of open disks and collection of open squares are equivalent bases for topologies in the plane. In each case the topology generated by the base is the usual topology.

Remark. The following theorem gives a characterization for a topology T′ to be finer than a topology T in terms of bases for T and T′.

Theorem.19. Let T and T′ be topologies on a set X and let **B** and **B**′ be bases for T and T′ respectively. Then the following are equivalent.

(a) T′ is finer than T

(b) For each $x \in X$ and each B ∈ **B** such that $x \in B$, there is a member B′ of **B′** such that $x \in B'$ and $B' \subseteq B$.

Proof. (a) ⇒(b) Suppose T′ is finer than T . Let $x \in X$ and let B ∈**B** such that $x \in B$. Since $B \in T$ and T′ is finer than T. $B \in T'$. Since T′ is generated by **B′** there is a member B′ of **B**′ such that $x \in B'$ and $B' \subseteq B$.
(b) ⇒ (a). Let U ∈T and let $x \in \cup$. Since T is generated by **B**, there is a member B of **B** such that $x \in B$ and $B \subseteq U$. By condition (b), there is a member B′ of *B*′ such that $x \in B'$ and $B' \subseteq B$. Since $B' \subseteq B$ and $B \subseteq U$, $B' \subseteq U$. Therefore U is the union of the members of a subcollection of **B**′ and hence $U \in T'$.

Theorem. 20. If **S** is any non-empty family of sets, the family of all finite intersections of members of S is the base for a topology for the set

$$X = \cup\{S/S \in \mathbf{S}\} .$$

Proof. If **S** is a family of sets and let **B** be the family of finite intersections of members of **S**. Then the intersection of two members of B is again a member of B and then applying the result

"A family **B** of sets is a base for some topology for the set $X = \cup\{B ; B \in \mathbf{B}\}$ if and only for every $B_1, B_2 \in B$ and every $x \in B_1 \cap B_2$, there exists a $B \in B$ such that $x \in \mathbf{B} \subseteq B_1 \cap B_2$, that is the intersection of any two members of **B** is the union of members of **B**. hence **B** is a base for the topology.

Definition. Let (X, T) be a topological space. A collection B_* of open subsets of X is called a subbase for a topology T if and only if finite intersections of members of B_* form a base for T.

Example 1. Let R be the set of real numbers and let T be the usual topology. If B_* contains open intervals of the form $(-\infty, b)$ or (a, ∞) where a and b are either real or rationals, then B_* is a subbase for the topology since

$(-\infty, a) \cap (b, \infty) = (a, b)$ i.e. an open interval and we know that the base of usual topology is the collection of all open intervals.

Theorem. 21. Let X be a set and **B** be a collection of subsets of X such that $X = U\{S ; S \in \mathbf{B}\}$. Then there is a unique topology T on X such that **B** is a subbase for T.

Proof. Let $\mathbf{B}' = \{B ; \rho(X); B$ is the intersection of a finite number of members of $\mathbf{B}\}$

Let $T = \{U; \rho(X); U = \phi$, or there is a subcollection $\mathbf{B''}$ of B' such that $U = U\{B ; B \in B''\}\}$

It is sufficient to prove that T is a topology on X.

(0_1) By definition $\phi \in T$ and since

$$X = \cup\{S ; S \in B\}, X \in T.$$

(0_2) Let $U_x \in T$ for each α in the index set $\wedge$. Then there is a subcollection $\mathbf{B}_x$ of **B** such that

$$U_x = U\{B ; B \in \mathbf{B}_x\}$$

Hence $U\{U_x; \alpha \in \wedge\} = \bigcup_{\alpha \in \wedge}\{\bigcup_{B \in B_x} B\}$

and so $U\{U_\alpha ; \alpha \in \wedge\} \in T$.

(0_3) Suppose U_1, U_2 eT and $x \in U_1 \cap U_2$. Then there exists $B_1, B_2, \in \mathbf{B}'$ such that $x \in B_1 \cap B_2$, $B_1 \subseteq U_1$, $B_2 \subseteq U_2$. Since each of B_1 and B_2 is the intersection of a finite number of members of B. Therefore, there is a subcollection B'' of B' such that $U_1 \cap U_2 = U\{B ; B \in B''\}$ and hence

$U_1 \cap U_2 \in T$. Therefore T is a topology and it is clear that T is the unique topology that has **B** as a subbase.

Remark. (1) The topology generated above is called the topology generated by **B**. Thus one advantage of the concept of subbasis is that we can define a topology on a set X by simply choosing an arbitrary collection of subsets of X whose union is X.

Remark. (2) Let

$$X = \{a, b, c, d\} \text{ and}$$

$$T = \{\phi, \{a\}, \{a, c\}, \{a, d\} \{a, c, d\}, X\}$$

Then $B_* = \{\{a, c\}, \{a, d\}\}$ is a subbase for T since the family **B** of finite intersections of $\boldsymbol{B}_*$ is given by $\mathbf{B} = \{\{a\}, \{a, c\}, \{a, d\}, X\}$ which is a base for T.

Subspace Topology or Relative Topology

This topology was introduced by Hausdorff.

Definition. Let X^* be a subset of a topological space (X, T). Then subspace, induced or relative topology for X^* is the collection T^* of all sets which are intersections of X^* with members of T. (X^*, T^*) is called a subspace for (X, T) iff T^* is the induced topology. The sets which are open with respect to the subspace (X^*, T^*) will be called relatively open sets.

We now show that T^* as defined above is a topology for X^*

Theorem. 22. Prove that T^* is a topology for X^*.

Proof. To prove T^* is a topology for X^* we have to show that all the axioms for a topology are satisfied.

(0_1) since $X^* = X^* \cap X$ where $X \in T$

$\Rightarrow \quad X^* \in T^*$

Also $\phi = \phi \cap X$, where $X \in T$

$\Rightarrow \quad \phi \in T^*$

(0_2) Let $G_\lambda{}^* \in T^*$ for all λ. Then there exist sets G_λ belonging to T such that

$$G_\lambda{}^* = X^* \cap G_\lambda \text{ for each } \lambda.$$

Since $\bigcup_\lambda G_\lambda$ belongs to T and

$$\bigcup_\lambda G_\lambda{}^* = \bigcup_\lambda (X^* \cap G_\lambda) = X^* \cap (\bigcup_\lambda G_\lambda)$$

we have $\bigcup_\lambda G_\lambda{}^* \in T^*$.

(0_3) Now suppose that $G_1{}^*$ and $G_2{}^*$ belong to T^*, there must exist set G_1 and G_2 belonging to T such that

$$G_1{}^* = X^* \cap G_1 \text{ and } G_2{}^* = X^* \cap G_2.$$

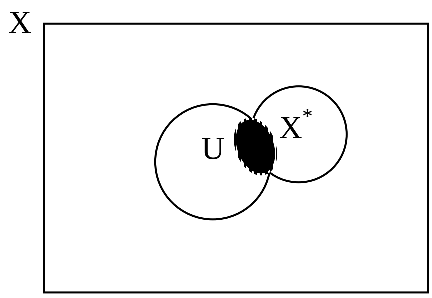
Since $G_1 \cap G_2$ also belongs to T and

$$G_1{}^* \cap G_2{}^* = (X^* \cap G_1) \cap (X^* \cap G_2)$$
$$= X^* \cap (G_1 \cap G_2)$$

It follows that $G_1{}^* \cap G_2{}^* \in T^*$

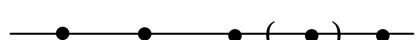
Hence T^* is a topology for X^*.

Remark. (1) Open sets in X^* are those sets written in the form $U \cap X^*$ where U is open in X. In the following fig, the shaded region represents an open set in X^*.



Example. (1) Consider the real line R as a subset of the plane R^2, by identifying a point $x \in R$ with the point $(x, 0) \in R^2$. Let T be the usual topology induced by the usual Euclidean metric on R^2. Then the topology induced by T on R is precisely the usual topology on R. Thus the subspace topology on R (regarded as a subset of R^2) is the usual topology on R.

(2) If T is the usual topology on R, then the subspace topology on the integers is the discrete topology



(3) If T is the discrete topology on a set X and $A \subseteq X$, then T_A is the discrete topology on A and the members of T_A are intersections of members of T with A .

(4) If T is the trivial topology on a set X and $A \subseteq X$, then T_A is the trivial topology on A.

(5) If (X, T) is a topological space and $A \subseteq B \subseteq X$, then the subspace topology on A determined by T is the same as the subspace topology on A Determined by T_B

Theorem. 23. Let (X, T) be a topological space and A be a subset of X. Let **B** be a basis for T. Then $B_A = \{B \cap A ; B \in \mathbf{B}\}$ is a basis for the subspace topology on A.

Proof. Let $U \in T$ and let $x \in U$. Then since **B** is a base for T, by definition, there is a member $B \in I$ such that $x \in B$ and $B \subseteq U$. Therefore $x \in B \cap A \subseteq U \cap A$. Hence by the result "let (X, T) be a topological space and suppose **B** is a subcollection of T such that for each $x \in X$ and each member U of T such that $x \in \cup$, There is an element B of **B** such that $x \in B$ and $B \subseteq U$. Then **B** is a basis for T." We get, $\mathbf{B}_A$ is the basis for the subspace topology on A.

Theorem. 24. Let (X, T) be a topological space and A be an open subset of X and T_A is the induced or relative topology on A determined by T. If $U \in T_A$, then $U \in T$.

Proof. Let $U \in T_A$, then there is an open subset V of X such that $U = A \cap V$. Since A and V are open in X, so is $A \cap V = U$. Hence $U \in T$.

Continuous Functions

Definition. A function f mapping a topological space (X, T) into a topological space (X^*, T^*) is said to be continuous at a point $x \in X$ if and only if for every open set G^* containing f(x), there is an open set G containing x such that

$$f(G) \subseteq G^*$$

We say that f is continuous on a set $E \subseteq X$ if and only if it is continuous at each point of E.

Example. (1) Let X = {1, 2, 3, 4} and

$$T = \{\phi, (1), (2), (1, 2), (2, 3, 4), X\}.$$

Define a mapping $f : X \to X$ by

$$f(1) = 2, f(2) = 4, f(3) = 2 \text{ and } f(4) = 3.$$

Show that (1) *f* is not continuous at 3 and f is continuous at 4 .

Analysis. The open sets containing $f(3) = 2$ are (2), (1, 2), (2, 3, 4) and X.

We see that

$$f^{-1}(2) = (1, 3), f^{-1}(1, 2) = (1, 3),$$
$$f^{-1}(2, 3, 4) = (1, 3, 4, 2) \text{ and}$$
$$f^{-1}(X) = (1, 2, 3, 4) = X \text{ itself.}$$

But we see that (1, 3) is not open as it does not belong to T. Hence *f* is not continuous at 3

Now we check continuity at the point 4. The open sets containing $f(4) = 3$ are the sets (2, 3, 4) and X. Now

$$f^{-1}(2, 3, 4) = (1, 3, 4, 2) = X \text{ and}$$
$$f^{-1}(X) = (1, 2, 3, 4) = X$$

is open. Hence *f* is continuous at 4.

(2) Let (X, T) be a discrete topological space and (Y, U) be any topological space. Then every function $f : X \to Y$ is necessarily continuous on X. For $f^{-1}(G)$, where G is open in Y is a subset of X and so open.

(3) Let X = (x, y, z) and T = {ϕ, (x), (y), (x, y), X} so that (X, T) is a topological space. Define $f : X \to X$ by $f(x) = x$, $f(y) = z$ and $f(z) = y$. Then by considering inverse images of the sets of T, we find that *f* is not continuous at x.

(4) Let T denote the usual topology on R and define $T : (R, T) \to (R, T)$ by $f(x) = x^3$. The collection **B** of all open intervals is a basis for T. Let (a, b) be an open interval. Then $f^{-1}(a, b) = (\sqrt[3]{a}, \sqrt[3]{b})$ is open so inverse image of every open set is open (equivalent condition for continuity of f, proved below). Hence f is continuous.

(5) Let (X, T) be a topological space and a singleton {a} be T-open. Suppose that (Y, T^*) is another topological space. Then the function $f : X \to Y$ is continuous at $a \in X$.

Remark. Composition of two continuous mappings is continuous.

Let the functions f and g be defined as follows

If $f : (X, T) \to (Y, T^*)$ and $g : (Y, T^*) \to (Z, T^{**})$

If f and g are continuous, then gof is also continuous. We note that if G^{**} is open in Z, then

$$(gof)^{-1} [G^{**}] = (f^{-1} o g^{-1}) (G^{**})$$
$$= f^{-1} [g^{-1}(G^{**})]$$

Since g is continuous, $g^{-1} (G^{**})$ is open in Y, and then since f is continuous, $f^{-1} [g^{-1}(G^{**})]$ is open in X. Therefore gof is continuous on X.

Theorem. 25. Let (X, T) and (X^*, T^*) be topological spaces and $f : X \to X^*$, then the following conditions are each equivalent to the continuity of f on X.

(1) The inverse image of every open set in X^* is an open set in X.

(2) The inverse image of every closed set in X^* is a closed set in X.

(3) $f(C(E)) \subseteq C^*(f(E))$ for every $E \subseteq X$.

Proof. (1) Suppose that f is continuous on X and G^* is an open set in X^*. If x is any point of $f^{-1}(G^*)$, and f is continuous at x, so there must exist an open set G containing x such that $f(G) \subseteq G^*$. Thus $G \subseteq f^{-1}(G^*)$ and hence $f^{-1}(G^*)$ is an open set in X.

Conversely if the inverse images of open sets are open, we may choose the set $f^{-1}(G^*)$ as the open set G required in the above definition.

(2) Suppose that f is continuous and G^* is closed in X^*. Then

$$f^{-1}(X^* - G^*) = X - f^{-1}(G^*)$$

Since f is continuous and $X^* - G^*$ is open, it follows that $X - f^{-1}(G^*)$ is open. Consequently $f^{-1}(G^*)$ is closed.

Conversely let G^{**} be an open subset of X^*, then

$$f^{-1}[X^* - G^{**}] = X - f^{-1} (G^{**}]$$

Since the left hand side is closed, it follows that $X - f^{-1}(G^{**})$ is closed which implies that $f^{-1}(G^{**})$ is open. Hence f is continuous.

(3) Suppose that f is continuous on X and E is a subset of X. Since $E \subseteq f^{-1} [f(E)]$ for any function, $E \subseteq f^{-1}[C^*(f(E))]$

But $C^*(f(E))$ is closed and therefore $f^{-1}[C^*(f(E))]$ is closed. Moreover this contains E. Therefore,

$$C(E) \subseteq f^{-1} [C^*(f(E))] \text{ and so}$$
$$f(C(E)) \subseteq f[f^{-1}(C^*(f(E))]$$
$$\subseteq C^*[f(E)]$$

conversely suppose that

$$f [C(E)] \subseteq C^*[f(E)] \qquad (1)$$

for all subsets $E \subseteq X$. let F^* be a closed set in X^*. Choose

$$E = f^{-1}(F^*), \text{ we obtain}$$
$$f(C(E)] = f[C(f^{-1}(F^*))]$$
$$\subseteq C^* [f(f^{-1}(F^*))] \qquad \text{[by (1)]}$$
$$\subseteq C^*(F^*) = F^*$$

since F^* is closed.

$\Rightarrow \quad C(E) \subseteq f^{-1}(F^*)$

$\Rightarrow \quad C[f^{-1}(F^*) \subseteq f^{-1}(F^*)$

$\Rightarrow \quad f^{-1}(F^*)$ is a closed set.

and so the inverse image of every closed set is a closed set.

Theorem. 26. Let $[T_\lambda ; \lambda \in \wedge]$ be an arbitrary collection of topologies on X and let (Y, V) be a topological space. If the mapping $f : X \to Y$ is T_λ–V continuous for all $\lambda \in \wedge$, then f is continuous with respect to the intersection topology

$$T = \cap\{T_\lambda ; \lambda \in \wedge\}$$

Proof. Let G be an open set in Y. Since f is T_λ–V continuous, therefore f^{-1} (G) is open in X that is $f^{-1}(G) \in T_\lambda$ for all $\lambda \in \wedge$

$\Rightarrow \quad f^{-1}(G) \in \cap\{T_\lambda ; \lambda \in \wedge\}$

$\Rightarrow \quad f^{-1}(G) \in T.$

$\Rightarrow \quad f$ is continuous with respect to T.

Theorem. 27. If f is a continuous mapping of (X, T) into (X^*, T^*), then f maps every connected subset of X onto a connected subset of X^*.

Proof. Let E be a connected subset of X and suppose that $E^* = f(E)$ is not connected. Then there must exist some separation $E^* = A^*/B^*$ where A^* and B^* are non-empty, disjoint, open subsets of E^*. Then f being continuous, both $f^{-1}(A^*)$ and $f^{-1}(B^*)$ are open in X. Clearly $A = f^{-1}(A^*) \cap E$ and $B = f^{-1}(B^*) \cap E$ are non-empty disjoint sets which are both open subsets of E. Thus E must have the separation E = A/B and so is not connected. Hence we get a contradiction. Thus E^* is connected.

Remark. Any continuous image of a compact topological space is compact

Homemorphism

The first systematic treatment of continuity and homemorphism was given by Hausdorff.

Definition. Let X and Y be two topological spaces and f be a mapping from X into Y. Then $f : X \to Y$ is called an open mapping if and only if $f(G)$ is open in Y whenever G is open in X.

Thus a mapping is open if and only if the image of every open set is an open set. Such mappings are also called interior mappings.

Similarly a mapping is closed if and only if image of every closed set is a closed set.

Remark. (1) Since there is no general containing relation between $f(E^C)$ and $[f(E)]^C$. We find that an open (closed) mapping need not be closed (open), even if continuous. For example let (X, T) be any topological space and let (X^*, T^*) be the space for which $X^* = [a, b, c]$ and

$T^* = [\phi, \{a\}, \{a, c\}, X^*\}$. The transformation which takes each point of X into the point a is continuous open map which is not closed. The transformation which takes each point of X into b is a continuous closed map which is not open.

Remark. (2) As we know that continuity of a function does not really depend upon the topology on the entire co-domain but rather on the relative topology on the range. This is no longer true for openness of a function. Consider the function $f : R \to R^2$ defined by $f(x) = (x, 0)$ for $x \in R$. Then f is not open with respect to the usual topologies on R and R^2. The range of f is the x-axis of R^2. If we regard f as a function from R to the x-axis (with relative topology), then it is open. Similarly, a restriction of an open function need not be open. These remarks also apply for closed functions.

Remark. (3) In order to show that a function is open, it is sufficient to show that it takes all members of a base for the domain space to open subsets of the co-domain. Using this fact, it it easy to show that projection functions from a product space to the co-ordinate spaces are open. It appears that they are also closed. But this is not the case. Consider the projection $\pi_1 : R^2 \to R$,

$\pi_1(x, y) = x$ (x, y∈R). Let H be the set $\{(x, y) \in R^2, xy = 1\}$. Then H is a closed subset of R^2 as its complement is open. However $\pi_1(H)$ is the set of all non-zero real numbers and it is not a closed subset of R.

Remark. (4) The following examples show that there is no direct relation between openness and continuity of a function.

(i) Let T denote the discrete topology on R and let μ denote the usual topology on R. Then $f : (R, T) \to (R, \mu)$ defined by $f(x) = x$ for each x∈R is continuous because if U∈μ, then $f^{-1}(U) \in T$. However f is not open because $\{1\} \in T$ whereas $\{1\} \notin \mu$.

(ii) The function $g : (R, \mu) \to (R, T)$ defined by $g(x) = x$ for all x∈R is open but not continuous.

Theorem 28. A mapping f of X into X^* is open if and only if $f[i(E)] \subseteq i^*[f(E)]$ for every $E \subseteq X$.

Proof. Suppose f is open and $E \subseteq X$. Since i(E) is an open set and f is an open mapping, $f(i(E))$ is an open set in X^*. Since $i(E) \subseteq E$, $f(i(E)) \subseteq f(E)$. Thus $f(i(E))$ is an open set contained in $f(E)$ and hence

$$f(i(E)) \subseteq i^*(f(E))$$

Conversely if G is an open set in X and $f(i(E)) \subseteq i^*(f(E))$ for all $E \subseteq X$, then $f(G) = f(i(G)) \subseteq i^*(f(G))$ and so $f(G)$ is an open set in X^*. Hence f is an open mapping.

Definition. Let X and Y be two topological spaces. Then a mapping $f : X \to Y$ is called a **homemorphism** if and only if it is bijective, continuous and open. Equivalently $f : X \to Y$ is a homemorphism if and only if it is bijective and bi-continuous. (By bi-continuous we mean that both f and f^{-1} are continuous)

Two topological spaces X and Y are said to be homeomorphic if there exists a homemorphism of X onto Y and in this case Y is called a homemorphic image of X.

A property of sets which is preserved by homemorphisms is called a **topological property**. The properties of a set being open, closed, connected, compact and dense in itself are topological properties but distances and angles are not the topological properties because they can be altered by suitable non-tearing deformations.

Remark. (1) If $f : X \to Y$ is a homemorphism, then X and Y are equivalent (as sets) since f is bijective. Also f and f^{-1} preserve open sets we may regard X and Y as equivalent topological spaces that is they may be thought of as indistinguishable from the topological point of view :

Remark. (2) Translations from $f : R \to R$ defined by $f(x) = x + a$ are homemorphisms. Completeness of metric spaces is not a topological property i.e. a complete metric space can be homemorphic to an incomplete metric space.

Theorem. 29. Metrizabiliy is a topological property.

Proof. Let (X, d) be a metric space and (Y, V) be a topological space and suppose $f : X \to Y$ is a homemorphism. Define $\rho : Y \times Y \to R$ by $\rho((y_1, y_2)) = d(f^{-1}(y_1)\, f^{-1}(y_2))$

We see that ρ is a metric on Y. Also the topology induced by ρ is T. Thus the property of metrizability is preserved under homemorphism.

Definition. A subset E of a topological space is called isolated if and only if no point of E is a limit point of E that is if

$$E \cap d(E) = \phi.$$

Theorem. 30. If f is a homemorphism of X onto X^*, then f maps every isolated subset of X onto an isolated subset of X^*.

Proof. Suppose E is an isolated subset of X and let $x^* \in f(E)$, There must then exist a point $x \in E$ such that $f(x) = x^*$. Since E is isolated, $x \notin d(E)$ and so there must exist an open set G containing x such that $E \cap G - \{x\} = \phi$. But f is a homemorphism and so $f(G)$ is an open set in X^* which contain $f(x) = x^*$. From the fact that f is one to one, it follows that

$$f(E \cap G - \{x\}) = f(E) \cap f(G) -(x^*\} = \phi$$

Thus $x^* \notin d(f(E))$ and so $f(E)$ must be isolated.

Theorem. 31. Let (X, T) and (X^*, T^*) be two topological spaces. A one to one mapping f of X onto X^* is homemorphism if

$$f(C(E)) = C(f(E)) \text{ for every } E \subseteq X.$$

Proof. Suppose first that f is a homemorphism and let $E \subseteq X$ and $G = f(E)$. Then

$$f^{-1}(G) = f^{-1}(f(E)) = E.$$

Since f is continuous

$$f(C(E)) \subset C(f(E)) \quad (1)$$

Moreover, since f^{-1} is continuous.

$$f^{-1} [C(G)] \subset C[f^{-1}(G)]$$

$$\Rightarrow \quad f^{-1} (C(f(E))] \subset C[f^{-1}(f(E))]$$

$$\Rightarrow \quad f^{-1} [C(f(E))] \subset C(E)$$

$$\Rightarrow \quad C(f(E)) \subset f(C(E)) \quad (2)$$

From (1 and (2), we have

$$f(C(E)) \subset C(f(E)) \subset f(C(E))$$

which proves the first part.

Conversely let us suppose that

$f(C(E)) = C(f(E))$. We shall show that f is homemorphism. Since f is bijective, it is sufficient to show that f and f^{-1} are continuous. We are given that

$$f(C(E)) = C(f(E))$$

$$\Rightarrow \quad f[C(E)] \subseteq C[f(E)]$$

$$\Rightarrow \quad f \text{ is continuous.}$$

Also $C[f(E)] \subseteq f[C(E)]$

Let $G = f(E)$. Then

$$f^{-1}(G) = f^{-1}(f(E)) = E.$$

Therefore,

$$C[f(f^{-1}(G))] \subset f[C(f^{-1}(G))]$$

$$\Rightarrow \quad f[f^{-1} (C(G))] \subset f[C(f^{-1}(G))]$$

$$\Rightarrow \quad f^{-1} [C(G)] \subseteq C(f^{-1} (G)]$$

which proves that f^{-1} is continuous.

2

CONNECTEDNESS

Connectedness was defined for bounded, closed subsets of R^n by Cantor in 1883, but this definition is not suitable for general topological spaces. In 1892, Camille Jordan (1838-1922) gave a different definition of connectedness for bounded closed subsets of $R^{n.}$ Then in 1911, N. J. Lennes extended Jordan's definition to abstract spaces Hausdorff's Grundzuge der Mengenlehre was the first systematic study of connectedness.

Connectedness represents an extension of the idea that an interval is in one piece. Thus from the intuitive point of view, a connected space is a topological space which consists of a single piece. This property is perhaps to the simplest which a topological space may have and yet it is one of the most important applications of topology to analysis and geometry.

On the real line, for instance intervals are connected subspaces and as we shall see they are the only connected subspaces. Connectedness is also a basic notion in complex analysis, for the regions on which analytic functions are studied are generally taken to be connected open subspace of the complex plane.

In the portion of topology which deals with continuous curves and their properties, connectedness is of great significance, for whatever else a continuous curve may be, it is certainly a connected topological space.

Spaces which are not connected are also interesting. One of the outstanding characteristics of the cantor set is the extreme degree in which it fails to be connected. Much the same is true of the subspace of the real line which consists of rational numbers. These spaces are badly disconnected.

Our purpose in this chapter is to convert these rather vague notions into precise mathematical ideas and also to establish the fundamental facts in the theory of connectedness which rests upon them.

Connected Spaces

Definition. A topological space is connected if it can not be expressed as the union of two non-empty disjoint open sets.

An equivalent formulation of this definition is that a set is said to be connected if and only if it has no separation.

Definition. Two subsets A and B form a separation or partition of a set E in a topological space (X, T) if and only if

(i) $E = A \cup B$
(ii) A and B are non-empty
(iii) $A \cap B = \phi$
(iv) Neither A contains a limit point of B nor B contains a limit of A.

If A and B form a separation of E, then we write $E = A/B$.

Remark. The requirements that A and B are disjoint sets and neither contains a limit point of the other may be combined in the formula

$$[A \cap C(B)] \cup [B \cap C(A)] = \phi$$

which is often called the Hausdorff-Lennes separation condition and such subsets are called separated.

Thus a set is said to be connected if and only if whenever it is written as the union of two non-empty disjoint sets, at least one of them must contain a limit point of the other. We now give examples of some spaces that are connected.

Example (1) Empty set and every set consisting of one point is a connected set.
Example (2) In the case of discrete topological space, only ϕ and the singleton are connected sets.
Example (3) Let X be a non-empty set and T be the trivial topology on X. Then (X, T) is connected.
Example (4) Every open interval is connected.
Example (5) Let T be the usual topology on R. Then (R, T) is connected.

Analysis. The proof is by contradiction. Suppose (R, T) is not connected. Then there exist disjoint open sets U and V such that $R = U \cup V$. Since $U = R - V$ and $V = R - U$, U and V are also closed. Let $a \in U$ and $b \in V$. We may assume w. l. o. g. that $a < b$. Let $W = U \cap [a, b]$. Since W is bounded, it has a least upper bound c. Since W is closed. $c \in W$. Since $W \cap V = \phi$, $c \neq b$. Also c is the least upper bound of W, $(c, b] \subseteq V$. Therefore $c \in \overline{V}$, but V is closed so $c \in V$. Therefore $c \in U \cap V$. This is contradiction because

$$U \cap V = \phi.$$

Theorem. 32. A topological space (X, T) is connected if and only if it can not be expressed as the union of two non-empty sets that are separated in X.
Proof. Suppose X is not connected. Then there are non-empty, disjoint open sets U and V such that $X = U \cup V$. Then U and V are closed so that $\overline{U} \cap V = U \cap V = \phi$ and $U \cap \overline{V} = U \cap V = \phi$. Therefore U and V are separated in X.

Suppose now that there are non-empty subsets A and B such that $X = A \cup B$ and $\overline{A} \cap B = A \cap \overline{B} = \phi$. Since $X = A \cup B$ and $\overline{A} \cap B = \phi$

$\Rightarrow$ $\overline{A} \subseteq A$ so that A is closed. Similarly B is closed. Therefore A and B are also open and hence X is not connected.

Remark. If A and B form a separation of the topological space (X, T), then A and B are both open and closed.

Theorem. 33. A topological space (X, T) is connected if and only if no non-empty proper subset of X is both open and closed.

Proof. Suppose X is not connected. Then there are non-empty disjoint open sets U and V such that $X = U \cup V$. Thus U is a non-empty proper subset of X that is both open and closed.

Suppose X has a non-empty proper subset U that is both open and closed. Then U and $X - U$ are non-empty disjoint open sets whose union is X. Therefore X is not connected.

The following result provides useful ways of formulating the definition of connectedness for subspaces of a topological space.

Theorem. 34. Let (X, T) be a topological space and let $A \subseteq X$. Then the following conditions are equivalent.

(a) The subspace (A, T_A) is connected.

(b) The set A can not be expressed as the union of two non-empty sets that are separated in X.

(c) There do not exist U, V $\in$T such that $U \cap A \neq \phi$, $U \cap V \cap A = \phi$ and $A \subseteq U \cup V$.

Proof. First we prove (a) $\Rightarrow$ b. In each case, we prove by contradiction.

(a) $\Rightarrow$ (b). Suppose the subspace (A, T_A) is connected and U and V are non-empty sets such that $A = U \cup V$ and $\overline{U} \cap V = U \cap \overline{V} = \phi$.

Then U and V are separated in A, so by Theorem. 32, A is not connected

(b) $\Rightarrow$ (c). Suppose there exist U, V $\in$T such that $U \cap A \neq \phi$, $V \cap A \neq \phi$, $U \cap V \cap A = \phi$ and $A \subseteq U \cup V$.

Then $U \cap A$ and $V \cap A$ are non-empty sets that are separated in X and A = $(U \cap A)$ U $(V \cap A)$

(c) $\Rightarrow$ (a). Suppose (A, T_A) is not connected.

Then there exist U′, V′, $\in T_A$ such that $U' \neq \phi \neq V'$ and $U' \cap V' = \phi$ and $A = U' \cup V'$. Thus there exist U, V$\in$T such that $U' = A \cap U$ and $V' = A \cap V$.

It is clear that $U \cap A \neq \phi$, $V \cap A \neq \phi$, $U \cap V \cap A = \phi$ and $A \subseteq U \cup V$.

Theorem. 35. A subspace of the real line R is connected if and only if it is an interval. In particular, R is connected.

Proof. Let X be a subspace of R. We first prove that if X is connected, then it is an interval. We do this by assuming that X is not an interval and by using this assumption to show that X is not connected. To say that X is not an interval is to say that there exist real numbers x, y, z such that $x < y < z$, x and z are in X, y is not in R. Thus

$X = \{X \cap (-\infty, y)\}$ U$\{X \cap (y, +\infty)\}$ is a disconnection of X, so X is disconnected.

Now we prove the second part. We show that if X is an interval, then it is necessarily connected. We first assume that X is disconnected. Let $X = A \cup B$ be a disconnection of X. Since A and B are non-empty, we can choose a point x in A and a point z in B. A and B are disjoint, so $x \neq z$, w. l. o. g. we assume that $x < z$. Since X is an interval, $[x, z] \subseteq X$ and each point in {x, z} is in either A or B. We now define y by

$$y = \sup ([x, z] \cap A).$$

It is clear that $x \leq y \leq z$, so y is in X. Since A is closed in X, the definition of y shows that y is in A. From this we conclude that $y < z$. Again by the definition of y, $y + \in$ is in B for every $\in > 0$ such that $y + \in \leq z$, and since B is closed in X, y is in B. We have proved that y is in both A and B, which contradicts our assumption that these sets are disjoint.

Theorem. 36. Any continuous image of connected space is connected.

Proof. Let $f : X \rightarrow Y$ be a continuous mapping of a connected space X into an arbitrary topological space Y. We must show that $f(X)$ is connected as a subspace of Y. Assume that $f(X)$ is disconnected. This means that there exist two open subsets G and H of Y whose union contains $f(X)$ and whose intersections with $f(X)$ are disjoint and non-empty. This implies that $X = f^{-1}(G) \cup f^{-1}(H)$ is a disconnection of X, which contradicts the connectedness of X.

Remark. Thus from the above theorem, it follows that property of connectedness is preserved by continuous mappings.

Theorem 37. If E is a subset of a subspace (X^*, T^*) of a topological space (X, T), then E is T^* connected if and only if it is T-connected.

Proof. In order to have a separation of E with respect to either topology, we must be able to write E as the union of two non-empty disjoint sets. If A and B are two non-empty disjoint sets, whose union is E, then

$$A, B \subseteq X^* \subseteq X$$

Calculating with the Hausdorff – Lennes separation condition, we find that

$$\begin{aligned}[A \cap C^*(B)] \cup [C^*(A) \cap B] &= [A \cap \{X^* \cap C(B)] \cup [\{X^* \cap C(A)\} \cap B] \\ &= [A \cap X^* \cap C(B)] \cup [X^* \cap C(A) \cap B] \\ &= [A \cap C(B)] \cup [B \cap C(A)]\end{aligned}$$

Thus if the condition is satisfied with respect to one topology, then it is satisfied with respect to the other.

Remark. The above theorem leads us to say that connectedness is an absolute property of sets that it does not depend on the space in which the set is contained except that the topology, of course must be the relative topology.

Theorem. 38. If E is a connected subset of a topological space (X, T), which has a separation X = A/B, then either $E \subseteq A$ or $E \subseteq B$.

Proof. Clearly

$$\begin{aligned}E = E \cap X &= E \cap (A \cup B) \\ &= (E \cap A) \cup (E \cap B)\end{aligned}$$

Since X = A/B, we have

$$\begin{aligned}&[(E \cap A) \cup C(E \cap B)] \cup [C(E \cap A) \cap (E \cap B)] \\ &\subseteq [A \cap C(B)] \cup [C(A) \cap B] = \phi\end{aligned}$$

since A and B are separations.

Thus if we assume that both $E \cap A$ and $E \cap B$ are non-empty, we have a separation for $E = (E \cap A)/(E \cap B)$. Hence either $E \cap A$ is empty so that $E \subseteq B$, or $E \cap B$ is empty so that $E \subseteq A$.

Cor. 1. If C is a connected set and $C \subseteq E \subseteq C(C)$, then E is a connected set.

Proof. Suppose that E is not a connected set, then it must have a separation E = A/B. By the above theorem, since C is a subset of E which has a separation, C must be contained in A or contained in B. Without loss of generality, let us suppose that $C \subseteq A$. From this it follows that $C(C) \subseteq C(A)$ and hence

$$C(C) \cap B \subseteq C(A) \cap B = \phi.$$

On the other hand, $B \subseteq E \subseteq C(C)$ and so $C(C) \cap B = B$, thus we must have $B = \phi$, which contradicts our hypothesis that E = A/B. Hence E is connected.

Cor. 2. If every two points of a set E are contained in some connected subset C of E, then E is a connected set.

Proof. If E is not connected, then it must have a separation E = A/B. Since A and B must be non-empty, let us choose points $a \in A$ and $b \in B$. From the hypothesis we know that a and b must be contained in some connected subset C contained in E. Then by the above theorem, either $C \subseteq A$ or $C \subseteq B$. Since A and B are disjoint, this is a contradiction to the fact that C contains points of A as well as of B. Hence E is connected.

Cor. 3. The union E of any family $\{C_\lambda\}$ of connected sets having a non-empty intersection is a connected set

Proof. Suppose E is not connected, then it must have a separation E = A/B, By hypothesis, we may choose a point $x \in \bigcap_{\lambda} C_\lambda$ which implies that $x \in C_\lambda$ for each λ. The point x must belong to either A or B and w.l.o.g. let us suppose $x \in A$. Then since $x \in C_\lambda$ for each λ, we have $C_\lambda \cap A \neq \phi$ for every λ. Now by the above theorem, each C_λ must be either a subset of A or a subset of B. Since A and B are disjoint and $C_\lambda \cap A \neq \phi$. We must have $C_\lambda \subseteq A$ for all λ. And so $E = \bigcup_{\lambda} C_\lambda \subseteq A$. From this we obtain the contradiction that $B = \phi$. Hence E is connected.

Cor. 4. If A and B are connected sets, then $A \cup B$ is connected.

Proof. Suppose $A \cup B$ is not connected, then it must have a separation $A \cup B = C/D$.

Since A is a connected subset of $A \cup B$, either $A \subseteq C$ or $A \subseteq D$. Similarly we have either $B \subseteq C$ or $B \subseteq D$. Now if $A \subseteq C$, $A \cup B \subseteq C$, $A \cup B \subseteq D$. But C and D are disjoint.
Hence contradiction.
Thus $A \cup B$ is connected.

Theorem. 39. If a connected set C has a non-empty intersection with both a set E and the complement of E in a topological space (X, T), then C has a non-empty intersection with the boundary of E.

Proof. We will show that if we assume that C is disjoint from b(E), we obtain the contradiction that C is not connected i.e.

$$C = (C \cap E)/(C \cap E^C).$$

From

$$\begin{aligned} C &= C \cap X \\ &= C \cap (E \cup E^C) \\ &= (C \cap E) \cup (C \cap E^C) \end{aligned}$$

we see that C is the union of two sets. These two sets are non-empty by hypothesis.

If we calculate $(C \cap E) \cap C(C \cap E^C)$

$$\begin{aligned} &\subseteq [\, C \cap C(E)] \cap C(E^C) \\ &= C \cap [C(E) \cap C(E^C)] \\ &= C \cap b(E) \end{aligned}$$

we see that the assumption that $C \cap b(E) = \phi$ leads to the conclusion that

$$(C \cap E) \cap (C \cap E^C) = \phi.$$

In the same way, we may show that $C[C \cap E] \cap [C \cap E^C] = \phi$ and we have a separation of C.

Theorem. 40. A topological space X is disconnected if and only if there exists a continuous mapping of X onto the discrete two point space {0, 1}.

Proof. If X is disconnected and $X = A \cup B$ is a disconnection, then we define a continuous mapping f of X onto {0, 1} by the requirement that $f(x) = 0$ if $x \in A$ and $f(x) = 1$ if $x \in B$. This is valid since A and B are disjoint and their union is X. Also A and B are non-empty and open, f is clearly onto and continuous.

On the other hand, if there exists such a mapping, then X is disconnected for if X were connected, then by the result that a continuous image of a connected space is connected, {0, 1} is connected but we know that a subspace of the real line R is connected if and if it is an interval. Thus X is connected would lead to a contradiction. Hence X is disconnected.

Theorem. 41. The product of any non-empty class of connected spaces is connected.

Proof. Let $\{X_i\}$ be a non-empty class of connected spaces and form their product $X = \prod_i X_i$. We assume that X is disconnected and we deduce a contradiction from this assumption. Now by the above theorem there exists a continuous mapping *f* of X onto the discrete two point space {0, 1}. Let a = $\{a_i\}$ be a fixed point in X and consider a particular index i_1. We define a mapping f_{i_1} of X_{i_1} into X by means of $f_{i_1}(x_{i_1}) = \{y_i\}$, where $y_i = a_i$ for $i \neq i_1$ and $y_{i_1} = x_{i_1}$. This is clearly a continuous mapping, so ff_{i_1} is a continuous mapping of X_{i_1} into {0, 1}. Since X_{i_1} is connected, ff_{i_1} is constant and

$$(ff_{i_1})(X_{i_1}) = f(a)$$

for every point x_{i_1} in X_{i_1}. This shows that *f*(x) = *f*(a) for all x′s in X which equal a in all co-ordinate spaces except X_{i_1}. By repeating this process with another index, i_2, etc, we see that *f*(x) = *f*(a) for all x′s in X. which equal a in all but a finite number of co-ordinate space. The set of all x′s of this kind is a dense subset of X, and so *f* is a constant mapping this contradicts the assumption that *f* maps X onto {0, 1} and this completes the proof.

Components

Definition. Let (X, T) be a topological space and x be a point of a subset E contained in X, then union of all connected sets containing x and contained in E, is called the component of E corresponding to x and will be denoted by C (E, x). Since union of connected sets is connected, C(E, x) is a connected set. Hence C(E, x) is the largest connected subset of E containing x. Thus a connected space clearly has only one component, namely the space itself.

Components corresponding to different points of E are either equal or disjoint, so that we may speak of the components of a set E without any reference to specific points. Every subset of a topological space has now been partitioned with disjoint subsets, its components.

Example. (1) Let (X, T) be a discrete space and $x \in X$ be arbitrary. Then {x} is a connected subset of X and also {x} is not a proper subset of any connected subset of X. Hence by definition, {x} is a component of X. Thus each point of a discrete space (X, T) is a component of X.

(2) If X is connected, then X has only one component, X itself.

(3) Every indiscrete space has only one component, namely the space itself.

(4) Let X = {a, b, c, d, e} consider the following topology on X.

$$T = \{X, \phi, \{a\}, \{c, d\}, \{a, c, d\}, \{b, c, d, e\}\}$$

The components of X are {a} and {b, c, d, e}. Any other connected subset of X, such as {b, d, e} is a subset of the one of the components.

Theorem. 42. The components of a topological space (X, T) are closed subsets of X.

Proof. Let C be the component of X. Since C is connected, its closure C(C) is also connected. Let a be a point of C and b be a point of C(C). Then the connected set C(C) contains both a and b. But by definition, C is the largest connected set containing a and so $b \in C$. Hence $C(C) \subseteq C$ and thus C is closed.

Theorem. 43. Let X be an arbitrary topological space, then we have the following

(1) Each point in X is contained in exactly one component of X.

(2) Each connected subspace of X is contained in a component of X.

(3) A connected subspace of X which is both open and closed is a component of X.

Proof. (1) Let x be a point in X. Consider the class $\{C_i\}$ of all connected subspaces of X which contain x. This class is non-empty, since {x} itself is connected. Since union of connected sets having a non-empty intersection is connected, $C = U_i\ C_i$ is a connected subspace of X which contains x. C is clearly maximal and therefore a component of X, because any connected, subspace of X which contains C is one of the C_i's and is thus contained in C. Finally, C is the only component of X which contains x. For if C^*, is another, it is clearly among the C_i's, and is therefore contained in C and since C^* is maximal as a connected subspace of X, we must have $C^* = C$.

(2) This is a direct consequence of the construction above and from this it follows that, a connected subspace of X is contained in the component which contains any one of its points.

(3) To prove (3), let A be a connected subspace of X which is both open and closed. By (2) above, A is contained in some component C. If A is a proper subset of C, then

$$C = (C \cap A) \cup (C \cap A')$$

is a disconnection of C. This contradicts the fact that C, being a component is connected and we conclude that A = C. Thus a connected subspace of X which is both open and closed is a component of X.

Locally Connected

Definition. A topological space (X, T) is said to be locally connected if and only if for every point $x \in X$ and every open set G containing x, there exists a connected open set G^* containing x and contained in G

Thus a space is locally connected if and only if the family of all open connected sets is a base for the topology for the space. We know that local compactness is implied by compactness local connectedness, however neither implies, nor is implied by connected as shown below.

Remark. (1) A locally connected set need not be connected. For example, a set consisting of two disjoint open intervals is locally connected but not connected

(2) A connected subset of the plane which is not locally connected.

For each positive integer n, let us denote by E_n, the line segment connecting the origin to the point $< 1, \frac{1}{n} >$. Each of these line segments is connected and all contain the origin, so their union is connected (since the intersection is the origin being a common point). If we let X be the point <1, 0> and Y be the point $< \frac{1}{2}, 0 >$, then X and Y are both limit points of the set consisting of all the E_n's

Then $E = \{X\} \cup \{Y\} \cup (U_n\ E_n\}$

is a connected set. This set is not locally connected, however since an open set containing X if sufficiently small will not contain a connected open set containing X.

Example. (1) Let (X, D) be a discrete topological space. We know that for every $x \in X$, the singleton {x} is connected and is such that {x} contains x and is contained in an open connected set {x}. Hence a discrete topological space is locally connected.

(2) Every Banach space is locally connected

(3) The union of two disjoint open intervals on the real line is a simple example of a space which is locally connected but not connected.

Remark. Removal of a point from a connected set leaves a set which is not connected.

Theorem. 44. A space X is locally connected if and only if for every open set U of X, each component of U is open in X.

Proof. Suppose that X is locally connected and let U be an open set in X and further let C be a component of U. If x is a point of C, we can choose a connected open neighbourhood V of x such that $V \subset U$. Since V is connected, it must be entirely in the component C of U. Therefore C is open in X.

Conversely, suppose that components of open sets in X are open. Given a point x of X and a neighbourhood U of x, let C be the component of U containing x. Since C is a connected subset of U. Thus for every open set U containing x, there is a connected open set C containing x such that $C \subset U$. Hence X is locally connected.

3

COMPACTNESS AND CONTINUOUS FUNCTIONS

Like many other notions in topology, the concept of compactness for a topological space is an abstraction of an important property possessed by certain sets of real numbers. The property we have in mind is expressed by Heine-Borel theorem, which asserts the following.

"If X is a closed and bounded subset of the real line R, then any class of open subsets of R whose union contains X has a finite subclass whose union also contains X." If we regard X as a topological space, as a subspace of R, this theorem can be thought of saying that any class of open subsets of X whose union is X has a finite subclass whose union is also X.

The Heine-Borel theorem has a number of profound and far reaching applications in analysis. Many of these guarantee that continuous functions defined on closed and bounded sets of real numbers are well behaved. For instance, any such function is automatically bounded and uniformly continuous. In contrast to this, we note that the function *f* defined on the open unit interval (0, 1) by $f(x) = \frac{1}{x}$ is neither bounded nor uniformly continuous.

Thus although the motivation for this concept is the Heine-Borel theorem, but the motivating theorem does not hold at all in general topological space. First of all the entire concept of boundedness is missing since we have no notion of distance. Secondly a compact subset need not be closed as the example of any proper subset of an indiscrete space shows. Despite these problems, the concept of compactness is of fundamental importance in topology.

Compact Sequentially compact

Countably Compact

Bolzano-Weiersteass Property.

Compact Topological Spaces

Definition. A collection $\{A_\lambda\}$ of sets is said to be a covering of a set E if

$$E \subseteq \bigcup_{\lambda} A_\lambda.$$

If all the sets of a covering are open, then it is called an open covering.

If some finite subcollection of a covering of a set E is also a covering of E, then we say that the covering is reducible to a finite sub-covering. A subcollection of an covering which is itself a covering is called a subcovering.

Definition. Let (X, T) be a topological space. A collection $\{G_i\}$ of open subsets of X is said to be an open covering of X if

$$X = \bigcup_{i} G_i$$

A subset E of a topological space (X, T) is said to be compact if and only if every open covering of E is reducible to a finite subcovering of E i.e. every open covering of E has a finite subcovering.

Example. (1) If X is a finite set and T is a topology on X, then (X, T) is compact.

(2) If T is the finite complement topology on any set X, then (X, T) is compact.

Now we give some examples of spaces which are not compact.

(1) If X is an infinite set and T is the discrete topology on X, then (X, T) is not compact.

(2) The open interval (0, 1) is not compact because $\left\{\left(\frac{1}{n},1\right); n \in N\right\}$ is an open cover that does not have a finite sub-cover.

(3) The real line is not compact because $\{(-n, n); n \in N\}$ is an open cover that does not have a finite subcover.

(4) The real line with the lower limit topology is not compact since $\{[n, n+1) ; n \in Z\}$ is an open cover that does not have a finite subcover.

Definition. A topological property is said to be weakly hereditary if whenever a space has it, so does every closed subspace of it.

Theorem. 1. If E is a subset of a subspace (X^*, T^*) of a topological space (X, T), then E is T^* compact if and only if it is T-compact.

Proof. Suppose E is T^*-compact and $\{G_\lambda\}$ is some T-open covering of E. We shall show that $\{G_\lambda\}$ has a finite subcovering. The family of sets $\{X^* \cap G_\lambda\}$ clearly forms a T^*–open covering of E since

$$E = X^* \cap E = X^* \cap (\bigcup_{\lambda} G_\lambda)$$

$$= \bigcup_{\lambda} (X^* \cap G_\lambda)$$

Since E is T^*–compact, there is a finite subcovering of E such that

$$E \subseteq \bigcup_{i=1}^{n} [X^* \cap G_i] \subseteq \bigcup_{i=1}^{n} G_i$$

Thus we have shown that every open covering of E has a finite subcovering and hence E is T-compact.

Conversely suppose that E is T-compact and $\{G^*_\lambda\}$ is some T^*–open covering of E. From the definition of induced topology, each $G_\lambda^* = X^* \cap G_\lambda$ for some T–open set G_λ.

The family $\{G_\lambda\}$ is clearly a T–open covering of E and so there must be some finite subcover of E such that $E \subseteq \bigcup_{i=1}^{n} G_i$

But then we have,

$$E = X^* \cap E$$

$$\subseteq X^* \cap (\bigcup_{i=1}^{n} G_i)$$

$$= \bigcup_{i=1}^{n} (X^* \cap G_i) = \bigcup_{i=1}^{n} G_i^*$$

which is a finite subcovering of E from $\{G_\lambda^*\}$

Hence E is T^*–compact.

Remark. The property of compactness is not Heriditary. It is weakly Heriditary. We do however the following result.

Theorem 2. Every closed subset of a compact space is compact.

Proof. Let G be an open covering of a closed subset E of a compact space (X, T), then

$$G^* = G \cup E^C$$

is an open covering of X. Since (X, T) is compact, every open covering of X has a finite subcovering. If we remove E^C from the finite subcovering of X, we will have a finite subcovering of E chosen from G. Hence E is compact.

Example. The cantor set C is the intersection of closed subsets of R. Therefore C is closed subset of R and hence of I = [0, 1]. Since I is compact by the above theorem, thus C is compact being the closed subset of a compact space.

Remark. The following example shows that a compact subset of a topological space is not necessarily closed.

For example let T be the finite complement topology on R and let $A = \{x \in R, x \text{ is rational}\}$

Then A is compact but not closed.

But if the space is Hausdorff, then every compact subset is closed.

First of all we give some definitions.

Definition. A topological space (X, T) is a T_1–space provided that for each pair x, y of distinct points of X, there exist open sets U and V such that $x \in U$, $y \notin U$, $y \in V$ and $x \notin V$.

Definition. A topological space (X, T) is a Hausdorff space provided that if x and y are distinct members of X then there exist disjoint open sets U and V such that $x \in U$ and $y \in V$.

Theorem. 3. Let (X, T) be a Hausdorff space, let A be a compact subset of X and $p \in X - A$. Then there exist disjoint open sets U and V such that $A \subseteq U$ and $p \in V$.

Proof. Since X is Hausdorff and A is a subset of X, for each $x \in A$, there are disjoint open sets U_x and V_x such that $x \in U_x$ and $p \in V_x$. Then $\mu = \{U_x ; x \in A\}$ is an open cover of A. Since A is compact subset of X, there is a finite subcollection $U_{x_1}, U_{x_2}, ..., U_{x_n}$ of μ that cover A. For each i = 1, 2,…,n, U_{x_i} and the corresponding V_{x_i} are disjoint. Therefore $U = \bigcup_{i=1}^{n} U_{x_i}$ and $V = \bigcap_{i=1}^{n} V_{x_i}$ are disjoint open sets such that $A \subseteq U$ and $p \in V$.

Cor.1. Every compact subset of a Hausdorff space is closed.

Proof. Let A be a compact subset of a Hausdorff space (X, T) and let $p \in X - A$. Then by Theorem 3, there is a neighbourhood U of p such that $U \subseteq X - A$. Therefore $X - A$ is open and so A is closed. Hence every compact subset of Hausdorff space is closed.

Thus we have proved that a subset A of a compact Hausdorff space is compact if and only if it is closed.

Theorem. 4. Any continuous image of a compact set is compact.

Proof. Let $f : X \rightarrow Y$ be a continuous mapping of a compact set X into an arbitrary topological space Y mapping a compact set E into f(E). We must show that f(E) is compact in Y.

Suppose that $\{G_\lambda^*\}$ is an open covering of f(E).

Since $E \subseteq f^{-1}(f(E))$

$$\subseteq f^{-1} [\bigcup_{\lambda} G_\lambda^*]$$

$$\subseteq [\underset{\lambda}{\cup} f^{-1} [G_\lambda^*]$$

Since f^{-1} (G_λ^*) is open in X. It follows that $f^{-1}(G_\lambda^*)$ is an open covering of E. Since E is compact, there must be some finite subcovering of E i.e.

$$E \subseteq \bigcup_{i=1}^{n} f^{-1}(G_\lambda^*)$$

But then $$f(E) \subseteq f(\bigcup_{i=1}^{n} f^{-1}(G_i^*))$$

$$= \bigcup_{i=1}^{n} f(f^{-1}(G_i^*))$$

$$\subseteq \bigcup_{i=1}^{n} G_i^*$$

and so we have a finite subcover for $f(E)$.

Hence $f(E)$ is compact.

Theorem 5. A topological space is compact if and only if every class of closed sets with empty intersection has a finite subclass with empty intersection.

Proof. First assume that topological space (X, T) is compact, then a class of open sets is an open covering $\Rightarrow X = \cup G_i$

$$\Rightarrow \quad X^C = \cap G_i^C$$

$$\Rightarrow \quad \cap G_i^C = \phi$$

Since G_i's are open $\Rightarrow G_i^C$'s are closed.

Thus the class of closed sets has empty intersection.

Since this open covering is reducible to a finite subcovering

$$\Rightarrow \quad X = \bigcup_{i=1}^{n} G_i \quad \Rightarrow X^C = (\bigcup_{i=1}^{n} G_i)^C$$

$$\Rightarrow \quad \phi = \bigcap_{i=1}^{n} G_i$$

Thus every class of closed sets with empty intersection has a finite subclass with empty intersection.

Similarly the other part follows.

Theorem. 6. Let (X, T) be a topological space and let *B* be a basis for T. Then (X, T) is compact if and only if every cover of X by members of *B* has a finite subcover.

Proof. Let (X, T) be a compact space and let μ be a cover of X by members of *B*. Then μ is an open cover of X and hence it has a finite subcover since X is compact.

On the other hand suppose every open cover of X by members of *B* has a finite subcover. We shall show that X is compact. Let μ be an open cover of X. For each $\cup \in \mu$, there is a subcollection B_* of *B* such that $U = \cup\{B ; B \in B_*\}$. Since every member of U is union of members of subcollection of B_* B.

Now, $\{B ; B \in B_*$ for some $\cup \in \mu\}$ is an open cover of X by members of *B* and hence it has a finite subcover μ^*. For each $B \in \mu^*$, choose a member U_B of μ such that $B \subseteq U_B$. Then $\{U_B ; B \in \mu^*\}$ is a finite subcollection of μ that covers X. Hence X is compact.

Theorem. 7. Every continuous real valued function on a compact space is bounded and attains its extrema.

Proof. Let X be a compact space and suppose $f : X \rightarrow R$ is continuous. First we show f is bounded. For each $x \in X$, let J_x be the open interval $(f(x)-1, f(x) +1)$ and let $V_x = f^{-1}(J_x)$.

By continuity of f, V_x is an open set containing x. Note that f is bounded on each V_x. Now the family $\{V_x ; x \in X\}$ is an open cover of X and by compactness of X, admits a finite subcover say $\{ V_{x_1}, V_{x_2}, ..., V_{x_n} \}$.

Let $M = \max \{f(x_1), f(x_2), \ldots, f(x_n)\} + 1$ and

let $m = \min \{f(x_1), f(x_2), \ldots, f(x_n)\} - 1$

Now for any $x \in X$, there is some i such that $x \in V_{x_i}$. Then $f(x_i) - 1 < f(x) < f(x_i) + 1$ and so $m < f(x) < M$ showing that f is bounded. It remains to show that f attains its bounds.

Let L, M be respectively the supremum and infimum of f over X. If there is no point x in X for which $f(x) = L$, then we define a new function $g : X \to R$ by $g(x) = \dfrac{1}{L - f(x)}$ for all $x \in X$. Then g is continuous since f is so. However g is unbounded, for given any $R > 0$, there exists x such that $f(x) > L - \dfrac{1}{R}$ and hence $g(x) > R$. But this contradicts the above part of the theorem and hence f attains the value L. Similarly f attains the infimum M.

Theorem. 8. Let (X, T) be a compact locally connected space. Then (X, T) has a finite number of components.

Proof. Suppose (X, T) has infinite number of components. Then since each component of X is open. Thus the collection of components of X is an open cover of X that does not have a finite subcover. Since X is compact, this is a contradiction. Hence a compact locally connected space has a finite number of components.

Remark. The following is the classical theorem of Heine-Borel Lebesgue.

Theorem. 9. A subset of the Euclidean n-space is E_n compact if and only if it is closed and bounded.

Proof. Let A be a compact subset of E_n. Since E_n is Hausdorff and A is a compact subset of Hausdorff space E_n, A is closed. Because of compactness. A can be covered by a finite family of open spheres of radius one and because each of these is bounded, A is bounded. Thus every compact subset of E_n is closed and bounded.

To prove the converse suppose that A is closed and bounded subset of E_n. Let B_i be the image of A under the projection into the ith co-ordinate space and note that each B_i is bounded because the projection decreases distances.

Then $A \subset \Pi \{B_i, i = 0, 1, \ldots, n-1\}$ and this set is a subset of a product of closed bounded intervals of real numbers. Since A is a closed subset of the product and the product of compact spaces is compact, the proof reduces to showing that a closed interval [a, b] is compact relative to the usual topology. Let μ be an open cover of [a, b] and let C be the supremum of all members x of [a, b] such that some finite subfamily of μ covers [a, x]. Choose U in μ such that $C \in U$ and choose a member d of the open interval (a, c) such that $[d, c] \subset U$. There is a finite subfamily of μ which covers [a, b] and this family with U adjoined covers [a, c] . Unless c = b the same finite subfamily covers [a,d] an interval to the right of c, which contradicts the choice of c . Hence the result.

Finite Intersection Property

Definition. A family of sets will be said to have the finite intersection property if and only if every finite subfamily of the family has a non-empty intersection.

Theorem. 10. A topological space (X, T) is compact if and only if any family of closed sets having the finite intersection property has a non-empty intersection.

Proof. Suppose that (X, T) is compact and $\{F_\lambda\}$ is a family of closed sets whose intersection is empty that is $\bigcap_\lambda F_\lambda = \phi$

$$\Rightarrow \quad (\bigcap_\lambda F_\lambda)^C = \phi^C = X$$

$$\Rightarrow \quad X = \bigcup_\lambda F_\lambda^C$$

and therefore, $\bigcup_\lambda F_\lambda^C$ is a covering of X, but (X, T) is given to be compact, therefore there exists finite subcovering of X i.e.

$$X = \bigcup_{i=1}^{n} F_i$$

But then $\phi = X^C = (\bigcup_{i=1}^{n} F_i^C)^C = \bigcap_{i=1}^{n} F_i$

so that the family $\{F_\lambda\}$ can not have the finite intersection property. Thus we arrive at a contradiction.

Hence $\quad \bigcap_\lambda F_\lambda \neq \phi$

Conversely suppose that (X, T) is not compact. This means that there exists an open covering $\{G_\lambda\}$ of X which has no finite subcovering. To say that there is no finite subcovering means that complement of the union of any finite number of members of the cover is non-empty,

i.e. $\quad (\bigcup_{\alpha=1}^{n} G_\alpha)^C \neq \phi$

$\Rightarrow \quad \bigcap_{\alpha=1}^{n} G_\alpha^{C} \neq \phi$ by De-Morgan's Law thus $\{G_\lambda^{C}\}$ is then a family of closed sets with finite intersection property. Since $\{G_\lambda\}$ is a covering of X, we have

$$\phi = X^C = (\bigcup_\lambda G_\lambda)^C = \bigcap_\lambda G_\lambda^C$$

Thus this family of closed sets with the finite intersection property has an empty intersection which contradicts our hypothesis.

Hence (X, T) is compact.

Countably Compact and Sequentially Compact

Definition. A topological space (X, T) has the Bolzano-Weieistrrass property provided that every infinite subset of X has a limit point.

Definition. A topological space (X, T) is countably compact provided every countable open cover of X has a finite subcover.

Theorem. 11. A topological space (X, T) is countably compact if and only if every countable family of closed subsets of X with the finite intersection property has a non-empty intersection.

Proof. Suppose (X, T) is countably compact. Let $A = \{A_\alpha; \alpha \in \wedge\}$ b a countable family of closed subsets of X with the finite intersection property. Suppose $\bigcap_{\alpha \in \wedge} A_\alpha = \phi$. Let

$$\mu = \{X - A_\alpha \,;\, \alpha \in \wedge\}$$

Since $\bigcup_{\alpha\in\wedge} (X - A_\alpha) = X - \bigcap_{\alpha\in\wedge} A_\alpha = X - \phi = X.$

If μ is an open cover of X. Since X is countably compact, there exists a finite number α_1, $\alpha_2, \ldots, \alpha_n$ of members of $\wedge$ such that

$$\{X - A_\alpha \,;\, i = 1, 2, \ldots, n\} \text{ covers X.}$$

Thus $$X = \bigcup_{i=1}^{n} (X - A_\alpha) = X - \bigcap_{i=1}^{n} A_\alpha \text{ and hence}$$

$$\bigcap_{i=1}^{n} A_\alpha = \phi. \text{ This is a contradiction.}$$

Hence $$\bigcap_{\alpha \in \wedge} A_\alpha \neq \phi.$$

Suppose every countable family of closed subsets of X with the finite intersection property has a non-empty intersection. Let

$\mu = \{U_\alpha ; \alpha \in \wedge \}$ be a countable open cover of X.

Suppose μ does not have a finite subcover.

Let $A = \{X - U_\alpha ; \alpha \in \wedge\}$. Then A is a countable family of closed subsets of X. Let T be a finite subset of $\wedge$. Since μ does not have a finite subcover,

$$\bigcap_{\alpha \in \Gamma} (X - U_\alpha) = X - \bigcup_{\alpha \in \Gamma} U_\alpha \neq \phi.$$

Therefore A has the finite intersection property.

Hence $\bigcap_{\alpha \in \wedge} (X - U_\alpha) \neq \phi$ since we have assumed that every countable family of closed subsets of X with the finite intersection property has a non-empty intersection. But this is contradiction since

$$\bigcap_{\alpha \in \wedge} (X - U_\alpha) = X - \bigcup_{\alpha \in \wedge} U_\alpha = X - X = \phi$$

Therefore (X, T) is countably compact.

Theorem. 12. Every countably compact topological space has the Bolzano-Weierstrass property.

Proof. Let (X, T) be a countably compact space and A be an infinite subset of X. Since A is infinite. A contains a countably infinite set B = $\{x_i, i \in N\}$. We may assume that if $i \neq j$, then $x_i \neq x_j$. The proof is by contradiction.

Suppose B has no limit point. Then for each $n \in N$,

$$C_n = \{x_i \in B \; ; \; i \geq n\} \text{ is a closed set.}$$

Further, $\{C_n \; ; \; n \in N\}$ has the finite intersection property. Therefore by the result "A topological space (X, T) is countably compact if and only if every countable family of closed subsets of X with the finite intersection property has a non-empty intersection", $\bigcap_{n=1}^{\infty} C_n \neq \phi$. But if $x_K \in B$, then $x_K \notin C_{K+1}$, and hence

$$x \notin \bigcap_{n=1}^{\infty} C_n. \text{ Therefore } \bigcap_{n=1}^{\infty} C_n = \phi$$

and we have a contradiction. Thus B has a limit point and $B \subseteq A$, A has a limit point, thus every infinite subset of X has a limit point and hence (X, T) has the Bolzano-Wierstrass property.

Remark. The converse of the above theorem is not true in a general topological space. It holds for T_1 space.

Definition. A topological space (X, T) is a T_1–space provided that for each pair x, y of distinct points of X, there exist open sets U and V such that $x \in U$, $y \notin U$ and $y \in V$, $x \notin V$.

Theorem. 13. Let (X, T) be a T_1–space. Then X is countably compact if and only if it has the Bolzano-Weiersttrass property.

Proof. Suppose (X, T) has the Bolzano-Weierstrass property. The proof that X is countably compact is by contradiction. Suppose $\{U_n ; n \in N\}$ is a countable open cover of X that has no finite subcover. Then for each $n \in N$,

$$C_n = X - \bigcup_{i=1}^{n} U_i \text{ is a non-empty closed set.}$$

For each $n \in N$, let $p_n \in C_n$ and let

$A = \{p_n ; n \in N\}$ If A is finite, there exists $p \in A$ such that $p_n = p$ for an infinite number of $n \in N$. Thus for each $n \in N$, $p \in C_n$.

This is a contradiction since $\{U_n ; n \in N\}$ covers X. Suppose that A is infinite. Then by hypothesis, there is a point $p \in X$ that is a limit point of A.

Since X is a T_1–space and p is a limit point of A, then every neighbourhood of p contains an infinite number of members of A. Therefore for each $n \in N$, p is a limit point of $A_n = \{p_i \in A; i > n\}$. For each $n \in N$, $A_n \subseteq C_n$. Since for each $n \in N$, C_n is closed, $p \in C_n$. Once again this is a contradiction, since $\{U_n ; n \in N\}$ covers X. Therefore X is countably compact.

The other part is already proved in the above theorem.

Theorem. 14. A compact topological space is countably compact.

Proof. We assume that the topological space X is compact and show that every infinite subset of X has a limit point in X i.e. it is countably compact.

Let E be an infinite subset of X with no limit point in X. So there must exist an open set G_x containing x such that

$$E \cap G_x - \{x\} = \phi$$

Clearly $E \cap G_x$ contains at the most one point x itself. Since the family $\{G_x\}_{x \in X}$ forms an open covering of the compact space X, therefore there must be some finite subcovering i.e.

$$X = \bigcup_{i=1}^{n} G_{x_i}$$

From this, it follows that

$$E = E \cap X = E \cap \left(\bigcup_{i=1}^{n} G_{x_i}\right)$$

$$= \bigcup_{i=1}^{n} (E \cap G_{x_i})$$

Since $E \cap G_{x_i}$ contains at the most one point, Therefore E being the finite union of such sets contains at most n elements and so is finite. This leads to the contradiction. Hence every infinite subset of X must have at least one limit point.

Definition. A topological space (X, T) is **sequentially compact** provided every sequence in X has a subsequence that converges.

There is no direct relationship between compactness and sequential compactness. We give two examples.

Example of a compact space that is not sequentially compact

For each $\alpha \in R$, let $X_\alpha = I$ and $X = \prod_{\alpha \in R} X_\alpha$. Then (X, T) is compact but not sequentially compact.

Example of a sequentially compact space that is not compact.

Let $(\Omega, \leq)$ be an uncountable well ordered set with a maximal element W_1 having the property that if $x \in \Omega$ and $x \neq W_1$, then $\{y \in \Omega ; y \leq x\}$ is countable. Let T be the order topology on Ω and let $\Omega_0 = \Omega - \{W_1\}$. Then

$$(\Omega_0, T_{\Omega_0}) \text{ is sequentially compact but not compact.}$$

Theorem. 15. Every sequentially compact space is countably compact.

Proof. Suppose (X, T) is a topological space that is not countably compact. Let μ be a countable open cover that does not have a finite subcover. Choose $x_1 \in X$. For each $j > 1$, let $U_j \in \mu$ that contains a point x_j that is not in $\bigcup_{i=1}^{j-1} U_j$. We claim, the sequence $<x_n>$ does not have a subsequence that converges. Let $x \in X$. Then there exists K such that $x \in U_K$. Then $x_j \notin U_K$ for any $j > K$. Thus no subsequence of $<x_n>$ converges to x. Since x is an arbitrary point, no subsequence of $<x_n>$ converges. Therefore (X, T) is not sequentially compact. Hence the result.

Locally Compact

Paul Alexandroff and Heinrich Tietze independently introduced the concept of local compactness.

Definition. A topological space (X, T) is said to be locally compact if and only if every point of X has at least one neighbourhood whose closure is compact that is if for each $x \in X$, there is an open set O containing x such that C(0) is compact.

i.e. A topological space (X, T) is said to be locally compact if and only if each element $x \in X$ has a compact nbd.

Remark. Every compact space is locally compact. In fact nbd of each point is the whose space X and C(X) = X which is given to be compact. But converse need not be true i.e. every locally compact space need not be compact.

Example. (1) Consider a Discrete topological space (X, D) where X is an infinite set. Then X is not compact since the collection of all singleton sets is an infinite open cover of X which is not reducible to a finite subcovering. On the other hand if x is an arbitrary element of X. Since every subset of X is open and therefore the neighbourhood {x} of x is open. Evidently {x} is a compact subset of x since {x} has a finite subcovering. Then x has a compact nbd and so X is locally compact.

Example. 2. Consider the real line R with usual topology. Observe that each point p $\in$R is interior to a closed interval e.g. $[p-\delta, p+\delta]$ and that the closed interval is compact by the Heine-Borel theorem. Hence R is locally compact. On the other hand R is not compact for example the class

$$A = \{\ldots(-3, -1), (-2, 0), (-1, 1), (0, 2), (1, 3),\ldots\}$$

is an open cover of R but contains no finite subcover.

Theorem. 16. Every closed subset of a locally compact topological space is locally compact.

Proof. Let y be an arbitrary point of the closed subset Y of (X, T). Since X is locally compact at y, there must exist a nbd N of y such that C(N) is compact. Now C(N) $\supset$ N. Therefore C(N) is T-closed compact nbd of y. Let

$$M = C(N) \cap Y$$

Since C(N) and y are T-closed, it follows that M is T-closed. Also M is closed in C(N) and Y hence M is compact in C(N) being a closed subset of a compact set C(N). It follows that M is compact in X and consequently in Y. Therefore Y is locally compact.

Remark. The next example shows that the continuous image of a locally compact space need not be locally compact.

Example. Let $A = \{\frac{1}{n}; n \in N\}$ and let $B = \{B \in P(R)$; B is an open interval that does not contain 0 or there is a positive number x such that $B = (-x, x) - A\}$. Then *B* is a basis for the topology T on R and the topological space (R, T) is a Hausdorff space. Also $A = \{\frac{1}{n}; n \in N\}$ is a closed subset of R and $0 \notin A$.

We know that (R, T) is not locally compact. Let μ be the discrete topology on R. Then (R, μ) is locally compact. The function $f : (R, \mu) \to (R, T)$ defined by $f(x) = x$ for each $x \in R$ is continuous. Thus the continuous image of a locally compact space need not be locally compact. But we have the following more general result.

Theorem. 17. Let (X, T) be a locally compact space and (Y, μ) be a topological space and let $f : X \to Y$ be an open continuous function from X onto Y. Then (Y, μ) is locally compact.

Proof. Let $y \in Y$ and $x \in f^{-1}(y)$. But X is locally compact. There is an open set V and a compact set C such that $x \in V$ and $V \subseteq C$. Let $U = f(\text{int}(C))$ and $K = f(C)$. Since $x \in V \subseteq C$ and V is open, $x \in \text{int}(C)$.

Since f is open, U is open and $x \in \text{Int}(C)$, $y \in U$. Also f is continuous and C is compact, and we know that continuous image of a compact space is compact, $f(C) = K$ is compact. But $U \subseteq K$. Thus (Y, μ) is locally compact at y. But y is an arbitrary point of y. Thus (Y, μ) is locally compact.

One Point Compactification

The one point compactification of a topological space X is the set $X^* = X \cup \{\infty\}$, where ∞ is any object not belonging to X with the topology T^* whose members are of the following type

(i) U where U is an open subset of X.

(ii) X^*-C, where C is a closed compact subset of X.

We now check that **the collection is infact a topology on X^***

The empty set is a set of type (i) and the space X^* is a set of type (ii). To check that intersection of two open sets is open, there are three cases.

$U_1 \cap U_2$ is of type (i)

$(X^* - C_1) \cap (X^* - C_2) = X^* - (C_1 \cup C_2)$ is of type (ii)

$U_1 \cap (X^* - C_1) = U_1 \cap (X - C_1)$ is of type (i) because C_1 is closed in X.

Similarly the union of any collection of open sets is open

$\cup U_\alpha = U$ is of type (i)

$\cup(X^* - C_\beta) = X^* - (\cap C_\beta) = X^* - C$ is of type (ii)

$(\cup U_\alpha) \cup (\cup(X^* - C_\beta)) = U \cup (X^* - C)$

$= X^* - (C - U)$

which is of type (ii) because C–U is a closed subset of C and is therefore compact.

We now show that X^* is compact. Let $\mathcal{C}$ be the collection of open sets in X^* covering X^*. This collection must contain an open set of the type X^*-C, since none of the open sets of X contain the point ∞. Take all the members of $\mathcal{C}$ different from X^*-C and intersect them with X, they form a collection of open sets in X covering C. Because C is compact, finitely many of them cover C. Let

$C \subseteq \bigcup_{i=1}^{n} (X \cap G_i)$, then the class

$\{X^* - C, X \cap G_1, X \cap G_2, \ldots, X \cap G_n\}$ covers all of X^* and since the covering is finite, X^* is compact.

Finally we show that X is a subspace of X^* and $C(X) = X^*$. Given any open set of X^*, its intersection with X is open in X since $U \cap X = U$ and $(X^* - C) \cap X = X - C$ both of which are open in X. Conversely any set open in X is a set of type (i) and therefore open in X^*. This proves our first part. Now we show that $C(X) = X^*$ since X is not compact, each open set $X^* - C$ containing the point ∞ intersects X. Therefore ∞ is a limit point of X, so that

$$C(X) = X^*.$$

Theorem. 18. The one point compactification X^* of a topological space X is a Hausdorff space if and only if X is a locally compact Hausdorff space.

Proof. Suppose X is a locally compact Hausdorff space and x, y are distinct points of X^*. If neither x nor y is equal to the ideal point ∞, then they both belong to X and there must be disjoint sets containing them which are open in X and so open in X^* as desired. We must then consider the case where one of the points, say y is ∞ while other is in X. By local compactness of X, there must be some open set G containing x such that C(G) is compact and hence also closed since every compact subset of hausdorff space is closed. Thus $X - C(G)$ is an open set in X whose complement is closed and compact in X. By the definition of one point compactification

$$\{\infty\} \cup (X - C(G))$$

is an open set in X^* containing ∞ which is disjoint from G, an open set in X^* containing x.

Conversely suppose that X^* is a Hausdorff space since X is a subspace of X^*, and the property of Hausdorffness is Hereditary, X is a Hausdorff space. Let x be a fixed point of X. Since x and ∞ are distinct points of the hausdorff space X^*, there must exist disjoint open sets G_x^* and G_∞^* in X^* such that

$$x \in G_x^* \text{ and } \infty \in G_\infty^*$$

However an open set containing ∞ must be of the form $G_\infty^* = \{\infty\} \cup G$ where G is an open set in X whose complement (in X) is compact. Since $\infty \notin G_x^*$, G_x^* is an open set in X containing x whose closure is contained in X–G and hence is compact. Thus X is locally compact.

4

SEPARATION AXIOMS(I) AND COUNTABILITY AXIOMS

A topological space may be very sparsely endowed with open sets as we know some spaces have only two, the empty set and the full space. In a discrete space, on the other hand, every set is open. Most of the familiar spaces of geometry and analysis fall somewhere in between these two artificial extremes. The so called separation properties enables us to state with precision that a given topological space has a rich supply of open sets to serve whatever purpose we have in mind.

The separation properties are important because the supply of open sets possessed by a topological space is intimately linked to its supply of continuous functions and since continuous functions are of central importance in topology, we naturally wish to guarantee that enough of them are present to make our discussions fruitful. If for instance, the only open sets in a topological space are the empty set and the full space, then the only continuous function present are constants and very little of interest can be aid about these. In general, more open sets there are, the more continuous function, a space have. Discrete spaces have continuous functions in the greatest possible abundance, for all functions are continuous. The separation properties make it possible for us to be sure that our spaces have enough continuous functions.

The separation axioms are of various degrees of strengths and they are called T_0, T_1, T_2, T_3 and T_4 axioms in ascending order of strength, T_0 being the weakest separation axiom. T_0-property was introduced by A. N. Kolmogorov and T_1–property was introduced by Frechet in 1907. Hausdorff introduced the T_2–property in 1923. The separation properties were also known as Trennungsaxiomen.

Basic Properties of Separation Axioms (T_0, T_1, T_2)

Definition. A topological space X is called a T_0 space if and only if it satisfies the following axiom of Kolmogorov.

[T_0] If x and y are distinct points of X, then there exists an open set which contains one of them but not the other.

It is easy to see why the T_0–axiom is the weakest separation axiom. For if a space X is not T_0, then there would exist two distinct points x, y in X such that e every open set in X either contains both x and y or else contains neither of them. In such a case, x and y may as well be regarded as topological identical and any topological statement about one of them will imply a corresponding statement about the other and vice-versa. For example a sequence in X will converge to x if and only if it converges to y. Thus T_0 condition is the minimum requirement if we want to distinguish between x and y topologically.

Examples. (1) Every metric space is T_0

(2) A topological space consisting of two point a, b with the topology T = {ϕ, X, {a}} is a T_0–space. This space is also known as connected double space.

(3)Let T be the topology on R whose members are ϕ, R and all sets of the form (a, ∞) for a ∈ R. In this space for x, y∈R with x < y, there exists an open set containing y but not x for example the open interval (x, ∞) although there exists no open set which contains x but not y. Hence (R, T) is a T_0–space.

(4) A topological space with discrete topology is a T_0–space. Since if x, y∈X, then {x} contains x and is an open set but it does not contain y.

(5) A topological space with indiscrete topology is not T_0. Infact there is only one open set X which contains all the points.

(6) Let X be the set of real number x where $0 \le x < 1$ and T be the lower limit topology that is the open sets in X are null set and the sets $0 \le x < k$, where $0 < k \le 1$. Then (X, T) is T_0.

Remark. Let T and T^* be two topologies defined on X and let (X, T) be T_0–space. If $T \subseteq T^*$, then (X, T^*) is also T_0–space.

Theorem. 19. The property of a space being a T_0-space is preserved under one to one, onto, open mapping and hence is a topological property.

Proof. Let (X, T_1) be a T_0-space and *f* be one to one open continuous mapping of (X, T_1) into another topological space (Y, T_2). We must show that (Y, T_2) is also a T_0-space.

Let y_1 and y_2 be distinct elements of Y. Since *f* is one-one, onto mapping, there exist $x_1, x_2 \in X$ such that $f(x_1) = y_1$ and $f(x_2) = y_2$. Again since (X, T_1) is a T_0-space, there exists an open set G containing one of x_1 and x_2 but not the other. w.l.o.g. suppose that $x_1 \in G$ and $x_2 \notin G$. Since *f* is open, the set *f*(G) is open in T_2. However *f*(G) is such that

$$f(x_1) \in f(G) \Rightarrow y_1 \in f(G)$$

and
$$f(x_2) \notin f(G) \Rightarrow y_2 \notin f(G)$$

as desired.

Theorem. 20. Every subspace of a T_0-space is a T_0-space. (Heriditary property).

Proof. Let (X, T) be a T_0-space and (Y_1, T*) be a subspace of (X, T). Let y_1 and y_2 be two distinct points of (Y, T^*). Since $Y \subset X$, y_1 and y_2 are also distinct points of X. But X is a T_0-space, there must exist a T-open set G such that $y \in G$ and $y_2 \notin G$. Then $G \cap Y$ is an open set in Y which contains y_1 but not y_2. (Y, T^*) is a T_0-space.

Theorem. 21. A topological space (X, T) is a T_0-space if and only if for every distinct arbitrary points x, $y \in X$, the closures of {x} and {y} are distinct.

Proof. Suppose that $x \neq y$ implies that $C(\{x\}) \neq C(\{y\})$ and that x and y are distinct points of X. Since the sets C({x}) and C({y}) are not equal, there must exist some point z of X which is contained in one of them but not the other. w.l.o.g let us suppose that $z \in C(\{x\})$ and $z \notin C(\{y\})$. If we had $x \in C(\{y\})$, then we would have $x \in C[C(\{y\})] = C(\{y\})$ and so $z \in C(\{x\}) \subseteq C(\{y\})$, which is a contradiction.

Hence $x \notin C(\{y\})$ and so $[C(\{y\})]^C$ is an open set containing x but not y showing that X is a T_0-space.

Conversely let us suppose that X is a T_0-space and that x and y are two distinct points of X. Hence there exists an open set G containing x but not y. Clearly G^c is a closed set containing y but not x. From the definition of C({y}), as the intersection of all closed sets containing {y}, we have $y \in C(\{y\})$, but $x \notin C(\{y\})$, because of G^C.

Hence $C(\{x\}) \neq C(\{y\})$

Definition. A topological space X is a T_1-space if and only if it satisfies the following separation axiom of Frechet.

[T_1] If x and y are two distinct points of X, then there exist two open sets, one containing x but not y and the other containing y but not x.

Remark. Clearly T_1-space is always a T_0-space but the converse need not be true. For example, let N be the set of natural numbers and let

$T = \{\phi$, N, all the subsets of N of the form $\{1, 2,\ldots, N\}\}$ be topology on X, then the subsets are of the type {1, 2}, {1, 3}. This space is clearly T_0 but not T_1 as we see from {1, 2}, {1, 3}, 1 is contained in both the sets.

Similarly in the case of connected double space (X, T), where X = {a, b} and $T = \{\phi, X, \{a\}\}$, we see that it is T_0 but not T_1 as there exist two open sets {a, b and {a}, which both have a .

(2) (R, T), where T is usual topology is a T_1-space.

(3) Let T be the finite complement topology on R. Then (R, T) is a T_1-space

Theorem. 22. A topological space is a T_1-space if and only if every degenerate (consisting of single element) set is closed.

Proof. If x and y are distinct points of a space X in which subsets consisting of exactly one point are closed, then $\{x\}^C$ is an open set containing y but not x, while $\{y\}^C$ is an open set containing y but not x, while $\{y\}^C$ is an open set containing x but not y. Thus X is a T_1-space.

Conversely let us suppose that X is a T_1-space and that x is a point of X. By Frechet axiom if $y \neq x$, then there exists an open set G_y containing y but not x that is $y \in G_y \subseteq \{x\}^C$. But then

$$\begin{aligned}\{x\}^C &= U\{y \,;\, y \neq x\} \\ &\subseteq U\{G_y \,;\, y \neq x\} \\ &\subseteq \{x\}^C\end{aligned}$$

and so $\{x\}^C$ is the union of open sets and hence is itself open. Thus $\{x\}$ is closed set for every $x \in X$.

Theorem. 23. Let (X, T) be a T_1-space and f be a 1–1 open mapping of (X, T) onto another topological space (Y, V). Then (Y, V) is also a T_1-space.

Proof. Suppose y_1 and y_2 are two distinct points of Y. Since f is bijective, there exists x_1 and x_2 in X such that $f(x_1) = y_1$, $f(x_2) = y_2$.

Since (X, T) is a T_1-space, there exists open sets G and H such that $x_1 \in G$, $x_2 \notin G$, $x_2 \in H$ but $x_1 \notin H$. Since f is open, $f(G)$ and $f(H)$ are also open in Y and further

$$y_1 = f(x_1) \in f(G) \text{ but } y_2 = f(x_2) \notin f(G)$$

and

$$y_2 = f(x_2) \in f(H) \text{ but } y_1 = f(x_1) \notin f(H)$$

which proves that (Y, V) is a T_1-space.

Remark. The above theorem proves that the property of a space being a T_1-space is a topological property.

Theorem. 24. Every subspace of a T_1-space is a T_1-space (Hereditary Property).

Proof. Let (X, T) be a T_1-space and (X^*, T^*) be a subspace of (X, T). Let x^* and y^* be two distinct points of X^*. Since (X, T) is T_1-space, there exists open sets G and H in X such that $x^* \in G$ but $y^* \notin G$ and $y^* \in H$ but $x^* \notin H$. But then T^*-open set $X^* \cap G$ contains x^* but not y^* and $X^* \cap H$ contains y^* but not x^*. Hence (X^*, T^*) is also a T_1-space.

Theorem. 25. Prove that every finite T_1-space is discrete.

Proof. Let (X, T) be a finite T_1-space. Then every degenerate set is closed. Then since finite union of closed sets being closed. We see that T contains all the subsets of X. Hence (X, T) is discrete.

Theorem. 26. Prove that in a T_1-space X, a point x is a limit point of a set E if and only if every open set containing x contains an infinite number of distinct point of E.

Proof. The sufficiency of the condition is obvious. To prove the necessity, suppose that there is an open set G containing x for which $G \cap E$ was finite. If we let

$G \cap E - \{x\} = \bigcup_{i=1}^{n} \{x_i\}$, then each set $\{x_i\}$ would be closed since in a T_1-space, every degenerate set is closed and the finite union $\bigcup_{i=1}^{n} \{x_i\}$ would also be a closed set.

But then
$$\left(\bigcup_{i=1}^{n} \{x_i\}\right)^C \cap G$$

would be an open set containing x with

$$\left[\left(\bigcup_{i=1}^{n} \{x_i\}\right)^C \cap G\right] \cap E - \{x\}$$

$$= \left(\bigcup_{i=1}^{n} \{x_i\}\right)^C \cap \left(\bigcup_{i=1}^{n} \{x_i\}\right) = \phi$$

$\Rightarrow$ x would not be a limit point of E.

Remark. It follows from the above theorem that a finite set in a T_1-space can not have a limit point.

We can also prove it as follows.

A finite set is a union of finite number of degenerate sets each of which has the null set as their derived set. So the derived set of a finite set in a T_1-space is a null set.

Theorem. 27. A T_1-space X is countably compact if and only if every countable open covering of X is reducible to a finite subcover.

Proof. Let a T_1-space X be countably compact and suppose $(G_n\}_{n\in N}$ is a countable open covering of the countable compact space X which has no finite subcover. This means that $\bigcup_{i=1}^{n} G_i$ does not contain X for any $n\in N$. If we let

$F_n = (\bigcup_{i=1}^{n} G_i)^C$, Then each F_n is a non-empty closed set contained in the preceding one. From each F_n, let us choose a point x_n and let $E = \bigcup_{n\in N} \{x_n\}$. The set E can not be finite because there would then be some point in an infinite number and hence all of the sets F_n and this would contradict the fact that the family $\{G_n\}_{n\in N}$ is a covering of X which has no finite subcover. Since E must be infinite, we may use the countable compactness of X to obtain a limit point x of E. But by the result "In a T_1-space X, a point x is a limit point of a set E if and only if every open set containing x contains an infinite number of distinct points of E", every open set containing x contains an infinite number of points of E and so x must be a limit point of each of the sets $E_n = \bigcup_{i>n} \{x_i\}$. For each n, however E_n is contained in the closed set F_n and so x must belong to F_n for every $n\in N$. This again contradicts the fact that the family $\{G_n\}_{n\in N}$ is a covering of X. Hence the condition is necessary.

Conversely let us suppose that a T_1-space X is not countably compact that is E is an infinite subset of X such that E has no limit point. Since E is infinite, we may choose an infinite sequence of distinct points x_n from E. The set $A = \bigcup_{n\in N} \{x_n\}$ has no limit point since it is a subset of E and so in particular, each point x_n is not a limit point of A. This means that for every $n\in N$, there exists an open set G_n containing x_n such that

$$A \cap \{G_n\} - \{x_n\} = \phi$$

From the definition of A, we see that $A \cap G_n = \{x_n\}$ for every $n\in N$. Since A has no limit point, it is a closed set and hence A^C is open. Then the collection $A^C \cup \{G_n\}_{n\in N}$ is countable open covering of X which has no finite subcover since the set G_n is needed to cover the point x_n for every $n\in N$ as $x_n\in X$ and x_n's are infinite in number. Thus the condition is sufficient.

Theorem. 28. If *f* is a continuous mapping of the T_1-space X into the topological space X^*, then *f* maps every countably compact subset of X onto a countably compact subset of X^*.

Proof. Suppose E is countably compact subset of X and $\{G_n^*\}$ is countable open covering of *f*(E) we need only show that there is a finite subcovering of *f*(E) since we note from the theorem, that the condition of this theorem is always sufficient, as *f* is continuous, $\{f^{-1}(G_n^*)\}$ is a countable open covering of E. In the induced topology, $\{E \cap f^{-1}(G_n^*)\}$ is a countable open covering of the countably compact T_1-space E. By the above theorem, there exists some finite subcovering $\{E \cap f^{-1} (G_i^*)\}_{i=1}^{K}$ and clearly the family $\{G^*{}_i\}_{i=1}^{K}$ is the desired finite subcovering of *f*(E).

Definition. A topological space X is said to be a T_2-space or hausdorff space if and only if for every pair of distinct points x, y of X, there exist two disjoint open sets one containing x and the other containing y.

Examples. (1) Every discrete topological space is Hausdorff. Also no indiscrete space containing at least two points is Hausdorff.

(2) Usual topological space (R, U) is an Hausdorff space.

(3) and $T_1 = \{\phi, \{a\}, \{b\}, \{c\}, \{a, b\}, \{b, c\}, \{c, a\}, X\}$

$T_2 = \{\phi, \{a\}, \{b, c\}, X\}$

Clearly T_1 is an Hausdorff topology but the topology T_2 is not Hausdorff.

(4) All metric spaces are Hausdorff.

Remark. Every T_2-space is T_1, but the converse need not be true. An infinite set with cofinite topology is T_1 but not T_2. In fact, no two open sets in are disjoint unless one of them is empty.

Theorem. 29. Every singleton of a Hausdorff space is closed.

Proof. Let (X, T) be a Hausdorff topological space and $x \in X$. Let y be an arbitrary point of X distinct from x. Since X is Hausdorff, there exists an open set G_y containing y such that $x \notin G_y$. It follows that y is not a limit point of {x}. Consequently

$$d(\{x\}) = \phi.$$

$$C\{x\} = \{x\} \cup d\{x\} = \{x\} \cup \phi = \{x\}$$

$$\Rightarrow \quad \{x\} \text{ is closed.}$$

Theorem. Let T and T^* be two topologies on a set X such that T^* is finer than T. If (X, T) is Hausdorff, then (X, T^*) is also Hausdorff.

Proof. Let x and y be two arbitrary points of X, since (X, T) is Hausdorff, there exist disjoint T-open sets G and H such that $x \in G$ and $y \in H$. Since $T \subset T^*$, the sets G and H are also T^*-open such that $x \in G$ and $y \in H$ and $G \cap H = \phi$. Hence it is a T_2-space.

Theorem. 30. The property of being a T_2-space is Cogradient or Heriditary property.

Proof. Let (X, T) be a T_2 space and suppose that (X^*, T^*) is a subspace of X. Let x, y be two distinct points in X^*. Since x and y are also points of X which is given to be a T_2-space, there exists two disjoint open sets G and H such that G contains x and H contains y. Then the sets $G \cap X^*$ and $H \cap X^*$ are disjoint open sets in X^* containing x and y respectively. Hence X^* is a Hausdorff space.

Theorem. 31. Every compact subset E of a Hausdorff space X is closed.

Proof. We shall prove that E is closed by showing that E^C is open. Let x be a fixed point of E^C. By Hausdorff property for each point $y \in E$, there exist two disjoint open sets G_x and G_y such that $x \in G_x$ and $y \in G_y$. The family of sets $\{G_y; y \in E\}$ is an open covering of E. Since E is compact, there must be some finite subcovering $\{G_{y_i}\}_{i=1}^{n}$. Let $\{G_{x_i}\}_{i=1}^{n}$ be the corresponding open sets containing x and let $G = \bigcap_{i=1}^{n} G_{x_i}$. Then G is an open set containing x since it is the intersection of a finite number of open sets containing x. Further we see that

$$G = \bigcap_{i=1}^{n} G_{x_i} \subseteq \bigcap_{i=1}^{n} (G_{y_i})^C = \left(\bigcup_{i=1}^{n} G_{y_i}\right)^C \subseteq E^C.$$

Thus each point in E^C is contained in an open set which is itself contained in E^C. Hence E^C is an open set and so E must be closed.

Theorem. 32. If f is one to one continuous mapping of the compact topological space X onto the T_2-space X^*, then f is open and so f is a homeomorphism.

Proof. Let G be an open set in X so that X−G or X/G is closed. But every closed subset of a compact space is compact. Therefore X/G is compact. Also we know that if f is a continuous mapping of (X, T) into (X^*, T^*), then f maps every compact subset of X onto a compact subset of X^*. Thus f(X/G) is compact in the Hausdorff space X^*. Also by the result "Every compact subset of a Hausdorff space is closed", we have f(X/G) is closed. Thus $X^*/f(X/G)$ is open. Since f is one to one and onto,

$$X^*/f(X/G) = f(G) \text{ which is open.}$$

Definition. Let (X, T) be a topological space and $x \in X$. Let $<x_n>$ be a sequence of points in X, then the sequence has limit x or converges to x written as lim $x_n = x$ or $x_n \to x$ if and only if for every open set G containing x there exists an integer N(G) such that $x_n \in G$ whenever n >N(G).

A sequence will be called convergent if and only if there is at least one point to which it converges. Every subsequence of a convergent sequence is also convergent and has the same limits. The convergence of a sequence and its limits are not affected by a finite number of alternations in the sequence, including the adding or removing of a finite number of terms of the sequence.

It is the failure of limits of sequences to be unique that makes this concept unsatisfactory in general topological spaces for example let T be the trivial topology on a set X and let $<x_n>$ be a sequence in X and x be any member of X, then $<x_n> \to x$.

However in a Hausdorff space, a convergent sequence has a unique limit as the following theorem shows.

Theorem. 33. In a Hausdorff space, a convergent sequence has a unique limit.

Proof. Suppose a sequence $<x_n>$ converges to two distinct points x and x^* in a Hausdorff space X. By the Hausdorff property, there exists two disjoint open sets G and G^* such that $x \in G$ and $x^* \in G^*$. Since $x_n \to x$, there exists an integer N such that $x_n \in G$ whenever n>N. Also $x_n \to x^*$, there exists an integer N^* such that $x_n \in G^*$ whenever $n>N^*$. If m is any integer greater than both N and N^*, then x_m must be in both G and G^* which contradicts that G and G^* are disjoint.

Remark. The converge of this theorem is not true.

Theorem. 34. If $<x_n>$ is a sequence of distinct points of a subset E of a topological space X which converges to a point $x \in X$ then x is a limit point of the set E.

Proof. If x belongs to the open set G, then there exists an integer N(G) such that , $x_n \in G$ for all n > N(G). Since the points x_n′s are distinct, at most one of them equals x and so $E \cap G - \{x\} \neq \phi$, this implies that x is a limit point of E.

Remark. The converse of this theorem is not true, even in a Hausdorff space.

Theorem. 35. If f is a continuous mapping of the topological space X in to the topological space X^*, and $<x_n>$ is a sequence of points of X which converges to the point $x \in X$, then the sequence $< f(x_n)>$ converges to the point $f(x) \in X^*$.

Proof. If $f(x)$ belongs to the open set G^* in X^*, then $f^{-1}(G^*)$ is an open set in X containing x, since f is continuous and we know that if f is continuous, then inverse image of every open set is open, then there must exist an integer N such that $x_n \in f^{-1}(G^*)$, whenever n>N. Thus we have $f(x_n) \in G^*$ when n > N and so $f(x_n) \to f(x)$. Hence $<f(x_n)>$ converges to the point $f(x) \in X^*$.

Remark. The converse of this theorem is also not true even in a Hausdorff space, that is the mapping f for which $x_n \to x$ implies $f(x_n) \to f(x)$, may not be continuous. For example, let X be the Hausdorff space of ordinals less than or equal to the first uncountable ordinal r with order topology. The real valued function f, defined by setting $f(\alpha) = 0$ if $\alpha < r$ and $f(r) = 1$ is not continuous at r, even though it does preserve convergent sequences.

Theorem. 36. An infinite Hausdorff space X contains an infinite sequence of non-empty disjoint open sets.

Proof. If X has no limit point, then X must have the discrete topology since singletons are closed in a Hausdorff space. Thus any infinite sequence of distinct points of X would serve as desired sequence.

Suppose, then that x is a limit point of X. Choose x_1 to be any point of X different from x. Since X is Hausdorff, there exists two disjoint open sets G_1 and V_1 such that $x_1 \in G_1$ and $x \in V_1$. Since x is a limit point of X belonging to the open set V_1, there exists some point $x_2 \in X \cap V_1 - \{x\}$. Again since X is Hausdorff, there exist two disjoint open sets G_2^* and V_2^* such that $x_2 \in G_2^*$ and $x \in V_2^*$. If we let

$$G_2 = G_2^* \cap V_1 \text{ and } V_2 = V_{2=}^* \cap V_1,$$

then G_2 and V_2 are disjoint open sets contained in V_1 and hence disjoint from G_1 containing x_2 and x respectively.

We will now proceed by using an inductive argument. Since we have already defined the points $\{x_K\}$ and the open sets $\{G_K\}$ and $\{V_K\}$ with the properties that $x_K \in G_K \subseteq V_{K-1}$, $x \in V_K \subseteq V_{K-1}$ and $G_K \cap V_K = \phi$ for all $K \le n$. Now x is a limit point of X belonging to the open set V_n and so there exists some point $x_{n+1} \in X \cap V_n - \{x\}$. Since X is Hausdorff, there exist two disjoint open sets G^*_{n+1} and V^*_{n+1} such that $x_{n+1} \in G^*_{n+1}$ and $x \in V^*_{n+1}$. If we let $G_{n+1} = G^*_{n+1} \cap V_n$ and $V_{n+1} = V^*_{n+1} \cap V_n$.

Then G_{n+1} and V_{n+1} are two disjoint open sets contained in V_n (and hence disjoint from G_n) containing x_{n+1} and x respectively. Since the sets $\{V_n\}$ are monotonic decreasing, we see that G_{n+1} is not only disjoint from G_n but is also disjoint from G_K for $k \le n$. Since $x_n \in G_n$, the infinite sequence $<G_n>$ defined by induction is the desired sequence of non-empty, disjoint open sets.

First and second countable spaces

The first axiom of countability, the second axiom of countability and seperability are countability axioms.

First axiom of countability

The first axiom of countability was introduced by Hausdorff. A topological space X is a first axiom space or first countable space if it satisfies the following first axiom of countability.

$[C_1]$ For every point $x \in X$, there exists a countable family $\{B_n(x)\}$ of open sets containing x such that whenever x belongs to an open set G,

$$B_n(x) \subseteq G \text{ for some n}$$

The family $\{B_n(x)\}$ is called a countable open base at x.

Examples. (1) Let T be the lower limit topology on R. Then (R, T) is first countable.

(2) If T is the usual topology on R. Then

$$\left\{\left(-\frac{1}{n},\frac{1}{n}\right), n \in N\right\} \text{ is a local basis at } 0.$$

(3) Every metric space is first countable.

Remark. Let T be the finite complement topology on R. Then (R, T) is not first countable.

Theorem. 37. The property of being a first axiom space is hereditary.

Proof. Let $p \in Y$. Since $Y \subset X$, $p \in X$. But (X, T) is first countable, so there exists a countable T-local base.

$B_p = \{B_n \,;\, n \in N\}$ at p. Set

$B_p^* = \{Y \cap B_n \,;\, n \in N\}$ is a T^*-local base at p. B_p is clearly countable. Hence the result.

Theorem. 38. Let (X, T) be a first axiom space. Then there exists a monotone decreasing local base at every point of X.

Proof. Let $\{B_n(x)\}$ be a local base at x.

We set $B_1^*(x) = B_1(x)$

$B_2^*(x) = B_1(x) \cap B_2(x)$

$B_3^*(x) = B_1(x) \cap B_2(x) \cap B_3(x)$

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………………………………….

$B_n^*(x) = \cap \{B_K(x) \,;\, k \le n\}$

Then clearly $\{B_n^*(x)\}$ is a monotone decreasing countable open base at x.

Theorem. 39. Let X be an uncountable set and ∞ be a fixed point of X. Let be the family of subsets G such that either (1) $\infty \notin G$ or (ii) $\infty \in G$ and G^C is finite. Then (X, T) is compact, non-first axiom, Hausdorff topological space (This space is known as **Fort's space.)**

Proof. First of all we show that (X, T) is a topological space.

Theorem. 40. A topological space X satisfying the first axiom of countability is a Hausdorff space if and only if every convergent sequence has a unique limit.

Proof. We know that in a Hausdorff space, a convergent sequence has a unique limit. Therefore the necessary part of the theorem is proved.

Now suppose that X is not a Hausdorff space. Therefore there must exist two points x and y such that every open set containing x has a non-empty intersection with every open set containing y. Therefore if $\{B_n(x)\}$ and $\{B_n(y)\}$ are monotone decreasing countable open bases at x and y respectively. We must have $B_n(x) \cap B_n(y) \neq \phi$ for every n and so we may choose a point x_n belonging to this intersection for each n. If G_x and G_y are arbitrary open sets containing x and y respectively there must exist some integer N such that $B_n(x) \subseteq G_x$ and $B_n(y) \subseteq G_y$ for all $n > N$ by definition of a monotone decreasing base. Hence $x_n \to x$ and $x_n \to y$ so that we have a convergent sequence without a unique limit.

Theorem. 41. If x is a point and E is a subset of a T_1-space X satisfying the first axiom of countability, then x is a limit point of E if and only if there exists a sequence of distinct points in E converging to x.

Proof. We know that if $<x_n>$ is a sequence of distinct points of a subset E of a topological space X which converges to a point $x \in X$, then x is a limit point of the set E. Therefore the sufficiency of the theorem is proved. Now suppose that $x \in d(E)$ and let $\{B_n(x)\} = \{B_n\}$ be a monotone decreasing countable open base at x. Since x belongs to the open set B_n, the set $B_n \cap E - \{x\}$ must be infinite since we know that "In a T_1-space X, a point x is a limit point of a set E if and only if every open set containing x contains an infinite number of distinct points of E". By induction we may choose a point x_n in this set different from each previously chosen x_K with $k < n$. Clearly $x_n \to x$ since the sets $\{B_n\}$ form a monotone decreasing base at x.

Theorem. 42. Let *f* be a mapping of the first axiom space x into the topological space X^*. Then *f* is continuous at $x \in X$ if and only if for every sequence $<x_n>$ of points in x converging to x we have the sequence $<f(x_n)>$ converging to the point $f(x) \in X^*$.

Proof. Suppose first that *f* is continuous and that $<x_n>$ is a sequence of points of X which converges to the point $x \in X$. Then it is already known that the sequence $<f(x_n)>$ converges to the point $f(x) \in X^*$. Hence the condition is necessary.

Conversely suppose that *f* is not continuous at x so there must exist an open set G^* containing $f(x)$ such that $f(G) \cap (G^*)^C \neq \phi$ for any open set G containing x. Let $\{B_n\}$ be a monotone decreasing countable open base at x. Then $f(B_n) \cap (G^*)^C \neq \phi$ for each n and we may pick $x_n^* \in f(B_n) \cap (G^*)^C$. Since $x_n^* \in f(B_n)$, we may choose a point $x_n \in B_n$ such that $f(x_n) = x_n^*$. We now have $x_n \to x$ since the sets $\{B_n\}$ form a monotone decreasing base at x. The sequence $<f(x_n)> = <x_n^*>$ can not converge to $f(x)$, however since $x_n^* \in (G^*)^C$ for all n. Hence we arrive at a contradiction.

Second axiom of Countability

This axiom was introduced by Hausdorff.

Definition. A topological space (X, T) is a second axiom space or second countable if and only if it satisfies the following second axiom of countability.

[C_2] there exists a countable base for the topology T.

The real number system with usual topology is an example of a second axiom space since we may choose the family of all open intervals with rational end points as our countable base while R with lower limit topology is not second countable.

Theorem. Every second axiom or second countable space is first countable.

Proof. Let (X, T) be a second axiom space and let *B* be a countable base for T. Let p be an arbitrary point of X. If B_p consists of all those members of *B* which contain p, then it is clear that B_p is a local countable at p.

Remark. But the converse is not true i.e. every first countable space need not be second countable.

For example the discrete topology on any countable set has no countable base since each set consisting of exactly one point must belong to any base, even though there is a countable open base at each point x obtained by letting $B_n(x) = \{x\}$.

Theorem. 43. The property of being a second axiom space is Hereditary.

Proof. Let (X, T) be a second countable space and let B = $\{B_n ; n \in N\}$ be a countable base for T. Let (Y, T^*) be a subspace of (X, T). Clearly the collection $B^* = \{Y \cap B_n ; x \in N\}$ is a base for T^* which is clearly countable.

Theorem. 44. The property of being a second axiom space is a topological property.

Proof. Let (X, T) be a second countable space and let (Y, V) be its homeomorphic image under the homeomorphism $f : X \to Y$. Let $B = \{B_n ; n \in N\}$ be a countable base for T. we shall show that the collection $\{f(B_n) ; n \in N\}$ is a countable base for V.

Clearly, this collection is countable. Further, since f is open, each $f(B_n)$ is V-open. Let G be the V-open subset of Y. Then $f^{-1}(G)$ is a T-open set since f is continuous. Therefore $f^{-1}(G)$ is a union of B_λ ; $\lambda \in \wedge$ where $\wedge \subset N$. Therefore

$$G = f\,[U\{B_\lambda ; \lambda \wedge\}]$$

$$= U\,[f(B_\lambda) ; \lambda \in \wedge]$$

Thus any V-open set G can be expressed as the union of members of $\{f(B_n); n \in N\}$. Hence $\{f(B_n) ; n \in N\}$ is a countable base for (Y, V).

Definition. We shall call a point x, a condensation point of a set E in a topological space if and only if every open set containing x contains an uncountable number of points of E.

Theorem. 45. Let (X, T) be a second countable space and let A be an uncountable subset of X. Then some point of A is a condensation point of A.

Proof. Let $B = \{B_n ; n \in N\}$

be a countable base for T and suppose that no point of A is a condensation point of A then for each point $x \in A$, there exists can open set G containing x such that $G \cap A$ is countable. Since B is a base, we may choose B_{n_x}

$\in B_\alpha$ $[n_x \in N]$ such that $x \in B_{n_x} \subseteq G$ and so $B_{n_x} \cap A$ is countable. But we may write

$$A = U\,\{x; x \in A) \subseteq U\{B_{n_x} \cap A; x \in A\}$$

and there can be at most a countable number of different indices since for each $x \in A$, we must get a different B_{n_x} such that $B_{n_x} \cap A$ is countable. Also $x \neq y \Leftrightarrow \{x\} \neq \{y\}$

$\Leftrightarrow A \cap B_{n_x} \neq A \cap B_{n_y} \Leftrightarrow B_{n_x} \neq B_{n_y}$. Therefore there exists one-one correspondence between the elements of A and the members of subcollection *B.* Hence A is at most countable union of countable sets and so countable. This contradicts the hypothesis. Hence every uncountable subset of a second axiom space contains a condensation point.

Theorem. 46. In a second axiom space, every collection of non-empty disjoint open sets is countable.

Proof. Let (X, T) be a second axiom space. Suppose $\{B_n\}$ is a countable base for T. Also let **G** be a collection of non-empty open disjoint sets of X. Since the sets $\{B_n\}$ form a base, for each set G in the collection **G**, there must exist at least one integer n such that $B_n \subseteq G$. Since the members of the collection **G** are disjoint, different integers will be associated with different members. If we order the collection **G** according to the order of the associated integer for each member, we obtain a (possibly finite) sequence which contains all the members of **G**. Therefore **G** is countable.

2.8 Lindelof Theorems.

Lindelof spaces were first studied by Ernst Lindelof. He proved in 1903 that every second axiom space is Lindelof.

Definition. A topological space (X, T) is said to be Lindelof space if and only if every open covering of the space is reducible to a countable subcovering.

Every compact space is Lindeloff space but the converse need not be true for example let X be countable and (X, T) be discrete topological space. It is Lindeloff but not compact.

Theorem. 47. (Lindelof's Theorem). In a second axiom space, every open covering of a subset is reducible to a countable subcovering that is every second axiom space is Lindeloff.

Proof. Let (X, T) be a second countable or second axiom space and let $\{B_n\}$ be a countable base for T. Also let E be a subset of X which has **G** as an open covering. We wish to show that **G** is reducible to a countable subcovering.

Let N(g) be the countable collection of integer n such that $B_n \subseteq G$ for some $G \in \mathbf{G}$. with each integer $n \in N(g)$, we may then associate a set $G_n \in \mathbf{G}$ such that $B_n \subseteq G_n$ the family $\{G_n ; n \in N(g)\}$ is clearly a countable subcollection of **G** and we assert that it is a covering of E. Let $x \in E$, since **G** is a covering of E, $x \in G$ for some $G \in \mathbf{G}$. From the definition of base, we have $x \in B_n \subseteq G$ for some integer n. This means however that $n \in N(g)$ and so $x \in B_n \subseteq G_n$ which proves that

$$E \subseteq U\{G_n ; n \in N(g)\}$$

This implies that E has a countable subcovering.

Remark. The converse of the above theorem is not true that is every Lindelof space need not be second axiom.

Analysis. Let X be any uncountable set and let T = {ϕ, complements of finite sets}. We show first that the space (X, T) is Lindelof. Let ⊄ be any open cover of X and let G be any member of ⊄. Now G is the complement of a finite set say $A = (a_1, a_2, \ldots, a_n)$. To cover these n points, we need at most members of C. This shows that (X, T) is Lindelof.

We shall now show that (X, T) is not second axiom or second countable. If possible let there exists countable base *B* for T. Let $x \in X$, then we claim that

$$\cap \{G ; G \text{ is open} ; x \in G\} = \{x\}$$

because the complement of every other point is an open set containing x. Now let $H = X - \{y\}$. Then H is an open set containing x and so the above intersection can not be {y} but {x}.

Now for each open set G containing x, we find out $B \in B$ such that $x \in B \subset G$. As G runs through all open sets containing x, B runs through those members of *B* which contain x. Hence the intersection of all those members of *B* which contains x is {x}.

Let **D** be the collection of all those members of *B* which contain x that is

$$\cap \{D ; D \in \mathbf{D}\} = \{x\}.$$

Taking complements of this countable intersection, we obtain the union of countable number of finite sets.

$$\cup\{D^C, D \in \mathbf{D}\} = \{x\}^C$$

The R. H. S. is uncountable but the left hand side is countable which is a contradiction.

Hence (X, T) is not second axiom space.

Theorem. 48. A continuous and onto image of a Lindelof space is a Lindelof space.

Proof. Let (X, T) be a Lindelof space and let f be a mapping $f : (X, T) \rightarrow (X, T^*)$ which is continuous and onto. Let

$$\subset = \{A_\lambda ; \lambda \in \wedge\} \text{ be a } T^*\text{-open cover of Y}$$

so that $Y = \cup\{A_\lambda ; \lambda \in \wedge\}$

Hence $f^{-1}[Y] = f^{-1}[U\{A_\lambda ; \lambda \in \wedge\}]$

$$= U[f^{-1}(A_\lambda); \lambda \in \wedge]$$

Since f is continuous, each $f^{-1}[A_\lambda]$ is a T-open subset of X. Since (X, T) is Lindeloff, this cover is reducible to a countable subcover, say $[f^{-1}(A\lambda_i)]_{i \in N}$ so that

$$X = U\{f^{-1}[A\lambda_i] ; i \in N\}$$

Hence $f(X) = f[\cup\{f^{-1}(A\lambda_i)\}_{i \in N}]$

$$= U[f\{f^{-1}(A\lambda_i)\}]$$

since f is onto, $f(X) = Y$ and $f[f^{-1}(A\lambda_i)] = A\lambda_i$

and therefore $Y = U\{A\lambda_i ; i \in N\}$

which proves that (Y, T^*) is Lindelof.

Theorem. 49. Every closed subspace Y of a Lindeloff space (X,T) is Lindeloff.

Proof. Let (X, T) be a Lindelof space and $Y \subset X$

Let $C = \{G_\lambda^* ; \lambda \in N\}$

Be a T^*-open cover of Y so that

$$Y = U\{G_\lambda^* ; \lambda \in \wedge\}$$
$$= U\{G_\lambda \cap Y ; \lambda \in \wedge\}$$

since $G_\lambda^* = G_\lambda \cap Y$ for all $\lambda \in \wedge$ and where G_λ is T-open

$$= U[G_\lambda; \lambda \in \wedge] \cap Y$$

Therefore $Y \subseteq U\{G_\lambda ; \lambda \in \wedge\}$

Also X – Y is T-open, then the collection

$$\{G_\lambda ; \lambda \in \wedge\} U \cup \{X - Y\}$$

is a T-open cover of X since Y is closed. But by hypothesis (X, T) is Lindeloff. This open cover has a countable subcover. Now two cases arise.

Case I. When this subcover contains X –Y as one of its member. In such a case dropping this out from the subcover gives us a countable family of T-open sets, say $G_1, G_2, \ldots, G_n$ that cover Y so that

$$Y = U\{G_n ; n \in N\}$$

Case II. If this subcover does not contain X–Y, in that case let this subcover be denoted by $\{G\lambda_i; i \in N\}$. Then $\{G\lambda_i \cap Y; i \in N\}$ is a countable subcover of C that covers Y.

2.9. Separable Space

Definition. A subset E of a topological space X will be called dense in X if and only if

$$C(E) = X.$$

Definition. A topological space (X, T) is said to be separable if and only if there exists a countable dense subset of X.

For example the space (R, U) is separable since the set Q of rational numbers is a countable dense subset of R that if C(Q) = R

On the other hand, let X be uncountable and (X, T) be discrete. Then (X, T) is not separable since X is the only set whose closure is X itself.

Definition. A space (X, T) is said to be Hereditary separable if and only if each subspace of the space is separable.

Theorem. 50. Every second axiom space is Hereditary separable.

Proof. Since every subspace of a second axiom space is second axiom, it is sufficient to prove that every second axiom space is separable.

Let (X, T) be a second axiom space. Let $B = \{B_n ; n \in N\}$ be a countable base for it. For each $n \in N$, are choose a point $b_n \in B_n$ and thus obtain a set $B = \{b_n ; n \in N\}$. Clearly B is countable. We shall show that B is dense in X.

Let x be an arbitrary point of X and let G be an open set containing x. Since B is a base for T, there exists at least one $B_{n_o} \in B$ such that $x \in B_{n_o} \subset G$.

By the definition of B, $b_{n_o} \in B$ is such that $b_{n_o} \in B_{n_o} \subset G$. Thus G contains a point of B. So every open set containing x contains a point of B and so $x \in C(B)$. Thus we have shown that

$$x \in X \quad \Rightarrow x \in C(B) \quad \Rightarrow X \leq C(B)$$

Also $\quad C(B) \subseteq X$

Therefore $\quad X = C(B)$

Hence X is separable.

Remark. The converse of this theorem is not true i.e. every separable space need not be second axiom. For example, let X be uncountable and T = {ϕ, all subsets A of X such that A cor. X−A is finite. We can check that (X, T) is not second axiom space. Let A be any infinite countable subset of X. Then C(A) = X which implies that X is separable.

Theorem. 51. A continuous onto image f of a separable space X is separable.

Proof. Let A be a countable dense subset of X. Then clearly f(A) is also countable. We wish to show that f(A) is dense in Y where $f : X \to Y$.

Let y be an arbitrary point of Y. Since Y is onto image of X, there exists some $x \in X$ such that $f(x) = y$. let G be a T^*-open set in Y containing y so that $y = f(x) \in G$. But $f(x) \in G \Rightarrow x \in f^{-1}(G)$ since f is continuous. $f^{-1}(G)$ is T-open neighbourhood of x. Now A is dense in X implies that C(A) = X

$\Rightarrow \quad f^{-1}(G) \cap A \neq \phi$

and therefore $\quad f[f^{-1}(G) \cap A] \neq \phi$

or $\quad f(f^{-1}(G)) \cap f(A) \neq \phi$

$\Rightarrow \quad G \cap f(A) \neq \phi$

This shows that every neighbourhood of y intersects f(A) that is $y \in C[f(A)]$. Thus we have shown

$y \in Y \Rightarrow \quad y \in C[f(A)]$

$\Rightarrow \quad Y \subseteq C[f(A)]$

Also $\quad C[f(A)] \subseteq Y$

Hence $\quad Y = C[f(A)]$

Which proves that Y is separable.

5

SEPARATION AXIOMS (PART II)

Regular Spaces

Regular spaces were first studied by Victoris in 1921.

Definition. A topological space X is regular if and only if it satisfies the following axiom of Vietoris

(R) If F is a closed subset of X and x is a point of X not in F, then there exist two disjoint open sets, one containing F and the other containing x.

Definition. A T_3-space is a regular space which is also a T_1-space.

Remark. Although every T_3-space is obviously a T_2-space, a regular space need not be a T_2-space and a T_2-space need not be a regular T_3-space.

Example. Let $X = \{a, b, c\}$
and $T = \{\phi, (b, c), (a), X\}$

Then X, (a), (b, c) and ϕ are closed (being the complements of open sets) It can be seen that (X, T) is regular but not T_3-space (In fact it is not T_1 and T_2 also) ($\because$ of (b, c))

Example. Let $A = \{\frac{1}{n}, n \in N\}$ and **B** = {B ∈ P(R), B is an open interval that does not contain 0 or there is a positive number x such that B = {(–x, x) –A}. Then **B** is a basis for a topology T on R and the space (R,T) is a Hausdorff space. Also $A = \{\frac{1}{n}, n \in N\}$ is a closed subset of R and $0 \notin A$. However, if U and V are open sets such that $A \subseteq U$ and $0 \in V$, then $U \cap V \neq \phi$. Hence (R, T) is not regular.

Theorem. 1. A topological space X is regular if and only if for every point $x \in X$ and open set G containing x there exists an open set G* such that $x \in G^*$ and $C(G^*) \subseteq G$.

Proof. Suppose X is regular and the point x belongs to the open set G. Then F = X − G is a closed set which does not contain x. Since (X, T) is regular, there exist two open sets G_F and G_0^* such that $F \subseteq G_F$, $x \in G^*$, and $G_F \cap G^* = \phi$. Since $G^* \subseteq G_F^C$, and therefore

$$\begin{aligned} C(G^*) &\subseteq C(G_F^C) \\ &= G_F^C \quad (\because G_F^C \text{ is closed}) \\ &\subseteq F^C = G \end{aligned}$$

Conversely suppose that the condition holds and x is a point not in the closed set F. Then x belongs to the open set F^C and by hypothesis there must exist an open set G* such that $x \in G^*$ and $C(G^*) \subseteq F^C$. Clearly G* and $[C(G^*)]^C$ are disjoint open sets containing x and F respectively.

$$\Rightarrow \quad (C(G^*))^C \supseteq (F^C)^C = F .$$

Theorem 2. A T_1-space (X, T) is regular if and only if for each $p \in X$ and each closed set C such that $p \notin C$, there exist open sets U and V such that

$$C \subseteq U,\ p \in V \text{ and } \bar{U} \cap \bar{V} = \phi .$$

Proof. Suppose (X, T) is a regular space. Let $p \in X$ and C be a closed set such that $p \notin C$. Then X–C is a neighbourhood of p and hence by Theorem 1, there is a neighbourhood W of p such that $\bar{W} \subseteq X–C$. Again by Theorem 1, there is a neighbourhood V of p such that $\bar{V} \subseteq W$. Let $U = X– \bar{W}$, since $\bar{W} \subseteq X – C$, $C \subseteq X – \bar{W} \subseteq U$.

Further

$$\bar{V} \cap \bar{U} \subseteq W \cap (X– \bar{W}) = \phi .$$

Therefore U and V are desired open sets.

Remark. Since in a Hausdorff, every singleton set is closed for each $p \in X$. It follows that a compact Hausdorff space is regular.

Remark. Regular space satisfies Hereditary and topological property.

Theorem. 3. Regularity is a topological property.

Proof. Let (X, T) be a regular space and let (Y, T*) be homemorphic image of (X, T) under a map *f*. Let F be a T*-closed set and y is a point of Y which is not in F. Since *f* is one to one onto function, there exists $x \in X$ such that $f(x) = y$. Now *f* being continuous, $f^{-1}(F)$ is closed in X. Since $y \notin F$, we have $f^{-1}(y) \notin f^{-1}(F) \Rightarrow x \notin f^{-1}(F)$ (since $x = f^{-1}(x)$). Thus $x \in X$, such that $x \notin f^{-1}(F)$ which is T-closed. Now regularity of (X, T) implies that the exist two disjoint open sets G and H such that $x \in G$ and $f^{-1}(F) \subset H$, $G \cap H = \phi \Rightarrow f(x) \in f(G)$ and $f(f^{-1}(F)) \subset f(H)$ and $f(G \cap H) = f(\phi) \Rightarrow y \in f(G)$ and $F \subseteq f(H)$ and $f(G) \cap f(H) = \phi$ Since *f* is open, we have *f*(G) and *f*(H) are open sets of Y whenever, G and h are open in X. Hence there exists two disjoint open sets *f*(G) and *f*(H) in (Y, T*) such that $y \in f(G)$ and $F \subseteq f(H)$. Hence (Y, T*) is regular.

Theorem 4. Regularity is a Hereditary property.

Proof. Let (X, T) be a regular space and (X*, T*) be a subspace of it. Suppose that F* is closed set in X* and $x^* \in X^*$ such that x* is not in F*. Then $C(F^*) = C(F) \cap X^*$ where F is closed in X. Since F* is closed, therefore $F^* = C(F^*) = C(F) \cap X^*$. Then $x^* \notin F^* \Rightarrow x^* \notin C(F) \cap X^* \Rightarrow x^* \notin C(F)$ or $x^* \notin X^* \Rightarrow x^* \notin C(F)$ for $x^* \in X^*$. But C(F) is closed set, since the closure of any set is closed and (X, T) is regular, $x^* \in X$ and $x^* \notin C(F)$, $\exists$ two disjoint open sets G and H such that $x^* \in G$ and $C(F) \subseteq H$. Then $C(F) \cap X^* \subseteq H \cap X^*$ and $x^* \in G \cap X^*$ and moreover $(H \cap X^*) \cap (G \cap X^*) = (G \cap H) \cap (X \cap X^*) = \phi \cap X^* = \phi$. Thus it follows that to each point $x^* \in X^*$ and a closed set F*, $\exists$ disjoint open sets $G \cap X^*$ and $H \cap X^*$ such that $x^* \in G \cap X^*$ and $F^* = C(F) \cap X^* \subseteq H \cap X^*$ Hence (X*, T*) is regular.

Normal Spaces

Normal spaces were introduced by Victoris in 1921 and by Tietz in 1923.

Definition. A topological space X is said to be normal if and only if it satisfies the following axiom of Urysohn:

[N] If F_1 and F_2 are two disjoint closed subsets of X, then there exist two disjoint open sets, one containing F_1 and the other containing F_2.

Definition. A T_4-space is a normal space which is also a T_1-space.

Example. Let $X = \{a, b, c\}$

and $T = \{\phi, X, (b, c), (a)\}$

Clearly (X, T) is a normal space as {a} and {b, c} are closed as well as open disjoint sets such that $\{a\} \subseteq \{a\}$ and $\{b, c\} \subseteq \{b, c\}$. As already proved it is not T_1. Therefore (X, T) is not T_4-space.

Theorem. 5. A topological space (X, T) is normal if and only if for any closed set F and open set G containing F, there exists an open set G* such that $F \subseteq G^*$ and $C(G^*) \subseteq G$.

Proof. Suppose (X, T) is normal and the closed set F is contained in the open set G. Then $K = X - G$ is a closed set which is disjoint from F. Since the space is normal, there exist two open sets G^* and G_K such that $F \subseteq G^*$ and $K \subseteq G_K$ and $G^* \cap G_k = \phi$. Since $G^* \subseteq X - G_k = G_k^C$, we have

$$\begin{aligned} C(G^*) &\subseteq C(X - G_K) \\ &= X - G_K \quad [\because X - G_K \text{ is closed}] \\ &\subseteq X - K = G. \ [\because K \subseteq G_K \Rightarrow K^C \supseteq G_K^C] \end{aligned}$$

Conversely, suppose that the condition holds and let F_1 and F_2 be two disjoint closed subsets of X. Then F_1 is contained in the open set $X - F_2$, and by hypothesis, there exists an open set G* such that $F_1 \subseteq G^*$ and $C(G^*) \subseteq X - F_2$. Clearly G* and $X - C(G^*)$ are the desired disjoint open sets containing F_1 and F_2 respectively.

Remark. Although the property of normality is topological but it is not hereditary, which follows from the following theorem.

Theorem. 6. Normality is a topological property.

Proof. Let (Y, T*) be an homemorphic image of a normal space (X, T) under the homemorphism *f*. Let G* and H* be two disjoint T* closed sets in Y. Since *f* is continuous, $f^{-1}(G^*)$ and $f^{-1}(H^*)$ are T-closed sets in (X, T) satisfying $f^{-1}(G^*) \cap f^{-1}(H^*) = f^{-1}(G^* \cap H^*) = f^{-1}(\phi) = \phi$. The space (X, T) being normal, there exist two disjoint open sets G and H such that $f^{-1}(G^*) \subseteq G$ and $f^{-1}(H^*) \subseteq H$. Then $f[f^{-1}(G^*)] \subseteq f(G)$ and $f[f^{-1}(H^*)] \subseteq f(H)$, that is $G^* \subseteq f(G)$ and $H^* \subseteq f(H)$. Since *f* is an open mapping, *f*(G) and *f*(H) are open sets in Y. Moreover $f(G) \cap f(H) = f(G \cap H) = f(\phi) = \phi$. Thus we have shown that if G* and H* are two disjoint closed subsets of Y, then there exists two disjoint open sets *f*(G) and *f*(H) such that $G^* \subseteq f(G)$ and $H^* \subseteq f(H)$. Hence (Y, T*) is normal.

Theorem. 7. (Urysohn's Lemma.) A topological space X is normal iff for every two disjoint closed subsets F_1 and F_2 of X and closed interval [a, b] of reals, there exists a continuous mapping $f : X \to [a, b]$ such that $f[F_1] = \{a\}$ and $f[F_2] = \{b\}$.

Proof. Since the mapping h defined by $h(x) = (b-a)\, x + a$ is a continuous mapping of [0, 1] onto [a, b], it suffices to prove the result for the closed interval [0, 1].

Suppose first that F_1 and F_2 are closed subsets of X such that $F_1 \cap F_2 = \phi$ and g a continuous mapping from X to [0, 1] satisfying the conditions $g(F_1) = \{0\}$ and $g(F_2) = \{1\}$. We want to prove that (X, T) is normal. To prove it, let

$$G = f^{-1}\left(\left[0, \frac{1}{2}\right)\right), \quad H = f^{-1}\left[\left(\frac{1}{2}, 1\right)\right]$$

we shall show that G and H are disjoint open subsets of X such that $F_1 \subseteq G$ and $F_2 \subseteq H$.

Since $\left[0, \frac{1}{2}\right)$ and $\left(\frac{1}{2}, 1\right]$ are open subsets of [0, 1]. (In fact they are open sets of the lower limit topology) and f is a continuous map, it follows that G and H are subsets of X.

Further $\qquad f(F_1) = \{0\} \Rightarrow F_1 \subseteq f^{-1}(\{0\}) \qquad (1)$

and $\qquad \{0\} \subset \left[0, \frac{1}{2}\right) \Rightarrow f^{-1}\{0\} \subset f^{-1}\left[0, \frac{1}{2}\right)$

$\Rightarrow \qquad f^{-1}(\{0\}) \subset G \qquad (2)$

From (1) and (2), we have $F_1 \subset G$.

Similarly $\qquad f(F_2) = \{1\} \Rightarrow F_2 \subseteq f^{-1}(\{1\}) \qquad (3)$

and $\qquad \{1\} \subset \left(\frac{1}{2}, 1\right] \Rightarrow f^{-1}(\{1\}) \subset f^{-1}\left\{\left(\frac{1}{2}, 1\right]\right\} \qquad (4)$

From (3) and (4), it follows that $F_2 \subset H$.

Also

$$G \cap H = f^{-1}\left(\left[0, \frac{1}{2}\right)\right) \cap f^{-1}\left(\left(\frac{1}{2}, 1\right]\right)$$

$$= f^{-1}\left[\left[0, \frac{1}{2}\right) \cap \left(\frac{1}{2}, 1\right]\right]$$

$$= f^{-1}(\phi) = \phi$$

$\Rightarrow$ (X, T) is normal space.

Conversely suppose that (X, T) is a normal space. We need only construct a continuous mapping g X$\to$[0, 1] such that for every pair, F_1, F_2 of disjoint closed subsets of X, we have $g(F_1) = \{0\}$ and $g(F_2) = \{1\}$. Then f = hog will be the required mapping.

We will first define a collection $\{G_r$; r rational$\}$ of open sets such that $C(G_r) \subseteq G_s$, $r < s$, in the following way.

Let $\qquad G_r = \phi$ for all $r < 0$

$\qquad G_r = X$ for all $r > 1$.

Next, we define $G_1 = X - F_2$ which is an open set containing F_1 with the desired property. By the characterization of normality, G_1 contains an open set G_0 containing F_1 such that $C(G_0) \subseteq G_1$.

Now let $\{r_n\}_{n \in N}$ be a set of all rationals in [0, 1] with $r_1 = 0$ and $r_2 = 1$. For each $n \geq 3$, we will inductively define the open set G_{r_n} by taking the largest r_i and the smallest r_j such that i, j < n and $r_i < r_n < r_j$ and then using the characterization of normality to obtain the open set G_{r_n} with the property that

$$C(G_{r_i}) \leq G_{r_n} \text{ and } C(G_{r_n}) \leq G_{r_j}$$

In fact let us suppose, we started with the standard way of arranging the elements of $\{r_n\}$ in an infinite sequence :

$$\{r_n\} = \left\{0, 1, \frac{1}{2}, \frac{1}{3}, \frac{2}{3}, \frac{1}{4}, \frac{3}{4}, \frac{1}{5}, \frac{2}{5}, \frac{3}{5} ......\right\}$$

After defining G_0 and G_1, we would define $G_{1/2}$. So we set $r_n = \frac{1}{2}$. Then the largest r_i and the smallest r_j such that i, j < n and $r_i < r_n < r_j$ are 0 and 1 respectively. Thus we have

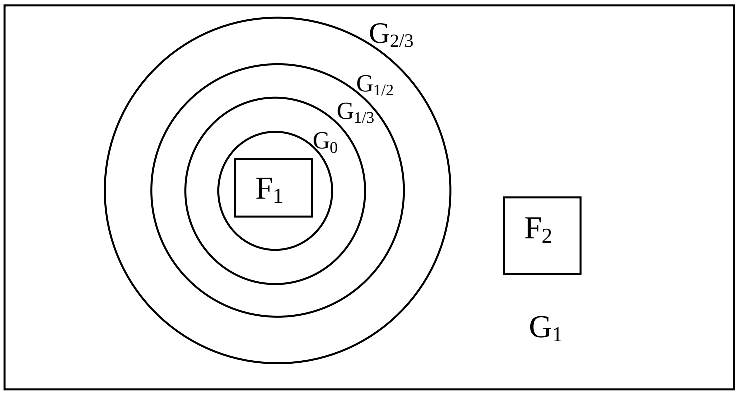
$$C(G_0) \leq G_{1/2} \text{ and } C(G_{1/2}) \subseteq G_1$$

Then we would fit $G_{1/3}$ between G_0 and $G_{1/2}$ and $G_{2/3}$ between $G_{1/2}$ and G_1 and so on.

We now define the desired mapping g by setting

$$g(x) = \text{Inf } \{r \,;\, x \in G_r\}$$

If $x \in F$, then $x \in G_r$ for every $r \geq 0$. Therefore



$$g(x) = \text{inf } \{\text{all non negative rational numbers}\}$$
$$= 0$$

which implies that $g(F_1) = \{0\}$

If $x \in F_2$, then $x \in G_r$ for no $r \leq 1$. Therefore

$$g(x) = \text{Inf } \{\text{all rationals greater than one}\}$$
$$= 1$$

which yields $g(F_2) = \{1\}$

We now show that g is continuous. For this purpose, we first prove

(i) $x \in C(G_p) \Rightarrow g(x) \leq p$

(ii) $x \notin G_p \Rightarrow g(x) \geq p$.

To prove first, note that if $x \in C(G_p)$, then $x \in G_s$ for every s . p. Therefore the set $\{r \,;\, x \in G_r\}$ contains all rationals greater than p. We thus have

$$g(x) = \text{Inf } \{r \,;\, x \in G_r\} \leq p$$

To prove (ii), note that if $x \notin G_p$, then x is not in G_s for any $s < p$. Therefore the set $\{r \,;\, x \in G_s\}$ contains no rational less than p and thus

$$g(x) = \text{Inf } \{r \,;\, x \in G_r\} \geq p$$

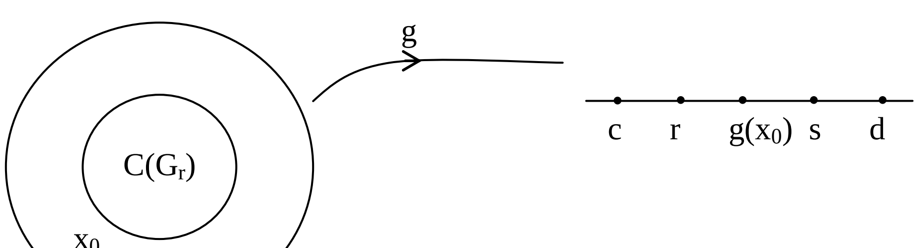
Now we come to the continuity of g. Given a point x_0 of X and an open interval (c, d) in **R** containing the point $g(x_0)$, we wish to find a nbd G of x_0 such that $g(G) \subseteq (c, d)$. Choose rational number r and s such that

$$c < r < g(x_0) < s < d$$

we assert that the open set

$$G = G_s - C(G_r)$$

is the desired nbd of x_0.



G_s

First we show that $x_0 \in G$. Necessarily $x_0 \in G_s$ because $x_0 \notin G_r \Rightarrow$ by (ii) that $g(x_0) \geq s$. Also $x_0 \notin C(G_r)$ because $x_0 \in C(G_r) \Rightarrow$ by (i) that $g(x_0) \leq r$. But $g(x_0) < s$. Hence $x_0 \in G$.

Secondly we show that $g(G) \subseteq (c, d)$. Let $x \in G$. Then $x \in G_s \subset C(G_s)$ so that $g(x) \leq s$ by (i). And $x \notin C(G_2)$, so that $x \notin G_r$ and $g(x) \geq r$ by (ii) Thus $g(x) \in [r, s] \subset (c, d)$ as desired.

Theorem 8. If the open set G has a non-empty intersection with a connected set C in a T_4-space X, Then either C consists of only one point or the set $C \cap G$ has cardinality greater than or equal to cordiality of reals.

Proof. Let us choose a point $x \in C \cap G$. Now if $C \cap G = \{x\}$, then $C = \{x\} \cup (C - \{x\})$ would be a separation of C in the T_1-space X unless C consists of only one point.

Now if $y \in C \cap (G - \{x\})$, The sets $\{x\}$ and $F = \{y\} \cup G^C$ would be disjoint, closed subsets of the normal space X. By Urysohn's lemma, there exists a continuous mapping

$$f : X \rightarrow [0, 1]$$

such that $f\{x\} = [0]$ and $f[F] = \{1\}$

Since f is continuous, $f[C]$ is a connected subset of [0, 1].

Since $f(x) = 0, f(y) = 1$

$f[C]$ must be all of [0, 1] and so have cardinality C.

Now $C \cap G^C \subseteq F$

so $f(C \cap G^C) = \{1\}$ has finite cardinality.

However
$$\begin{aligned} f[C] &= f(C \cap X) = f(C \cap (G \cup G^C)) \\ &= f[(C \cap G) \cup (C \cap G^C)] \\ &= f(C \cap G) \cup f(C \cap G^C) \end{aligned}$$

Thus $f(C \cap G)$ must have cardinality C and so $C \cap G$ must have cardinality greater than or equal to C.

Theorem 9. (Tietze Extension Theorem)

A topological space X is normal if and only if for every real-valued continuous function f of a closed subset F of X into a closed interval [a, b], there exists a real valued continuous mapping f^* of X into [a, b] such that $f^*/F = f$.

Proof. Suppose that for every real-valued continuous mapping f of a closed subset F of X into [a, b], there exists a continuous extension of f over X. We shall show that (X, T) is normal. To prove it, let F_1 and F_2 be two closed subsets of X such that $F_1 \cap F_2 = \phi$ and let [a, b] any closed interval. We define a mapping

$$f : F_1 \cup F_2 \rightarrow [a, b]$$
$$f(x) = a \text{ if } x \in F_1$$
$$f(x) = b \text{ if } x \in F_2$$

This mapping is certainly continuous over the subspace $F_1 \cup F_2$. Because if H be any closed subset of [a, b], then

$$f^{-1}(H) = \begin{cases} F_1 \text{ if } a \in H \text{ and } b \notin H \\ F_2 \text{ if } b \in H \text{ and } a \notin H \\ F_1 \cup F_2 \text{ if } a \in H \text{ and } b \notin H \\ \varphi \text{ if } a \notin h \text{ and } b \notin H \end{cases}$$

Since F_1 and F_2 are disjoint, we have

$$(F_1 \cup F_2) \cap F_1 = F_1 \text{ and } (F_1 \cup F_2) \cap F_2 = F_2$$

so that F_1 and F_2 are closed in $F_1 \cup F_2$. Of course $F_1 \cup F_2$ and ϕ are closed in $F_1 \cup F_2$. Hence f is a continuous map over the subspace $F_1 \cup F_2$. Hence by hypothesis, f can be extended to a continuous map g over X. This means that there exists a continuous map

$$g : X \to [a, b]$$

such that $g(x) = a$ if $x \in F_1$

and $g(x) = b$ if $x \in F_2$

The mapping g now satisfies the condition stated in Urysohn's lemma and hence (X, T) is normal.

Conversely let (X, T) be a normal space and let f be a real valued continuous map of the closed subset F into the closed interval [a, b] which for numerical convenience we take [–1, 1]. To show that there exist a continuous extension of f over X we begin by defining a map

$$f_0 : F \to [-1, 1]$$

by setting

$$f_0(x) = f(x)\ \forall\ x \in F \tag{1}$$

let $G_0 = f_0^{-1}\left(\left[-1, -\frac{1}{3}\right]\right)$, $H_0 = f_0^{-1}\left(\left[\frac{1}{3}, 1\right]\right)$

Since $\left[-1, -\frac{1}{3}\right]$ and $\left[\frac{1}{3}, 1\right]$ are closed in [–1, 1] and f_0 is continuous, it follows that G_0 and H_0 are closed in F and so also closed in X. Furthermore

$$G_0 \cap H_0 = f_0^{-1}\left(\left[-1, -\frac{1}{3}\right]\right) \cap f_0^{-1}\left(\left[\frac{1}{3}, 1\right]\right)$$

$$= f_0^{-1}\left(\left[-1, -\frac{1}{3}\right] \cap \left[\frac{1}{3}, 1\right]\right)$$

$$= f_0^{-1}(\phi) = \phi$$

Thus G_0, H_0 are disjoint closed subsets of X. Since X is normal, from Urysohn's lemma, we have a continuous map

$$g_0 : X \to \left[-\frac{1}{3}, \frac{1}{3}\right]$$

such that

$$g_0[G_0] = \left\{-\frac{1}{3}\right\} \text{ and } g_0(H_0) = \left\{\frac{1}{3}\right\}$$

we next define a map

$$f_1 : F \to \left[-\frac{2}{3}, \frac{2}{3}\right]$$

by setting

$$f_1(x) = f_0(x) - g_0(x)$$

Since f_0 and g_0 are continuous, f_1 is also a continuous map. Further, the range of f_1 is contained in $\left[-\frac{2}{3}, \frac{1}{3}\right]$ as shown below:

If $x \in G_0$, then

$$-1 \le f_0(x) \le -\frac{1}{3} \text{ and } g_0(x) = -\frac{1}{3}$$

so that

$$0 \ge f_0(x) - g_0(x) \ge -1 - \left(-\frac{1}{3}\right)$$

i.e.
$$0 \ge f_0(x) - g_0(x) \ge -\frac{2}{3}$$

Similarly, if $x \in H_0$, then

$$+\frac{1}{3} \le f_0(x) \le 1 \text{ and } g_0(x) = \frac{1}{3}$$

so that

$$0 \le f_0(x) - g_0(x) \le 1 - \frac{1}{3}$$

i.e.
$$0 \le f_0(x) - g_0(x) \le \frac{2}{3}$$

Finally if, $x \notin G_0 \cup H_0$, then $-\frac{1}{3} < f_0(x) < \frac{1}{3}$ and $-\frac{1}{3} < g_0(x) < \frac{1}{3}$, so that

$$-\frac{1}{3} - \frac{1}{3} < f_0(x) - g_0(x) < \frac{1}{3} - \left(-\frac{1}{3}\right)$$

that is
$$-\frac{2}{3} < f_0(x) - g_0(x) < \frac{2}{3}$$

Now let

$$G_1 = f_1^{-1}\left(\left[-\frac{2}{3}, - \frac{1}{3}.\frac{2}{3}\right]\right)$$

$$H_1 = f_1^{-1}\left(\left[\frac{1}{3}.\ \frac{2}{3},\ \frac{2}{3}\right]\right)$$

Dealing with G_1 and H_1os we did with G_0 and H_0, we can show that G_1 and H_1 are disjoint closed subsets of X. By Urysohn's lemma, there exists a continuous map

$$g_1 : X \to \left[-\frac{1}{3}\left(\frac{2}{3}\right), \frac{1}{3}\left(\frac{2}{3}\right)\right]$$

such that

$$g_1(G_1) = \left\{-\frac{1}{3}\left(\frac{2}{3}\right)\right\} \text{ and } g_1(H_1) = \left\{\frac{1}{3}.\frac{2}{3}\right\}$$

we next define a map

$$f_2 : F \to \left[-\left(\frac{2}{3}\right)^2, \left(\frac{2}{3}\right)^2\right]$$

by setting

$$f_2(x) = f_1(x) - g_1(x)$$

$$= f_0(x) - g_0(x) - g_1(x) \ \forall \ x \in F.$$

As before, f_2 is a continuous map whose range is contained in the closed interval

$$\left[-\left(\frac{2}{3}\right)^2, \left(\frac{2}{3}\right)^2\right]$$

We continue the construction by induction. Suppose that for n = 0, 1, 2,…m−1, there exists a continuous map

$$g_n : X \to \left[-\frac{1}{3}\left(\frac{2}{3}\right)^n, \frac{1}{3}\left(\frac{2}{3}\right)^n\right]$$

we define a map

$$f_m : F \to \left[-\left(\frac{2}{3}\right)^m, \left(\frac{2}{3}\right)^m\right]$$

by setting

$$f_m = f_0(x) - \sum_{n=0}^{m-1} g_n(x) \ \ \forall \ x \in F$$

Now let

$$G_m = f_m^{-1}\left(\left[-\left(\frac{2}{3}\right)^m, -\frac{1}{3}\left(\frac{2}{3}\right)^m\right]\right)$$

$$H_m = f_m^{-1}\left(\left[\frac{1}{3}\left(\frac{2}{3}\right)^m, \left(\frac{2}{3}\right)^m\right]\right)$$

Since $\left[-\left(\frac{2}{3}\right)^m, -\frac{1}{3}\left(\frac{2}{3}\right)^m\right]$ and $\left[\frac{1}{3}\left(\frac{2}{3}\right)^m, \left(\frac{2}{3}\right)^m\right]$

are disjoint closed subsets of [−1, 1] and f_m is continuous map. It follows that G_m, H_m are disjoint and closed in F as so also closed in X. Since G_m and H_m are disjoint closed subsets of normal space X, by Urysohn's lemma, there exists a continuous map,

$$g_m : X \to \left[-\frac{1}{3}\left(\frac{2}{3}\right)^m, \frac{1}{3}\left(\frac{2}{3}\right)^m\right]$$

such that

$$g_m[G_m] = \left\{-\frac{1}{3}\left(\frac{2}{3}\right)^m\right\}$$

and $$g_m[H_m] = \left\{\frac{1}{3}\left(\frac{2}{3}\right)^m\right\}$$

we define a mapping

$$f_{m+1} : F \to \left[-\left(\frac{2}{3}\right)^{m+1}, \left(\frac{2}{3}\right)^{m+1}\right]$$

by setting

$$f_{m+1} = f_m(x) - g_m(x) = f_0(x) - \sum_{n=0}^{m} g_n(x) \ \forall \ x \in F$$

As in the case of f_1 , It can be shown that f_{m+1} is a continuous map whose range in contained in $\left[-\left(\frac{2}{3}\right)^{m+1}, \left(\frac{2}{3}\right)^{m+1}\right]$. Thus the induction is complete.

We now set

$$g(x) = \sum_{n=0}^{\infty} g_n(x) \ \forall \ x \in X$$

and show that g is continuous extension of f over X. We observe that

$$|g(x)| = |\sum_{n=0}^{\infty} g_n(x)| \le \sum_{n=0}^{\infty} |g_n(x)| \le \sum_{n=0}^{\infty} \frac{1}{3}\left(\frac{2}{3}\right)^n$$

$$= \frac{1/3}{1-2/3} = 1$$

So by Weierstrass's M–Test, the series $\sum_{0}^{\infty} g_n(x)$ converges uniformly and absolutely over X and since $g_n(x)$ is continuous, it follows that g is a continuous mapping of X into [–1, 1]. Finally, we see that

$$|f_m(x)| \le \left(\frac{2}{3}\right)^m \to 0 \text{ as } m\to\infty.$$

Since $$f_m(x) = f_0(x) - \sum_{n=0}^{m-1} g_n(x) \ \ \forall x \in F$$

we have

$$\lim_{m\to\infty} f_m(x) = f_0(x) - \lim_{m\to\infty} \sum_{n=0}^{m-1} g_n(x)$$

Hence

$$0 = f_0(x) - g(x) \ \forall \ x \in F.$$

that is, $g(x) = f_0(x) = f(x) \ \forall \ x \in F$ by (1)

It follows that g is a continuous extension of f over X.

Theorem 10. Every compact Hausdorff space is normal (T_4).

Proof. Let F and F* be two disjoint closed subsets of the compact Hausdorff space X. Since F and F* are closed subsets of a compact space, they are also compact. Since the space is hausdorff, for each point <x, x*> of points with $x \in F$ and $x^* \in F^*$, there exist disjoint open sets G(x, x*) and G*(x, x*) containing x and x* respectively. For each fixed point $x \in F$, the collection

$$\{G^*(x, x^*) : x^* \in F^*\}$$

forms an open covering of the compact set F*. Hence there must be a finite subcovering which we denote by

$$\{G^*(x, x_i^*) \ ; \ i = 1, 2,\ldots,n\}$$

Now let

$$G^*(x) = \bigcup_{i=1}^{n} G^*(x, x_i^*)$$

and $$G(x) = \bigcap_{i=1}^{n} G(x, x_i^*)$$

Then G(x) and G*(x) are open disjoint sets containing x and F* respectively. Now the collection

$$\{G(x) \ ; \ x \in F\}$$

forms an open covering of the compact set F. There must be a finite subcovering, which we denote by

$$\{G(x_i)\ ;\ i = 1, 2,\ldots m\}$$

If we let $G = \bigcup_{i=1}^{m} G(x_i)$ and the finite intersection. $G^* = \bigcap_{i=1}^{m} G^*(x_i)$, then G and G* are two disjoint open sets containing F and F* respectively.

Theorem. 11. Every regular Lindeloff space is normal.

Proof. Let F and F* be two disjoint closed subsets of the regular Lindeloff space X. Since F and F* are closed subsets of a Lindeloff space they themselves are Lindeloff. Then for each $x \in F$, there exists an open set G(x) such that $x \in G(x) \subseteq C(G(x)) \subseteq X - F^*$ [$\because X - F^*$ is open containing F] .
The collection $\{G(x);\ x \in F\}$ forms an open covering of the Lindeloff set F. There must be a countable subcovering which we denote by $\{G_i\}_{i \in N}$. Similarly, for each point $x \in F^*$, there must exist an open set G*(x) such that $x \in G^*(x) \subseteq C(G^*(x)) \subseteq X - F$. The collection $\{G^*(x),\ x \in F^*\}$ forms an open covering of the Lindeloff set F*. There must be a countable subcovering, which we denote by $\{G_i^*\}_{i \in N}$ now the sets

$$G = \bigcup_{n \in N} [G_n - \bigcup_{i \le n} C(G_i^*)]$$

and

$$G^* = \bigcup_{n \in N} [G_n^* - \bigcup_{i \le n} C(G_i)]$$

are disjoint open sets containing F and F* respectively. To show it, let

$$V_n = G_n - \bigcup_{i \le n} C(G_i^*)$$

and

$$w_n = G_n^* - \bigcup_{i \le n} C(G_i)$$

Now $\quad V_n \cap C(G_i^*) = \phi$ when $i \le n$.

and $\quad w_i \subset G_i^* \subset C(G_i^*)$

Hence $\quad V_n \cap w_i = \phi$ when $i \le n$

Similarly $w_j \cap V_n = \phi$ when $n \le j$

Hence $V_m \cap w_K = \phi \quad \forall\ m, K \in N$

Now $\quad V_n = G_n - U\{C(G_i^*),\ i \le n\}$

$\qquad = G_n \cap [X - U\ \{C(G_i^*\ ;\ i \le n)\}]$

It follows that V_n is an open set, being the intersection of two open sets.

Theorem 12. The property of a space being T_4 is Hereditary.

Proof. Let (X, T) be a T_4-space so that it is T_1 as well as a normal space. Let (Y, T*) be a subspace of (X, T). We shall show that (Y, T*) is also a T_4-space. Since the property of a space being T_1 is hereditary, it follows that (Y, T*) is a T_1-space. We now show that (Y, T*) is a normal space. Let L* and M* be two disjoint T* closed subsets of Y. If x is an arbitrary point of L* and y that of M*, then $x \ne y$. Now $\{x\} \ne \{y\}$. being degenerate (singleton) sets in T_1 are disjoint T-closed subsets of X. Hence by normality of X, there exist T-open subsets G_x and H_y of X such that

$$\{x\} \subset G_x,\ \{y\} \subset H_y$$

and $\quad G_x \cap H_y = \phi$

These relations imply

$$L^* \subset U(G_x;\ x \in L^*\},\ M^* \subset U\{H_y,\ y \in M\}$$

and $[\cup\{G_x ; x\in L^*\}] \cap [\cup\{H_y ; y\in M^*\}] = \phi$

set $G = \cup\{G_x ; x\in L^*\}, H = \cup \{H_y, y\in M^*\}$

then G and H are T-open subsets of X such that

$$L^* \subset G, M^*\subset H \text{ and } G\cap H = \phi$$

Since $L^* \subset Y$ and $M^* \subset Y$, these relations imply

$$L^* C\ G\cap Y, M^*\subset H\cap Y$$

and $(G\cap Y) \cap (H\cap Y) = \phi$

If we set $G \cap Y = G^*, H \cap Y = H^*$

Then G^*, H^* are T^*-open subsets of Y such that

$$L^*\subset G^*, M^* \subset H^* \text{ and } G^* \cap H^* = \phi$$

which proves that (Y, T^*) is normal also.

Theorem 13. Closed subspace of a normal space is normal. (Weakly- Hereditary)

Proof. Let (X, T) be a normal space and let (Y, T^*) be a closed subspace of X. Let L^*, M^* be disjoint T^* closed subsets of Y. Then there exist T-closed subsets L, M of X such that $L^* = L\cap Y$ and $M^* = M\cap Y$

Since Y is T-closed, it follows that L^*, M^* are disjoint T-closed subsets of X. Then by normality of X, there exist T-open subsets G, H of X such that $L^* \subset G$, $M^*\subset H$ and $G\cap H = \phi$ since $L^*\subset Y$ and $M^* \subset Y$, we have

$$L^* \subset G\cap Y, M^*\subset H\cap Y$$

and $(G\cap Y) \cap (H\cap Y) = \phi$

Setting $G\cap Y = G^*, H\cap Y = H^*$

We see that G^* and H^* are T^*-open subsets of Y such that $L^* \subset G^*$, $M^* \subset H^*$ and $G^* \cap H^* = \phi$. Hence (Y, T^*) is normal.

Completely Normal Spaces

Completely normal spaces were introduced by Tietz in 1923.

Definition. A topological space (X, T) is said to be completely normal if and only if it satisfies the following axiom of Tietz : "If A and B are two separated subsets of X, then there exist two disjoint open sets G and H such that $A \subset G$ and $B \subset H$.

Definition. A completely normal space which is also T_1 is called a T_5-space.

Theorem 14. Every completely normal space is normal and hence every T_5-space is a T_4-space.

Proof. Let (X, T) be a completely normal space. Let A and B two closed subsets of X such that $A \cap B = \phi$. Since A and B are closed, we have $C(A) = A$, $C(B) = B$, and

so $C(A) \cap B = \phi, A\cap C(B) = \phi$

Thus A and B are separated subsets of X. By complete normality, there exist open sets G and H such that $A \subset G$, $B\subset H$ and $G\cap H = \phi$. It follows therefore that (X, T) is normal. Also definition, T_5 is T_1-space also. Therefore T_5 is normal as well as T_1-space also. Hence T_5 is T_4-space also.

Theorem. 15. Complete normality is a topological property

Proof. Let (X, T) be a completely normal space and let (Y, T^*) be its homemorphic image under a homemorphism *f*. We shall show that (Y, T^*) is completely normal. Let A and B be any two separated subsets of Y such that

$$A\cap C(B) = \phi, \ B\cap C(A) = \phi.$$

Since f is continuous mapping, we have

$$C[f^{-1}(A)] \subset f^{-1}[C(A)] \text{ and } C[f^{-1}(B)] \subset f^{-1}[C(B)]$$

Hence
$$f^{-1}(A) \cap C[f^{-1}(B)] \subset f^{-1}(A) \cap f^{-1}[C(B)]$$
$$= f^{-1}[A \cap C(B)] = f^{-1}(\phi) = \phi$$

and
$$C[f^{-1}(A)] \cap f^{-1}(B) \subset f^{-1}[C(A)] \cap f^{-1}(B)$$
$$= f^{-1}[C(A) \cap B] = f^{-1}(\phi) = \phi$$

Thus $f^{-1}(A)$ and $f^{-1}(B)$ are two separated subsets of X. Since (X, T) is completely normal, there exist T-open sets G and H such that

$$f^{-1}(A) \subset G,\ f^{-1}(B) \subset H \text{ and } G \cap H = \phi$$

These relations imply that

$$A = f[f^{-1}(A)] \subset f(G)$$
$$B = f[f^{-1}(B)] \subset f(H) \qquad (\because \text{ of onto})$$

and $f(G) \cap f(H) = f(G \cap H) = f(\phi) = \phi$ $\qquad [\because f \text{ is } 1\text{–}1]$

Also since f is an open map, $f(G)$ and $f(H)$ are T*-open sets. Thus we have shown that for any two separated subsets A, B of Y, there exist T* open subsets

$$G_1 = f(G) \text{ and } H_1 = f(H)$$

such that $A \subset G_1$, $B \subset H_1$ and $G_1 \cap H_1 = \phi$
which completes the proof of the theorem.

Corollary. The property of a space being T_5-space is a topological property.

Proof. Since the property of a space being a T_1-space and of being a completely normal space both are topological, it follows that the property of a space being a T_5-space is also topological.

Theorem 16. Complete normality is a Hereditary property.

Proof. Let (X, T) be a completely normal space and (Y, T*) be any subspace of (X, T). We shall show that (Y, T*) is also completely normal. Let A, B be T*-separated subsets of (Y, T*), we have

$$A \cap C^*(B) = \phi \text{ and } B \cap C^*(A) = \phi$$

Also
$$C^*(A) = C(A) \cap Y \text{ and } C^*(B) = C(B) \cap Y$$

Hence
$$\phi = A \cap C^*(B) = A \cap [C(B) \cap Y] = (A \cap C(B) \cap Y$$
$$= A \cap C(B) \qquad \text{(i)}$$

Similarly
$$\phi = B \cap C^*(A) = B \cap [C(A) \cap Y] = (B \cap C(A) \cap Y$$
$$= B \cap C(A) \qquad \text{(ii)}$$

So A and B are T-separated. Hence by completely normality of X, there exist T-open sets G and H such that

$$A \subset G,\ B \subset H \text{ and } G \cap H = \phi$$

Since A and B are subsets of Y,

$$A \subset G \cap Y,\ B \subset H \cap Y$$

and

$$(G \cap Y) \cap (H \cap Y) = (G \cap H) \cap Y = \phi \cap Y = \phi$$

It follows therefore that (Y, T*) is completely normal.

Cor. The property of a space being T_5-space is a Hereditary property.

Proof. The property of a space being T_1 as well as the property of a space being completely normal is hereditary. Therefore, it follows that the property of a space being T_5 is hereditary.

Theorem. 17. A topological space X is completely normal if and only if every subspace of X is normal.

Proof. Let (X, T) be a completely normal space and (Y, T*) be any subspace of (X, T). We shall prove that (Y, T*) is normal. Let F_1 and F_2 be any pair of T* closed subsets of Y such that $F_1 \cap F_2 = \phi$. We shall denote the T* closure of F_1 and F_2 by $C^*(F_1)$ and $C^*(F_2)$ and their T-closure by $C(F_1)$ and $C(F_2)$ respectively.
Since F_1 and F_2 are T*-closed,

$$C^*(F_1) = F_1 \text{ and } C^*(F_2) = F_2$$

Also $C^*(F_1) = Y\cap C(F_1),\ C^*(F_2) = Y\cap C(F_2)$

Hence
$$\begin{aligned} F_1 \cap C(F_2) &= C^*(F_1) \cap C(F_2) \\ &= (Y\cap C(F_1)) \cap C(F_2) \\ &= (Y\cap C(F_1)) \cap (Y\cap C(F_2)) \\ &= C^*(F_1) \cap C^*(F_2) \\ &= F_1 \cap F_2 = \phi \end{aligned}$$

Similarly $F_2 \cap C(F_1) = \phi$

It follows that F_1 and F_2 are two separated subsets of X. By complete normality there exist T-open sets G and H such that $F_1 \subset G$. $F_2 \subset H$ and $G\cap H = \phi$. Then the sets $G_1 = Y\cap G$.
and $H_1 = Y\cap H$ are T* open sets such that

$$F_1 \subset G_1,\ F_2 \subset H_1 \text{ and}$$
$$\begin{aligned} G_1 \cap H_1 &= (Y \cap G) \cap (Y\cap H) \\ &= Y \cap (G\cap H) = Y \cap \phi = \phi \end{aligned}$$

$\Rightarrow$ (Y, T*) is normal.

Conversely suppose that (Y, T*) is normal. Let A and B by two separated subsets of X.

Let $Y = X-(C(A) \cap C(B))$.

Then Y is T-open subset of X. The sets

$$Y \cap C(A) \text{ and } Y\cap C(B)$$

are T*–closed such that

$$\begin{aligned} (Y\cap C(A)) \cap (Y\cap C(B)) &= Y \cap(C(A) \cap C(B)) \\ &= [X-(C(A)\cap C(B))] \cap [C(A) \cap C(B)] \\ &= \phi \end{aligned}$$

Hence by the normality of Y, there exist T*-open subsets G, H of Y such that

$$Y\cap C(A) \subset G,\ Y\cap C(B)\subset H \qquad (1)$$

and $G\cap H = \phi$

Again, since G and H are T*-open sets, there exist T-open sets G_1 and H_1 such that

$$G = Y\cap G_1,\ H = Y\cap H_1$$

and since Y is T-open, it follows that G and H are also T-open sets. Further, since

$$Y = X-[C(A)\cap C(B)]$$

and A and B are separated sets of X since

$$\begin{aligned} Y &= X - [C(A) \cap C(B)] \\ &= [X - C(A)] \cup [X-C(B)] \end{aligned}$$

But $X-C(A) \supset B$ and $X-C(B) \supset A$

$\Rightarrow$ $B \subset Y$ and $A \subset Y$.

Hence it follows from (1) that $A \subset G$, $B \subset H$ and $G \cap H = \phi$ (2)

Also A and B are separated since $B \subset X - C(A)$

Thus we have shown that for any pair A, B of separated subsets of X, there exist T-open subsets G and H satisfying (2). Hence (X, T) is completely normal.

Remark. Every completely normal space is normal but there are normal spaces which are not completely normal e.g. consider

$$X = \{a, b, c, d\}$$
$$T = \{\phi, (a), (a, c), (a, b, c), X\}$$

Here ϕ and X are the only disjoint closed subsets of X.
Then (X, T) is normal. But the subspace

$$Y = \{a, b, c\}$$

of X is not normal as can be seen below If T* is a relative topology in Y which is given by

$$T^* = \{\phi, (a), (a, b), (a, c), Y\}$$

Here (b) and (c) are disjoint T* closed subsets of Y but there are no disjoint open sets G and H such that (b) $\subset$ G and (c) $\subset$ H.

Hence the subspace (Y, T*) is not normal. It follows that the space (X, T) is not completely normal. This ex shows that normality is not hereditary.

Completely Regular Space and $T_{3\frac{1}{2}}$ Space

Completely regular spaces were introduced by Paul Urysohn in a paper that appeared in 1925. Their importance was established by Tychonoff in 1930.

Definition. A topological space (X, T) is completely regular if and only if it satisfies the following axiom.
"If F is a closed subset of X and x is a point of X not in F, then there exists a continuous mapping $f : X \to [0, 1]$ such that $f(x) = 0$ and $f(x) = 1$

Definition. A Tichonov space or $\left(T_{3\frac{1}{2}} \text{ space}\right)$ is completely regular space which is also a T_1-space.

Theorem. 18. Every completely regular space is regular and hence every Tichonov space is a T_3-space.

Proof. Let (X, T) be completely regular space. Let F be a closed subset of X and let x be a point of X not in F i.e. $x \in X - F$. By complete regularity there exists a continuous map $f : X \to [0, 1]$ such that such that $f(x) = 0$ and $f(F) = \{1\}$. Also [0, 1] with relativised usual topology is a Hausdorff space. Hence there exist open sets G and H of [0, 1] such that

$$0 \in G \text{ and } 1 \in H \text{ and } G \cap H = \phi$$

Since f is continuous, $f^{-1}(G)$ and $f^{-1}(H)$ are T-open subsets of X such that

$$f^{-1}(G) \cap f^{-1}(H) = f^{-1}[G \cap H] = f^{-1}[\phi] = \phi$$

Further

$$f(x) = 0 \in G \Rightarrow x \in f^{-1}(G)$$

and
$$f(F) = \{1\} \in H \Rightarrow F \subset f^{-1}(H)$$

Thus there exist disjoint T-open sets $f^{-1}(G)$ and $f^{-1}(H)$ containing x and F respectively. It follows that (X, T) is regular. Also since every Tychonov space is a completely regular and T_1-space, if follows that every Tychonov space is a T_3-space.

Theorem. 19. Every T_4-space is a Tychonov space. $T_4 \Rightarrow T_{3\frac{1}{2}}$.

Proof. Let (X, T) be a T_4-space. Then by definition, it is normal and T_1-space. Hence it sufficies to show that (X, T) is completely regular. Let F be a T-closed subset of X and bet x be a point of X such that $x \notin F$. Since the space is T_1, $\{x\}$ is a closed subset of X. Thus $\{x\}$ and F are closed subsets of X. Again since the space is normal, by Urysohn's lemma, there exists a continuous mapping $f : X \to [0, 1]$ such that

$$f[(x)] = \{0\} \text{ and } f(F) = \{1\}$$

i.e. $$f(x) = 0 \text{ and } f(F) = 1.$$

Cor. Every compact Hausdorff space is a Tychonov space.

Proof. Since every compact Hausdorff space is normal, it is also Tychonov space.

Theorem 20. Complete regularity is a topological property.

Proof. Let (X, T) be a completely regular space and let (Y, V) be a homemorphic image of (X, T) under a homemorphism f. To show that (Y, V) is completely regular. Let F be a V-closed subset of Y and let y be a point of Y such that $y \notin F$. Since f in one-one onto, there exists a point $x \in X$ such that $f(x) = y \Leftrightarrow x = f^{-1}(y)$. Again since f is a continuous map, $f^{-1}(F)$ is a T-closed subset of X. Further $y \notin F \Rightarrow f^{-1}(y) \notin f^{-1}(F) \Rightarrow x \notin f^{-1}(F)$. Thus $f^{-1}(F)$ is a T-closed subset of X and $x = f^{-1}(y)$ is a point of X such that $x \notin f^{-1}(F)$. Hence by complete regularity of X there exists a continuous mapping g of X into [0, 1] such that

$$g[f^{-1}(y)] = g(x) = 0 \text{ and } g[f^{-1}(F)] = \{1\}$$

i.e. $$(gof^{-1})\,(y)) = 0 \text{ and } (gof^{-1})\,(F) = \{1\}$$

Since f is homemorphism, f^{-1} is a continuous mapping of Y onto X. Also g is a continuous map of X into [0, 1]. It follows that gof^{-1} is a continuous map of Y into [0, 1]. Thus we have shown that for each V-closed subset F of Y and each point $y \in Y - F$, there exists a continuous map $h = gof^{-1}$ of Y into [0, 1] such that $h(y) = 0$ and $h(F) = \{1\}$. Hence (Y, V) is completely regular.

Remark. The property of a space being Tychonov space is a topological property.

Theorem. 21. Complete regularity is Hereditary property.

Proof. Let (X, T) be a complete regular space and let (Y, T*) be a subspace of X. Let F* be a closed subset of Y and y be a point of Y such that $y \notin F^*$. Since F* is T* closed, there exists a T-closed subset F of X such that $F^* = Y \cap F$. Also

$$y \notin F^* \Rightarrow y \notin Y \cap F$$
$$\Rightarrow y \notin F \qquad [\because y \in Y]$$

and $$y \in Y \Rightarrow y \in X.$$

This F is a T-closed subset of X and y is a point of X such that $y \notin F$. Hence by complete regularity of X, there exists a continuous map f of X into [0, 1] such that

$$f(y) = 0 \text{ and } f(F) = [1]$$

let g denote the restriction of f to Y. Then g is continuous mapping of Y into [0, 1]. Now by definition of g,

$$g(x) = f(x) \ \forall \ x \in Y.$$

Hence

$$f(y) = 0 \Rightarrow g(y) = 0$$

and since $f(x) = 1 \ \forall \ x \in F$ and $F^* \subset F$, we have

$$g(x) = f(x) = 1 \ \forall \ x \in F^*$$

so that $g(F^*) = \{1\}$

Thus we have shown that for each T*-closed subset F* of Y and each point $y \in Y - F^*$, there exists a continuous map g of Y into [0, 1] such that $g(y) = 0$ and $g(F^*) = \{1\}$.

Remark. The property of a space being Tychonov is hereditary.

Theorem 22. A normal space is completely regular if and only if it is regular.

Proof. Since every completely regular space is regular, we only need to prove that any normal, regular space is completely regular. Let F be a closed subset of X not containing the point x so that x belongs to the open set X–F. Since the space X is regular, there exists an open set G such that $x \in G$ and $C(G) \subset X-F$ so that

$$F \cap C(G) = \phi$$

Thus C(G) and F are disjoint closed subsets of a normal space X. Hence by Urysohn's lemma, there exists a continuous mapping $f : X \to [0, 1]$ such that

$$f(F) = \{1\} \text{ and } f[C(G)] = \{0\}$$

Also $x \in G$ and $f[C(G)] = \{0\}$ implies that $f(x) = 0$. Hence (X, T) is completely regular.

Example. Let (X, T) be a completely regular space. Prove that if F is a T-closed subset of X and $p \notin F$, then there exists a continuous mapping f of X into [0, 1] such that $f(p) = 1$ and $f(F) = \{0\}$.

Analysis. Since (X, T) is completely regular, there exists a continuous map g of X into [0, 1] such that

$$g(p) = 0 \text{ and } g(F) = \{1\}$$

Now consider the mapping $f : X \to [0, 1]$ defined by setting $f(x) = 1-g(x) \; \forall \; x \in X$.

Since constant functions are continuous, it follows that f is continuous.

Further $$\overline{f(p)} = 1-g(p) = 1-0 = 1$$

$$f(x) = 1-g(x) = 1-1 \; \forall \; x \in F$$
$$= 0$$

so that $f(F) = \{0\}$.

Example. Consider the topology T on R defined as follows T-consists of ϕ, R and all open says of the form $[-\infty, a)$ $a \in \mathbf{R}$

Show that

(i) (R, T) is normal.

(ii) Not regular

(iii) Not T_4

(iv) Is (R, T) complete regular.

Analysis. (i) Here the only disjoint closed subsets of R are of the form ϕ and $[a, \infty)$ and for each such pair, there exist disjoint open sets ϕ and R such that $\phi \subset \phi$ and $[a, \infty) \subset R$. It follows that (R, T) is normal.

(ii) Consider the closed set $F = [1, \infty)$ and the point 0. Here $0 \notin F$. But the only open set containing F is R which must intersect every open set containing zero. Hence there exists no open sets G and H such that

$$0 \in G, F \subset H \text{ and } G \cap H = \phi.$$

$\Rightarrow$ The space (R, T) is not regular.

(iii) The space (R, T) is not T_1 since no singleton subset of R is T-closed. It follows that the space is not T_4.

(iv) Since the space (R, T) is normal and not regular, it follows from the theorem that a normal space is completely regular iff it is regular. Hence this space is not completely regular.

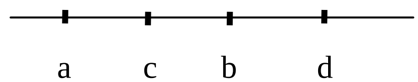
Remark. Let B be the class of open closed intervals in the real line R :

$$B = \{(a, b] \,;\, a, b \in R, a < b\}$$

Clearly R is the union of members of B since every real number belongs to some open-closed intervals. In addition. the intersection $(a, b] \cap (c, d]$ of any two open closed intervals is either empty or another open closed interval. e.g. if $a < c < b, < d$, then

$$(a, b] \cap (c, d] = (c, b]$$

as indicated in the diagram.



Thus the class T consisting of unions of open-closed intervals is a topology on R i.e. b is a base for the topology T on R. This topology T is called the upper limit topology on R. Similarly the class of closed-open intervals.

$$B^* = \{[a, b) \,;\, a, b \in R, a < b\}$$

is a base for a topology T* on R called the lower limit topology on R.

Example. If x and y are two distinct points of a Tychonoff space (X, T), then there exists a real valued continuous mapping *f* of X such that $f(x) \neq f(y)$

Analysis. Since (X, T) is a T_1-space, the singleton subset {y} is T-closed and since x and y are distinct point $x \notin \{y\}$. By complete regularity of X, there exists a real valued continuous mapping *f* of X such that

$$f(x) = 0 \text{ and } f[\{y\}] = \{1\}$$

$$\Rightarrow \quad f(x) = 0 \text{ and } f(y) = 1$$

But $0 \neq 1 \Rightarrow f(x) \neq f(y)$.

Remark: We have the following Hierarchy :

Metrizable $\Rightarrow$ completely normal $\Rightarrow$ normal

$\Rightarrow$ completely regular $\Rightarrow$ regular $\Rightarrow$ Hausdorff $\Rightarrow T_1 \Rightarrow T_0$.

6

EMBEDDING AND METRIZATION

Given a topological space (X, T), it is natural to ask whether there is a metric for X such that T is the metric topology. Such a metric metrizes the topological space and the space is said to be metrizable. Thus a metrizable space is a topological space whose topology is generated by some metric. There are topological spaces that are not metrizable for example let X be a set with atleast two members and let T be the trivial topology, then (X, T) is not metrizable.

Embedding

Definition. Let (X, T) and (Y, U) be topological spaces. An embedding or (sometimes called imbedding) of X into Y is a function $e : X \to Y$ which is a homemorphism when regarded as a function from (X, T) onto (e(X), U/e(X)) that is X is embedded in Y by *f* if and only if *f* is a homemorphism between X and some subspace of Y. Intuitively, to embed a space X into a space Y should mean that we can identify X with a subspace of Y, where identification is upto a homemorphism.

Example. The function $f : R \to R^2$ defined by $f(x) = (x, 0)$ for each $x \in R$ is an embedding of R in R^2.

Analysis. It is clear that *f* is one to one. Let T denote the usual topology on R^2 and let A $= \{(x, y) \in R^2, y = 0\}$. Then *f* maps R onto A. Since every metric space is first countable. In order to show that *f* is continuous, it is sufficient by the result that "Let *f* be a mapping of the first axiom space X into the topological space X*. Then *f* is continuous at $x \in X$ if and only if for every sequence $<x_n>$ of points in X converging to x, we have the sequence $<f(x_n)$ converging to the point $f(x) \in X^*$ " Now to show that if $<x_n>$ is a sequence in R that converges to x in R. Then $<f(x_n)>$ converges to $f(x)$ in A, is obvious. Thus f^{-1} is continuous. So if U denotes the usual topology on R, then $f : (R, \mu) \to (A, T_A)$ is a homemorphism

Remark. 1. If X is a subspace of a space Y, then the inclusion map $i : X \to Y$ defined by $i(x) = x$ for every $x \in X$ is an embedding. Thus inclusion maps are the most immediate examples of embeddings and as the definition implies, these are the only examples upto homemorphisms. An important problem in topology is to decide when a space X can be embedded in another space Y that is when there exists an embedding from X into Y. This is called the embedding problem. Theorems asserting the embeddability of a space into some other space which is more manageable than the original space are known as embedding theorems.

Theorem 1. Since a continuous bijection from a compact space onto a Hausdorff space is a homemorphism, every continuous, one to one function from a compact space into a Hausdorff space is an embedding.

Theorem 2. A function $e : X \to Y$ is an embedding if and only if it is continuous and one to one and for every open set V in X, there exists an open subset W of Y such that $e(V) = W \cap Y$.

Proof. The result follows directly from definitions of homemorphism and relative topology.

Definition. Let $\{Y_i ; i\in I\}$ be an indexed family of sets. Suppose X is a set and let for each $i\in I$, $f_i : X\to Y_i$ be a function. Then the function $e : X\to \prod_{i\in I} Y_i$ defined by $e(x)(i) = f_i(x)$ for $i\in I$, $x\in X$ is called the evaluation function of the indexed family $\{f_i ; i\in I\}$ of functions.

In other words, for each $x\in X$, the i-th co-ordinate of e(x) is obtained by evaluating the i-th function of f_i at x. This justifies the term evaluation function. Intuitively evaluation function is obtained by listing together the information given by various f_i's. To illustrate this, suppose X is the set of all students in a class and f_1, f_2, f_3,…, etc. are functions specifying respectively say, the age, the sex, the height etc of members of X. Then the evaluation function is like a catalogue which lists against each student all the information available about the student. For example, a typical entry in this catalogue might be

e(Mr. X. Y. Z) = (21, Male, 5 feet 6 inches,….)

The following Theorem Characterizes evaluation functions.

Theorem 3. Let $\{Y_i ; i\in I\}$ be a family of sets, X, a set and for each $i\in I$, $f_i : X\to Y_i$, a function then the evaluation function is the only function from X into ΠY_i whose composition with the projection $\pi_i : \Pi Y_i\to Y_i$ equals f_i for all $i\in I$.

Proof. Let $e : X\to \Pi Y_i$ be the evaluation function of the family $\{f_i ; i\in I\}$. Then for any $i\in I$ by very definition of e $\pi_i(e(x)) = e(x)(i) = f_i(x)$ and so $\pi_i \circ e = f_i$.

Conversely, suppose $e' : X\to \Pi Y_i$ satisfies that $\pi_i \circ e' = f_i$ for all $i\in I$. Let $x\in X$. Then for any $i\in I$,

$$e'(x)(i) = \pi_i (e'(x)) = f_i(x') = e(x)(i)$$

and so $e(x) = e'(x)$. But since $x\in X$ was arbitrary, this means that $e' = e$. Thus e is the only function from X into ΠY_i having the given property.

Theorem 4. The evaluation function defined above is continuous if and only if each f_i is continuous.

Proof. By the above prop, we have

$$\pi_i \circ e = f_i \text{ for all } i\in I.$$

Now by the result "Let (X, T) be the topological product of an indexed family of topological spaces $\{(X_i, T_i) ; i\in I\}$ and let Y be any topological space. Then a function $f : Y\to X$ is continuous with respect the product topology on X if and only if for each $i\in I$, the composition $\pi_i \circ f : y\to X_i$ is continuous, where $\pi_i : X\to X_i$ is the projection function". The evaluation map is continuous if and only if each f_i is continuous since projection mapping is always continuous.

Definition. An indexed family of functions $\{f_i : X\to Y_i ; i\in I\}$ where X, Y_i are topological spaces, is said to distinguish points from closed sets in X if for any $x\in X$ and any closed subset C of X not containing x, there exists $j\in I$ such that $f_j(x) \notin \overline{f_j(C)}$ in Y_j.

This definition reminds us of complete regularity because there too, a point was separated from a closed set by means of a real valued map as the next theorem shows.

Theorem 5. A topological space is completely regular iff the family of all continuous real valued functions on it distinguishes points from closed sets.

Proof. Let X be a topological space and let **F** be the family of all continuous real valued functions on X. Suppose first that X is completely regular. Let a point $x\in X$ and a closed subsets C of X, not containing x be given. Then there exists a continuous function $f : X\to[0, 1]$ such that $f(x) = 0$ and

$f(C) \subset \{1\}$. Then we can regard f as a function from X to R. Then $f \in \mathbf{F}$ and evidently $f(x) \notin \overline{f(C)}$ since {1} is a closed subset of R. So **F** distinguishes points from closed sets in X.

Conversely suppose that **F** distinguishes points from closed sets. Let $x \in X$ and C be a closed subset of X not containing x. Then there exists a map $f : X \to R$ such that $f(x) \notin \overline{f(C)}$. Now $\{f(x)\}$ and $\{\overline{f(c)}\}$ are disjoint closed subsets of R which is normal space. So there exists a continuous function $g : R \to [0, 1]$ which takes the values, 0 and 1 respectively on them. Let $h : X \to [0, 1]$ be the composite $g \circ f$. Then clearly $h(x) = 0$ and $h(y) = 1$ for all $y \in C$. Thus X is a completely regular space.

Theorem. (Embedding Lemma) Let $\{f_i ; X \to Y_i ; i \in I\}$ be a family of continuous functions which distinguishes points and also distinguishes points from closed sets. Then the corresponding evaluation map is an embedding of X into the product space $\prod_{i \in I} Y_i$. To prove this result, first we need to prove some Lemmas.

Lemma 1. The evaluation function of a family of functions is one-to-one if and only if that family distinguishes points.

Proof. For $i \in I$, let $f_i : X \to Y_i$ be a function, and let $e : X \to \Pi Y_i$ be the evaluation function. Let x, y be distinct points of X. Then $e(x) \neq e(y)$ if and only if there exists $j \in I$ such that $e(x)(j) \neq e(y)(j)$. But $e(x)(j) = f_j(x)$ by definition of e. Similarly $e(y)(j) = f_j(y)$.

So the condition that $e(x) \neq e(y)$ is equivalent to saying that there exists $j \in I$ (which may depend upon both x and y) such that $f_j(x) \neq f_j(y)$. Since x, y are arbitrary. Thus The evaluation function of a family of functions is one to one if and only if that family distinguishes points.

Lemma 2. Let $\{f_i ; X \to Y_i, i \in I\}$ be a family of functions which distinguishes points from closed sets in X. Then the corresponding evaluation function $e : X \to \prod_{i \in I} Y_i$ is open. When regarded as a function from X onto e(X).

Proof. Let V be an open subset of X. We have to show that e(V) is an open subset of e(X). A typical point of e(V) is of the form e(x) for some $x \in V$. Now $X - V$ is a closed subset of X not containing x. So by the hypothesis, there exists $j \in I$ such that $f_j(x) \notin \overline{f_j(X-V)}$. Let $G = Y_j - \overline{f_j(X-V)}$. Then G is an open subset of Y_j and so $\pi_j^{-1}(G)$ is an open subset of $\prod_{i \in I} Y_i$ we claim that

$$\pi_j^{-1}(G) \cap e(X) \subset e(V)$$

For suppose $y \in \pi_j^{-1}(G) \cap e(X)$. Then $y = e(z)$ for some $z \in X$. Also $\pi_j(y) \in G$ and so $\pi_j(e(z)) \in G$ whence $f_j(z) \in G$ since $\pi_j \circ e = f_j$. From this it follows that $z \in V$, since otherwise $f_j(z) \in f_j(X-V) \subset \overline{f_j(X-V)} = Y_j - G$. Thus $y = e(z) \in e(V)$ as was to be shown. Now the set $\pi_j^{-1}(G) \cap e(X)$ is open in the relative topology on e(X) and clearly it contains e(x), showing that e(V) is a neighbourhood of e(x) in e(X). But e(x) was a typical point of e(V). Then e(V) is a neighbourhood of each of its points in the relative topology on e(X). So e(V) is an open subset of e(X). Since this holds for all open sets V in X , we see that e, regarded as a function from X onto e(X) is open.

Remark. The converse of this lemma is false. Let X = R^2 and f_1, f_2 be the two projections on R. Then the evaluation function is simply the identity map which is open. But the family {f_1, f_2} does not distinguish points from closed sets in X.

Lemma 3. The evaluation function e : $X \to \prod_{i \in I} Y_i$ is continuous if and only if each $f_i : X \to Y_i$ is continuous.

Proof. Already proved.

Proof of the theorem. Let e : $X \to \Pi Y_i$ be the Lemma 3 evaluation function. Now continuity of the evaluation function e is continuous, follows from Lemma 3. That it is one to one follows from lemma 1 while lemma 2 shows that e is an open map when regarded as a function from X to e(X). Thus e is one to one, continuous and open from x to e(X). Hence e is an embedding.

Definition. The Cartesian product of closed unit intervals with the product topology is called a cube A cube is then the set I of all functions on a set A to the closed unit interval I with the topology of pointwise or co-ordinatewise, convergence. The cube is used as a standard sort of space and we want to describe those topological spaces which are homemorphic to subspaces of cubes.

Definition. The set I of all real sequences (a_1, a_2,…) such that $0 \le a_n \le \frac{1}{n}$ for every $n \in N$ is called Hilbert cube. Also I is a closed and bounded subset of R^∞

Tychonoff Embedding Theorem.

Theorem 7. A topological space is a Tychonoff space if and only if it is embeddable into a cube.

Proof. Let the topological space (X, T) be embedded into a cube. We shall prove that X is a Tychonoff space. Since we know that a Tychonoff space is T_1 and completely regular and we know that every cube is a Tychonoff space. The Tychonoff property is Hereditary. So every subspace of a cube is Tychonoff. Also Tychonoff property is topological property, it is preserved under homemorphism. Thus every space homemorphic to a subspace of a cube is a Tychonoff space.

Conversely suppose X is a Tychonoff space. Let **F** be the family of all continuous functions from X into [0, 1]. Then since X is completely regular **F** distinguishes points from closed sets in X. But since X is also T_1 all singletons are closed since we know that in a T_1-space each singleton is closed. Thus it follows that **F** distinguishes points as well. But by the embedding lemma "Let {f_i ; $X \to Y_i$; $i \in I$} be a family of continuous functions which distinguishes points and also distinguishes points from closed sets. Then the corresponding evaluation map is an embedding of X into the product space $\prod_{i \in I} Y_i$ ". We have, the evaluation map e : $X \to [0, 1]^{\mathbf{F}}$ is an embedding of X into the cube $[0, 1]^{\mathbf{F}}$.

Metrizability

Definition :- A topological space (X, T) is metrizable if there is a metric d on X such that the topology induced by d is T .

Note the distinction between a metric space and metrizable space. A metrizable space is a topological space whose topology is generated by some metric and a metric space is a set with a metric on it. Of course the metric on a set X generates a topology on X and thus a metric space (X, d) determines a topological space (X, T). Given a metrizable space (X, T), there is a metric d on X such that the topology induced by d is T. As the metric d is not unique, there are many metrics that generate T.

Remark :- The following example shows that there are topological spaces that are not metrizable.

Example. Let X be a set with at least two members and let T be the trivial topology on X. Then (X, T) is not metrizable.

Analysis. Let d be a metric on X and let μ be the topology generated by d. We show that (X, T) is not metrizable by showing that $\mu \neq T$. Let a and b denote distinct members of X. By the definition of a metric, there is a positive number r such that d(a, b) = r. Therefore $a \in B(a, r)$ but $b \notin B(a, r)$. Hence $B(a, r) \in b$, $B(a, r) \neq X$ and $B(a, r) \neq \phi$. But the only members of T are ϕ and X . Thus $\mu \neq T$.

Urysohn's Metrization Theorem

Theorem 8. Every second axiom T_3-space X is metrizable.

Proof. We shall show that any second axiom T_3-space X is homemorphic to a subset of the Hilbert cube. Since Hilbert cube is metrizable. This will prove that X is metrizable.

Let $\{G_n\}_{n\in N}$ be the non-empty sets in some denumerable base for X, made infinite by repetition if necessary. Now for each integer j, there is some point $x\in G_j$ and by regularity an open set G such that $x\in G \subseteq C(G) \subset G_j$. Since the collection $\{G_n\}$ forms a base, there must be some integer i such that $x\in G_i \subseteq G$. Thus for every integer j, there is an integer i such that $C(G_i) \subseteq G_j$. The collection of all such pairs of elements from the base is denumerable and suppose $<G_i, G_j>$ is the nth pair in some fixed ordering. The sets $C(G_i)$ and X/G_j are then disjoint closed subsets of X. Since the regular second axiom space X is completely normal and hence normal. Thus by Urysohn's Lemma, there exists some continuous mapping $f_n : X\to[0, 1]$ such that

$$f_n (C(G_i)) = \{0\} \text{ and } f_n(X/G_j) = \{1\} .$$

We may define a mapping f of X into the Hilbert cube by setting

$$f(x) = <2^{-n} f_n(x)>_{n\in N} \text{ for every } x \in X.$$

Since $0 \le f_n(x) \le 1$ for all $x\in X$ and integers n, $f(x)$ is clearly a uniquely determined element of the Hilbert cube for each $x\in X$. We shall now show that f is a homemorphism of X onto $f(X)$.

Let x and y be two distinct points of X. Since X is a T_1-space, there exists some integer j such that $x\in G_j$ but $y\in X/G_j$. As above there exists some integer i such that $x\in G_i$ and $C(G_i) \subseteq G_j$. Suppose $<G_i, G_j>$ is the nth pair in this ordering. The nth-coordinate of $f(x)$ is $2^{-n}f_n(x) = 0$. Since $x\in C(G_i)$ while the nth-coordinate of $f(y)$ is $2^{-n} f_n(y) = 2^{-n} \neq 0$ since $y\in X/G_j$. Thus $f(x) \neq f(y)$ and f is one to one.

Also let x* be a fixed point of X and $\in$ be arbitrary positive number. Let us choose first an index N = N($\in$) such that

$$\sum_{n=N+1}^{\infty} 2^{-2n} < \in^2/2.$$

For each n, such that $1 \le n \le N$, the mapping f_n is continuous and so there exists a basic open set G_n containing x* such that $|f_n(x^*) - f_n(x)| < \frac{\in}{\sqrt{2N}}$ whenever $x\in G_n$.

Let G = $\bigcap_{n=1}^{N} G_n$, which is then an open set containing x*. If x∈G, then x∈G_n for each n = 1, 2, …, N and so.

$$d_H(f(x^*), f(x)) = \sqrt{\sum_{n=1}^{\infty}\left[2^{-n} f_n(x^*) - 2^{-n} f_n(x)\right]^2}$$

$$= \sqrt{\sum_{n=1}^{N} 2^{-2n} \mid f_n(x)^* - f_n(x)\mid^2 + \sum_{n=N+1}^{\infty} 2^{-2n} \mid f_n(x^*) - f_n(x)\mid^2}$$

$$\leq \sqrt{\sum_{n=1}^{N} \mid f_n(x^*) - f_n(x)\mid^2 + \sum_{n=N+1}^{\infty} 2^{-2n}}$$

$$< \sqrt{N \cdot (\in / \sqrt{2N})^2 + \in^2 / 2} = \in$$

Thus *f* is continuous.

Finally suppose G is an open subset of X and y is an arbitrary point of *f*(G). Thus y = *f*(x) for some point x∈G. As above for some integer n, the nth pair <G_i. G_j> is such that

x∈G_i ⊆ C(G_i) ⊆ G_j ⊆ G. Hence
f_n(x) = 0 and f_n(X/G) = {1}. Thus for any
t ∈X/G, d_H(*f*(x), *f*(t)) ≥ 2^{-n} because of the difference in their nth co-ordinates that is

f(X/G) ∩ B(*f*(x), 2^{-n}) = φ

Hence y∈B (y, 2^{-n}) ∩ *f*(X) ⊆ *f*(G) and so *f*(G) is an open subset of *f*(X)

⇒ *f* is an open mapping.

Hence *f* is a homemorphism from X into the Hilbert cube and since Hilbert cube is metrizable. Thus X is metrizable. Hence every second axiom T_3-space is metrizable.

Remark. Since we know that every metrizable space whether second axiom or not is T_3. Thus we have proved that "A second axiom space is metrizable if and only if it is T_3."

7

PRODUCT TOPOLOGICAL SPACES

Definition. Let $\{X_\lambda\ ;\ \lambda\in\wedge\}$ be an arbitrary collection of sets indexed by $\wedge$. Then the cartesian product of this collection is the set of all mappings *f* of $\wedge$ into $\bigcup_{\lambda\in\wedge} X_\lambda$ such that $f(\lambda) \in X_\lambda$ for every $\lambda\in\wedge$. We shall denote an individual mapping in this collection by $<x_\lambda>_{\lambda\in\wedge}$ where x_λ is a point of X_λ such that $f(\lambda) = x_\lambda$.

Projection Mappings and Product Topology

Definition. For each $\beta\in\wedge$, the mapping $\pi_\beta : \prod_\lambda X_\lambda \to X_\beta$ assigning to each element $<x_\lambda>$ of $\prod_\lambda X_\lambda$ its βth co-ordinate,

$$\pi_\beta(<x_\lambda>) = x_\beta$$

is called the projection mapping associated with the index β.

Consider the set $\pi_\beta^{-1}(G_\beta)$ where G_β is an open subset of X_β. It consists of all points

$$p = \{a_\lambda\ ;\ \lambda \in\wedge\} \text{ in } \prod_\lambda X_\lambda \text{ such that}$$

$\pi_\beta(p) \in G_\beta$. In other words

$$\pi_\beta^{-1}(G_\beta) = \prod_\lambda Y_\lambda$$

where $Y_\beta = G_\beta$ and $Y_\lambda = X_\lambda$ whenever $\lambda \neq \beta$ that is

$$\pi_\beta^{-1}(G_\beta) = X_1 \times X_2 \times \ldots \times X_{\beta-1} \times G_\beta \times X_{\beta+1} \times \ldots.$$

Definition. For each λ in an arbitrary index set $\wedge$, let (X_λ, T_λ) be a topological space and let $X = \pi_\lambda X_\lambda$. Then the topology T for X which has a subbase the collection

$$\mathbf{B}_* = \{\ \pi_\beta^{-1}(G_\lambda)\ ;\ \lambda\in\wedge \text{ and } G_\lambda \in T_\lambda\}$$

is called the product topology or Tichonov topology for X and (X, T) is called the product space of the given spaces.

The collection $\mathbf{B}_*$ is called the defining subbase for T. The collection **B** of all finite intersections of elements of $\mathbf{B}_*$ would then form the base for T.

Remark. The projection mappings are continuous for G_β is T_β-open in $X_\beta \Rightarrow \pi_\beta^{-1}(G_\beta) \in \mathbf{B}_*$ which is a subbase for T and Therefore $\pi_\beta^{-1}(G_\beta)$ is T-open in

$$X = \prod_\lambda X_\lambda.$$

Theorem 1. Let A be a member of the defining base for a product space $X = \prod_{\lambda} X_\lambda$. Then the projection of A into any co-ordinate space is open.

Proof. Since A belongs to the defining base for X.

$A = \prod_{\lambda}\{X_\lambda ; \lambda \neq \alpha_1, \alpha_2,\ldots,\alpha_m\} \times G_{\alpha_1} \times\ldots\times G_{\alpha_m}$ where G_{α_i} is an open subset of X_{α_i}. So for any projection $\pi_\beta : X \to X_\beta$,

$$\pi_\beta(A) = \begin{cases} X_\beta & \text{if } \beta \neq \{\alpha_1, \alpha_2, ..., \alpha_m\} \\ G_\beta & \text{if } \beta \in \{\alpha_1, \alpha_2, ..., \alpha_m\} \end{cases}$$

In either case $\pi_\beta(A)$ is an open set.

Theorem 2. Every projection $\pi_\beta : X \to X_\beta$ on a product space $X = \prod_{\lambda} X_\lambda$ is open.

Let G be an open subset of X. For every point $p \in G$, there is a member A of the defining base of the product topology such that $p \in A \subset G$. Thus for any projection $\pi_\beta : X \to X_\beta$, $p \in G \Rightarrow \pi_\beta(p) \in \pi_\beta(A) \subset \pi_\beta(G)$

But $\pi_\beta(A)$ is an open set. Therefore every point $\pi_\beta(p)$ in $\pi_\beta(G)$ belongs to an open set $\pi_\beta(A)$ which is contained in $\pi_\beta(G)$. Hence $\pi_\beta(G)$ is an open set.

Remark. As each projection is continuous and open, but projections are not closed maps e.g consider the space R×R with product topology.

Let $H = \{(x, y) ; x, y \in R \text{ and } xy = 2\}$

Here H is closed in **R**×R but

$$\pi_1(H) = \mathbf{R} \sim (0)$$

is not closed with respect to the usual topology for R where π_1 is the projection in the first co-ordinate space R.

Tychonov Product Topology in Terms of Standard Subbases

Example. Consider the topology

$$T = \{\phi, X, (a), (b, c)\} \text{ on } X = (a, b, c)$$

and the topology $T^* = \{\phi, Y, (u)\}$ on $Y = (u, v)$ Determine the defining subbase $\mathbf{B}_*$ of the product topology on $X \times Y$.

Solution.

$$X \times Y = \{(a, u), (a, v) (b, u), (b, v), (c, u), (c, v)\}$$

is the product set on which the product topology is defined. The defining subbase $\mathbf{B}_*$ is the class of inverse sets $\pi_x^{-1}(G)$ and $\pi_y^{-1}(H)$ where G is an open subset of X and H is an open subset of Y. Computing, we have

$$\pi_x^{-1}(X)\ \pi_y^{-1}(Y) = X \times Y,$$

$$\pi_x^{-1}(\phi) = \pi_y^{-1}(\phi) = \phi$$

$$\pi_x^{-1}(a) = \{(a, u), (a, v)\}$$

$$\pi_x^{-1}(b, c) = \{(b, u), (b, v), (c, u), (c, v)\}$$

$$\pi_y^{-1}(u) = \{(a, u), (b, u), (c, u)\}$$

Hence the defining subbase $\mathbf{B}_*$ consists of the subsets of X × Y above. The defining base **B** consists of finite intersections of members of the defining subbase that is

$$\mathbf{B} = \{\phi, X \times Y, (a, u), \{(b, u), (c, u)\}$$
$$\{(a, u), (a, v)\}, \{(b, u), (b, v), (c, u), (c, v)\}$$
$$\{(a, u), (b, u), (c, u)\}\}$$

Theorem 3. Let (X_λ, T_λ) be an arbitrary collection of topological spaces and let $X = \prod_\lambda X_\lambda$. Let T be a topology for X. If T is the product topology for X, then T is the smallest topology for X for which projections are continuous and conversely also

Proof. Let π_λ be the λ-th projection map and let G_i be any T_λ-open subset of X_λ. Then since T is the product topology for X, $\pi_\lambda^{-1}(G_\lambda)$ is a member of the subbase for T and hence $\pi_\lambda^{-1}(G_\lambda)$ must be T-open. It follows that π_λ is T-T_λ continuous. Now let V be any topology on X such that π_λ is V-T_λ continuous for each $\lambda \in \wedge$. Then $\pi_\lambda^{-1}(G_\lambda)$ is V-open for every $G_\lambda \in T_\lambda$. Since V is a topology for X, V contains all the unions of finite intersections of members of the collection

$$\{ \pi_\lambda^{-1}(G_\lambda) ; \lambda \in \wedge \text{ and } G_\lambda \in T_\lambda\}$$

⇒ V contains T that is T is coarser than V thus T is the smallest topology for X such that π_λ is T−T_λ continuous for each $\lambda \in \wedge$.

Conversely. Let $\mathbf{B}_*$ be the collection of all sets of the form $\pi_\lambda^{-1}(G_\lambda)$ where G_λ is an open subset of X_λ for $\lambda \in \wedge$. Then by definition, a topology V for X will make all the projections π_λ continuous if and only if $\mathbf{B}_* \subset V$. Thus the smallest topology for X which makes all the projections continuous, is the topology determined by $\mathbf{B}_*$ as a subbase.

Theorem 4. A function $f : Y \rightarrow X$ from a topological space Y into a product space $X = \prod_\lambda X_\lambda$ is continuous if and only if for every projection $\pi_\beta : X \rightarrow X_\beta$, the composition mapping $\pi_\beta o f : Y \rightarrow X_\beta$ is continuous.

Proof. By the definition of product space, all projections are continuous. So if *f* is continuous, then $\pi_\beta o f$ being the composition of two continuous functions, is also continuous.

On the other hand, suppose every composition function $\pi_\beta o f : Y \rightarrow X_\beta$ is continuous. Let G be an open subset of X_β. Then by the continuity of $\pi_\beta o f$,

$$(\pi_\beta o f)^{-1}(G) = f^{-1} [\pi_\beta^{-1}(G)]$$

is an open set in Y. But the class of sets of the form $\pi_\beta^{-1}(G)$ where G is an open subset of X_β is the defining subbase for the product topology on X. Since their inverses under *f* are open subsets of Y, *f* is a continuous function.

Remark. The projections π_x and π_y of the product of two sets X and Y are the mappings of X×Y onto X and Y respectively defined by setting

$$\pi_x (<x, y>) = x \text{ and } \pi_y(<x, y>) = y$$

Theorem 5. If X and Y are topological spaces, the family of all sets of the form V×W with V open in X and W open in Y is a base for a topology for X×Y.

Proof. Since the set X×Y is itself of the required form, X×Y is the union of all the members of the family. Now let $<x, y> \in (V_1 \times W_1) \cap (V_2 \times W_2)$ with V_1 and V_2 open in X and W_1 and W_2 open in Y. Then

$$<x, y> \in (V_1 \cap V_2) \times (W_1 \times W_2) = (V_1 \times W_1) \cap (V_2 \times W_2)$$

with $V_1 \cap V_2$ open in X and $W_1 \cap W_2$ open in Y. Then the family is a base for topology for X×Y.

Theorem 6. Let C_i be a closed subset of a space X_i for $i \in I$. Then $\prod_{i\in I} C_i$ is a closed subset of $\prod_{i\in I} X_i$ w.r.t the product topology.

Proof. Let $X = \prod_{i\in I} X_i$ and $C = \prod_{i\in I} C_i$. We claim X – C is an open set in the product topology on X let $x \in X - C$. Then $C = \bigcap_{i\in I} \pi_i^{-1}(C_i)$ and so $x \notin C$ implies that there exists $j \in I$ such that $\pi_j(x) \notin C_j$. Let $V_j = X_j - C_j$ and let $V = \pi_j^{-1}(V_j)$. Then V_j is an open subset of X_j and so V is an open subset (in fact a member of the standard subbase) in the product topology on X. Evidently $\pi_j(x) \in V_j$ and so $x \in V$. Moreover $C \cap V = \phi$ since $\pi_j(C) \cap \pi_j(V) = \phi$. So $V \subset X - C$. Thus X–C is a neighbourhood of each of it point. So X–C is open and C closed in X.

Separation Axioms and Product Spaces

In this section, we shall be concerned with finding out whether a certain topological property carries over from co-ordinate spaces to their topological products. In other words, suppose X is the topological product of an indexed family $\{(X_i, T_i), i \in I\}$ of topological spaces. If each X_i has a topological property can we say that X also has it? This will depend upon the property itself and also upon how large the index set I is depending on it.

Theorem 7. Let X be topological product of an indexed family of spaces $\{(X_i, T_i); i \in I\}$. If the product is non-empty, then each co-ordinate space is embeddable in it.

Proof. Let X be the topological product of an indexed family of spaces $\{(X_i, T_i) ; i \in I\}$. Since X is non-empty, so in each X_i for $i \in I$. Now fix $j \in I$. We want to show that X_j can be embedded into X. For each $j \neq i$ in I, fix some $y_i \in X_i$. Now define $e : X_j \to X$ by

$$e(x)(i) = \begin{cases} y_i & \text{for } i \neq j \in I \\ x & \text{for } i = j \end{cases}$$

In other words, e(x) is that element of $\prod_{i\in I} X_i$ whose jth co-ordinate is x and all other co-ordinates are equal to the respective chosen y_i's. Evidently e is one to one since $e(x) = e(x')$ would in particular imply

$$e(x)(j) = e(x')(j) \text{ whence } x = x'.$$

Also the composite map $\pi_j o e$ is the identity map on X_j while for any $i \neq j$, π_j oe is the constant map taking all point of X_j to y_i. In either case the composite $\pi_i o e$ is continuous for all $i \in I$. So by the result "A function $f : Y \to X$ from a topological space Y into a product space $X = \prod_j X_j$ is continuous if and only if for every projection $\pi_\beta : X \to X_\beta$, $\pi_\beta o f : Y \to X_\beta$ is continuous", e is continuous. To show that e is an embedding, it only remains to show that e is open when regarded as a function from X_j onto $e(X_j)$. Note that $e(X_j)$ is the box [A box in the product space $X = \prod_j X_j$ is a subset B of X of the form $\prod_{i\in I} B_i$ such that $B_i \subset X_i$, $i \in I$, For $J \in I$, B_j is called the j-th side of the box B] $\prod_{i\in I} Y_i$ where $Y_j = X_j$ and $Y_i = \{y_i\}$ for $i \neq j \in I$. Now let V be an open subset of X_j. It is then easy to show that

$e(V) = \pi_j^{-1}(V) \cap e(X_j)$ as each side equals the box $\prod_{i\in I} Z_i$ where

$Z_j = V$ and $Z_i = \{y_i\}$ for $i \neq j \in I$. Now the set $\pi_j^{-1}(V)$ is open in X and so e(V) is an open in $e(X_j)$ in the relative topology. Thus e is an embedding.

Theorem 8. A topological product is T_0, T_1, T_2 or regular if and only if each co-ordinate space has the corresponding property.

Proof. Since all the properties T_0, T_1, T_2 or regular are hereditary, now if $X = \prod_{i\in I} X_i$, then each co-ordinate space X_j is homeomorphic to a subspace of X by the last theorem. (i.e. by Theorem 7). So whenever X is T_0, T_1, T_2 or regular, so will be each X_j.

Conversely, we first show that T_2-property is productive (i.e. A topological property is said to be productive if whenever $\{(X_i, T_i); i \in I\}$ is an indexed family of spaces having that property. The topological product $\prod_{i\in I} X_i$ also has it). Suppose X_i is a T_2 space for each $i \in I$, and let $X = \prod X_i$

Let x, y be distinct points of X. Note that x,y are both functions on the indexed set I and so to say that they are distinct means that they assume distinct values at some index. So there exists $j \in I$ such that $x(j) \neq y(j)$ i.e. $\pi_j(x) \neq \pi_j(y)$ in X_j (where $\pi_j : X \to X_j)_{j\in I}$. Since X_j is a Hausdorff space, there exists disjoint open sets U and V in X_j such that $x \in U$, $y \in V$. Now let $G = \pi_j^{-1}(U)$, $H = \pi_j^{-1}(V)$.

Then G, H are open subsets of X. Also $x \in G$, $y \in H$ and $G \cap H = \phi$ showing that X is a Hausdorff space. Similar argument applies to show that T_0 and T_1 are productive properties. Now we prove the converse part for regularity i.e. we will prove that if indexed family $\{(X_i, T_i)_{i\in I}\}$ of spaces is regular, then the topological product $X = \prod_{i\in I} X_i$ is regular. Suppose each X_i is regular and $x \in X$ and C be a closed subset of X not containing x. Then $X-C$ is an open set containing x . So there exists a member V of the standard base for the product topology such that $x \in V$ and $V \subset X - C$. Let $V = \prod_{i\in I} V_i$ where each V_i is open in X_i and $V_i = X_i$ for all $i \in I$ except possibly for $i = i_1, i_2,\ldots, i_n$ (say). Let $x_{i_r} = \pi_{i_r}(x)$ for $r = 1,2,\ldots,n$. So be regularity of each co-ordinate space, there exists an open set U_{i_r} in X_{i_r} such that $x_{i_r} \in U_{i_r}$ and $\overline{U}_{i_r} \subset V_{i_r}$. For $i \neq i_1, i_2,\ldots, i_n$, let $U_i = X_i$ and consider the box $U = \prod_{i\in I} U_i$. Clearly U is an open set in X and $x \in U$. Also U is contained in the box $\prod_{i\in I} \overline{U}_i$ which itself is contained in V. But the set $\prod_{i\in I} \overline{U}_i$ is closed in X and hence contains $\overline{U}$. Thus $x \in U$ and $\overline{U} \subset X - C$. Therefore U and $X - \overline{U}$ are mutually disjoint open subsets containing x and C respectively. This show that X is regular.

Remark. Normality is not a productive property; It is not even finitely productive (i.e. whenever X_i's are normal, the topological product $\prod_{i\in I} X_i$ is not normal (I finite). However, complete regularity is productive property as shown below.

Theorem 9. Let S be a subbase for a topological space X . Then X is completely regular if and only if for each $V \in S$, and for each $x \in V$, there exists a continuous function $f : X \to [0, 1]$ such that $f(x) = 0$ and $f(y) = 1$ for all $y \notin V$.

Proof. If X is completely regular, then certainly the condition holds for all open sets, and so in particular members of S. Conversely suppose each $V \in S$, satisfies the given condition. Let G be an open set in X and let $x \in G$. By definition of a subbase, there exist $V_1, V_2, \ldots, V_n \in S$ such that $x \in \bigcap_{i=1}^{n} V_i \subset G$. For each i = 1, 2,…, n, we have a map $f_i : X \to [0, 1]$ which vanishes at x and takes the value 1 outside V_i. Define

$$f : X \to [0, 1] \text{ by } f(y) = 1-(1-f_1(y))\,(1-f_2(y)) \ldots (1-f_n(y))$$

Clearly f is continuous and $f(x) = 0$ and f takes the value 1 outside $\bigcap_{i=1}^{n} V_i$ and hence outside G. This proves that X is completely regular.

Theorem 10. A product of topological spaces is completely regular if and only if each co-ordinate space is so.

Proof. From Theorem 7, since each co-ordinate space is embeddable in the topological product, and the fact that complete regularity is a hereditary property, i.e. if the space is completely regular, then every subspace is also completely regular. Thus if the topological product $X = \prod_{i \in I} X_i$ is completely regular, then each space X_i is completely regular.

Conversely suppose $X = \prod_{i \in I} X_i$ where each X_i is completely regular. Let $S = \{\pi_j^{-1}(V_j);\ j \in I,\ V_j$ open in $X_j\}$ be the standard subbase for the product topology and suppose $x \in \pi_j^{-1}(V_j)$. Then $\pi_j(x) \in V_j$ and so by the complete regularity of X_j, there exists a map $f : X_j \to [0, 1]$ such that $f(\pi_j(x)) = 0$ and f takes value 1 on $X_j - V_j$. Then the composite $f o \pi_j : X \to [0,1]$ vanishes at x and takes value 1 on $\pi_j^{-1}(X_j - V_j)$ i.e. on $X - \pi_j^{-1}(V_j)$. Hence the condition of last theorem (i.e. Theorem 9) is satisfied by every member of the subbase S. So X is completely regular.

Theorem 11. A topological product of spaces is Tychonoff if and only if each co-ordinate space is so.

Proof. Since by definition Tychonoff property means the combination of complete regularity with T_1, the result follows by merely putting together Theorem 8 and Theorem 10.

Theorem 12. $\prod_{\lambda} X_\lambda$ is Hausdorff if and only if each space X_λ is Hausdorff.

Proof. Suppose each space X_λ is Hausdorff and let $X = \langle x_\lambda \rangle_{\lambda \in \wedge}$ and $Y = \langle y_\lambda \rangle_{\lambda \in \wedge}$ be two distinct points of $\prod_{\lambda} X_\lambda$. Since X and Y are distinct, there must exist some index β such that $x_\beta \neq y_\beta$. Since X_β is a Hausdorff space, there must exist disjoint open subsets G_x and G_y containing x_β and y_β respectively. Clearly the sets $\pi_\beta^{-1}(G_x)$ and $\pi_\beta^{-1}(G_y)$ are disjoint open sets in the product space containing X and Y respectively since the projection mapping $\pi_\beta : \prod_{i \in I} X_i \to X_\beta$ is continuous. Hence if G_x and G_y are two disjoint open sets in the Hausdorff space X_β containing x_β and y_β, Then

$\pi_\beta^{-1}(G_x)$ and $\pi_\beta^{-1}(G_y)$ are disjoint open sets in the product space containing X and Y respectively. This proves that $\prod_\lambda X_\lambda$ is Hausdorff.

Conversely suppose that $\prod_\lambda X_\lambda$ is Hausdorff. We want to prove that each X_λ is Hausdorff. Since the property of a space being Hausdorff is Hereditary, we shall show that each X_λ is homeomorphic to a certain subset of the product space. This will prove that each X_λ is hausdorff. To do so, let $<z_\lambda>$ be a fixed point in the product space. For each β, consider the subset E_β of the product space consisting of all points $<x_\lambda>$ such that $x_\lambda = z_\lambda$ if $\lambda \neq \beta$, while x_β may be any point of X_β. Now let *f* be a π_β restricted to E_β. *f* is clearly one-to-one continuous mapping of E_β onto X_β (since the projections are continuous) we want to show that *f* is open.

Now a base for the open sets in the subspace E_β is the family of intersection of E_β with the basis elements for the product topology. But $E_\beta \cap \prod_\lambda Y_\lambda$ is either empty or consists of points of E_β for which $x_\beta \in Y_\beta$. The image under *f* is then either empty or Y_β and so open in either case. Thus *f* is homeomorphism.

Connectedness

Theorem 13. X×Y is connected if and only if X and Y are connected.

Proof. Suppose first that X×Y is connected. Since $\pi_x : X \times Y \to X$ is continuous and X×Y is connected. The projection mapping π_x maps the connected space into connected space. Hence X is connected. Similarly the continuous image Y of X×Y which is connected, is connected. Hence if X×Y is connected, then X and Y are also connected.

Conversely suppose that X and Y are connected. We shall prove that X×Y is connected. Let $<x, y>$ and $<x^*, y^*>$ be any two points of X×Y. Consider the mappings

$$f : \{x\} \times Y \to Y$$
$$g : X \times (y^*\} \to X$$

defined by

$$f(<x, y>) = y \text{ and } g(<x^*, y^*>) = x^*.$$

It can be shown that *f* and g are homeomorphisms. Since projections are continuous and open. It follows therefore that {x}×Y and X×{y*} are connected. They intersect in the point $<x, y^*>$. Hence by the result "The union E of any family $\{C_\lambda\}$ of connected sets having a non-empty intersection is a connected set". Their union is a connected set which contains the two points $<x, y>$ and $<x^*, y^*>$. Thus X×Y is connected.

Theorem 14. $\prod_\lambda X_\lambda$ is connected if and only if each X_λ is connected.

Proof. Suppose first that $\prod_\lambda X_\lambda$ is connected. Define $\pi_\beta : \prod_\lambda X_\lambda \to X_\beta$. Since the projection mapping is continuous and the continuous image of a connected set is connected. It follows that each X_λ is connected.

Conversely suppose that each X_λ is connected. To prove that $\prod_\lambda X_\lambda$ is connected, let C be a component of the product space and let $<z_\lambda ; \lambda \in \wedge>$ be a fixed point in the product space which belongs to C. Suppose further that $<x_\lambda\ ;\ \lambda\in\wedge>$ be an arbitrary point of $\prod_\lambda X_\lambda$ and $\prod_\lambda Y_\lambda$ is an arbitrary open set containing $<x_\lambda\ ;\ \lambda\in\wedge>$ such that Y_λ is open in x_λ and

$$Y_\lambda = X_\lambda \text{ if } \lambda \neq \beta_1, \beta_2,\ldots, \beta_n$$

Then the set

$$E_{\beta_1} \times E_{\beta_2} \times \ldots \times E_{\beta_n}$$

consisting of all point $<t_\lambda\ ;\ \lambda\in\wedge>$ such that

$$t_\lambda = z_\lambda \text{ if } \lambda \neq \beta_1, \beta_2,\ldots, \beta_n$$
$$t_\lambda = x_\lambda \text{ if } \lambda = \beta_1, \beta_2,\ldots, \beta_n$$

is homeomorphic to

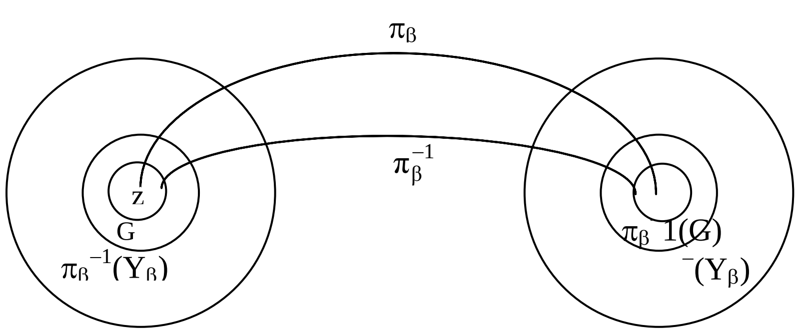
$$X_{\beta_1} \times X_{\beta_2} \times \ldots \times X_{\beta_n}$$

which being the Cartesian product of finite number of connected sets is connected. Therefore $E_{\beta_1} \times E_{\beta_2} \times \ldots \times E_{\beta_n}$ is connected. But we know that the component C of X is largest connected set in X. Thus $E_{\beta_1} \times E_{\beta_2} \times \ldots \times E_{\beta_n} \subset C$. However the point $<t_\lambda\ ;\ \lambda\in\wedge>$ lies in $\prod_\lambda Y_\lambda$ which was an arbitrary open set containing $<x_\lambda\ ;\ \lambda\in\wedge>$. Thus $<x_\lambda\ ;\ \lambda\in\wedge>$ is in the closure of C. Since the component is closed, therefore closure of C is equal to C itself. Therefore $<x_\lambda\ ;\ \lambda\in\wedge>$ belongs to C. But $<x_\lambda\ ;\ \lambda\in\wedge>$ was an arbitrary point of $\prod_\lambda X_\lambda$. Therefore $\prod_\lambda X_\lambda \subset C$. But $C \subset \prod_\lambda X_\lambda$. Hence $\prod_\lambda X_\lambda$ is equal to C which is connected. Hence $\prod_\lambda X_\lambda$ is connected.

Locally Connected

Theorem 15. $\prod_\lambda X_\lambda$ is locally connected if and only if each space X_λ is locally connected and all but a finite number are connected.

Proof. Suppose $\prod_\lambda X_\lambda$ is locally connected, and let $x_\beta \in X_\beta$ be contained in some open set Y_β. Choose some point $z = <z_\lambda>$ with $z_\beta = x_\beta$ and we have z belonging to the open set $\pi_\beta^{-1}(Y_\beta)$. By local connectedness, there must exist a connected open set G containing z and contained in $\pi_\beta^{-1}(Y_\beta)$. By local connectedness, there must exist a connected open set G containing z and contained in $\pi_\beta^{-1}(Y_\beta)$. Taking the β-th projection, we see that $z_\beta = x_\beta$ is contained in the connected open set $\pi_\beta(G)$ which is itself contained in Y_β and so X_β is locally connected.



Further, if z is any point of the product space, it must be contained in some connected open set G. By definition $z\in \prod_\lambda Y_\lambda \subseteq G$ where Y_λ is open in X_λ for all λ and $Y_\lambda = X_\lambda$ for all but a certain finite number of values of λ. But then the projections of G are connected and are equal to X_λ, except for that finite number of values of λ.

Now suppose that X_λ is locally connected for old λ and connected for $\lambda \neq \beta_1, \beta_2, \ldots, \beta_n$. Let $X = \langle x_\lambda \rangle_\lambda$ be an arbitrary point of $\prod_\lambda Y_\lambda$ where Y_λ is open in X_λ for all λ and $Y_\lambda = X_\lambda$ for $\lambda \neq \beta_1^*, \beta_2^*, \ldots, \beta_k^*$. Since $x_\lambda \in Y_\lambda$ for all λ and Y_λ is locally connected, there is a connected open set G_λ in X_λ such that $x_\lambda \in G_\lambda \subseteq Y_\lambda$. Consider the subset $\prod_\lambda Z_\lambda$ where $Z_\lambda = G_\lambda$ if $\lambda = \beta_1, \beta_2, \ldots, \beta_n, \beta_1^*, \ldots, \beta_k^*$ and $Z_\lambda = X_\lambda$ otherwise. But by the result "$\prod_\lambda X_\lambda$ is connected if and only if each X_λ is connected". This set is connected. Hence we have formed a connected open set containing X and contained in $\prod_\lambda Y_\lambda$.

Product Space as First Axiom Space

Theorem 16. Product space $\prod_\lambda X_\lambda$ is first axiom if and only if each space X_λ is first axiom and all but a countable number are indiscrete.

Proof. Suppose $\prod_\lambda X_\lambda$ is first axiom. Since each space X_β is homeomorphic to set E_β where each

$$E_\beta = \begin{cases} \langle x_\lambda \rangle \text{ such that } x_\lambda = z_\lambda \text{ if } \lambda \neq \beta \\ x_\beta \text{ is any point of } X_\beta \text{ and} \\ \langle z_\lambda \rangle \text{ is a fixed point of } X = \prod_\lambda X_\lambda \end{cases}$$

and the first axiom of countability is both hereditary and topological, each space is first axiom. Now if the space X_λ is not indiscrete, we may choose a point $x_\lambda \in X_\lambda$ which is contained in an open set $G_\lambda \neq X_\lambda$. If X_λ is indiscrete, we choose any point $x_\lambda \in X_\lambda$. Let $X = \langle x_\lambda \rangle$ and suppose $\{B_n\}_{n \in N}$ is a countable open base at X. For each integer n, the set B_n must contain a member of the Tichonov base of the form $\prod_\lambda Y_\lambda$ where Y_λ is open in X_λ for all λ and $Y_\lambda = X_\lambda$ for all but a finite number of values of λ ; $\lambda_1^n, \lambda_2^n, \ldots, \lambda_{k_n}^n$. The collection of all the expected values of λ ; $\{\lambda_i^n ; i = 1, 2, \ldots, k_n$ and $n \in N\}$ is countable. For any other value of λ, we choose $x_\lambda \in G_\lambda \neq X_\lambda$ if X_λ was not discrete, but $\Pi_\lambda^{-1}(G_\lambda)$ would then be an open set containing X which would not contain any B_n. Hence for all other values of λ, X_λ must be indiscrete.

Now suppose that X_λ is first axiom for all λ and indiscrete for $\lambda \notin \{\lambda_i\}_{i \in N}$. Let $X = \langle x_\lambda \rangle_\lambda$ be an arbitrary point in $\prod_\lambda X_\lambda$, and let $\{B_{n,\lambda}\}$ be a countable open base at x_λ. We note that $B_{n,\lambda} = x_\lambda$ for all n if $\lambda \notin \{\lambda_i\}$. The family $\{\pi_{\lambda_i}^{-1}(B_n, \lambda_i)$, i, $n \in N\}$ is a countable collection of open sets in the product space. The set of all finite intersections of members of this collection is also countable and it is already an open base at x, as desired.

Product Space as Compact Space

Theorem 17. $X \times Y$ is compact if and only if X and Y are compact.

Proof. Since the projection mappings are continuous and onto if $X \times Y$ is compact, then so are X and Y.

Conversely suppose that X and Y are compact. If G is any open covering of $X \times Y$, then each member of G is a union of basis elements of the form $V \times W$ with V open in X and W open in Y. We may restrict our attention to the covering $\{V_\lambda X W_\lambda\}_{\lambda \in \wedge}$ of $X \times Y$ by these basis elements where each $V_\lambda \times W_\lambda$ is contained in some member of G, since any finite subcovering of this basic open cover will lead immediately to a finite sub-covering chosen from the original cover G. For each $x \in X$, let $Y_x = \{x\} \times Y$ which is homeomorphic to Y and hence compact. Since $\{V_\lambda \times W_\lambda\}$ also covers Y_x , there must exist a finite subcovering $\{V_{x;\lambda_i} \times W_{x,\lambda_i}\}_{i=1}^{n}$ of Y_x by sets which have a non-empty intersection with Y_x. Let

$$G_x = \bigcap_{i=1}^{n(x)} V_{x,\lambda_i}$$

The G_x is an open set containing x and the above finite subcover actually covers $G_x \times Y$. Now $\{G_x ; x \in X\}$ covers X and so there is a finite subcover $\{G_{x,j}\}_{j=1}^{n}$. But then

$$\left[\left\{V_{x_j,\lambda_i} \times W_{x_j,\lambda_i}\right\}_{i=1}^{n(x_j)}\right]_{j=1}^{n}$$

covers $X \times Y$. Hence $X \times Y$ is compact.

Definition. Let X be any non-empty set. A filter H in the set X is a family of subsets of X satisfying the following axioms :

(1) $\phi \notin \mathbf{H}$

(2) If $F \in \mathbf{H}$, $G \in \mathbf{H}$, then $F \cap G \in \mathbf{H}$.

(3) If $F \in \mathbf{H}$ and $H \supset F$, then $H \supset \mathbf{H}$.

Definition. An ultrafilter in a set X is a filter in X which is maximal in the collection of all filters partially ordered by inclusion, that is a filter which is not properly contained in any other filter.

The topic will be discussed in detail in Chapter 8.

Theorem 18 (Tichonoff Product Theorem). $T_\lambda X_\lambda$ is compact if and only each space X_λ is compact.

Proof. Since the projections are continuous, so if $\prod_\lambda X_\lambda$ is compact, then each X_λ is compact.

Now suppose each X_λ is compact and let **H** be a family of closed subsets of $\prod_\lambda X_\lambda$ with the finite intersection property. The family **H** generates a filter which is contained in some ultrafilter **F.** Now consider the family

$$\{C(\pi_\lambda F_i \; ; F \in \mathbf{F} \}$$

of closed subsets of X_λ . This family has the finite intersection property because

$$\bigcap_{i=1}^{n} C(\pi_\lambda F_i) \supseteq \bigcap_{i=1}^{n} C(\pi_\lambda F_i) \supseteq \prod_\lambda \left(\bigcap_{i=1}^{n} F_i\right)$$

which is nonempty because **F** has the finite intersection property. Now since X_λ is compact, there exists some point x_λ which belongs to $C(\pi_\lambda F)$ for every $F \in$ **F** . Let $X = \langle x_\lambda \rangle_\lambda$ and show that X belongs to every set in **F .**

Now let $S = \prod_\lambda Y_\lambda$ with $Y_\beta = G_\beta$ an open set in X_β and $Y_\lambda = X_\lambda$ if $\lambda \neq \beta$ be an arbitrary member of the subbase for the product topology which contain X. But then G_β is an open set containing the point x_β which is in the closure of $\pi_\beta F$ for every $F \in$ **F .** Hence G_β must contain a point of $\pi_\beta (F)$ for every $F \in$ **F .** Then S must contains a point of F for every $F \in$ **F** and by the lemma, this implies that $S \in$ **F** .

Finally let x be contained in an arbitrary open set G. By definition, $x \in S_1 \cap \ldots \cap S_n \subseteq G$ for some finite number of sets S_i in the subbase. We have just shown that each $S_i \in$ **F** and so $S_1 \cap S_2 \cap S_n \in$ **F** and hence $G \in$ **F**. Going back to our original family **H** of closed sets with the finite intersection property, we note that each member of H is also a member of **F** and so $H \cap G \in$ **F** for every $H \in$ **H** and then $H \cap G \neq \phi$ for every $H \in$ **H** . Thus $x \in C(H) =$ H for every $H \in$ **H** $\Rightarrow \bigcap_{H \in \mathbf{H}} H \neq \phi$. So we have proved that every family of closed sets in the product topology which has finite intersection property has a nonempty intersection. Thus $\prod_{\lambda \in \wedge} X_\lambda$ is compact.

8

NETS AND FILTERS

Inadequacy of sequences : We are familiar with the fact that a function $f : R\to R$ is continuous at x_0 in R if whenever $<x_n>$ is a sequence converging to x_0 in R, then the sequence $<f(x_n)>$ converges to $f(x_0)$. We introduced topologies for the purpose of providing a general setting for the study of continuous functions. Two questions arise.

(a) Can we define sequential convergence in a general topological space ?

(b) If so, does the resulting notion describe the topology and hence the continuous functions? The answer to the first question is yes and the definition is as follows :

Definition. A sequence $<x_n>$ in a topological space X is said to converge to $x\in X$ written $\lim_{n\to\infty} x_n = x$ or $x_n\to x$ iff for each neighbourhood U of x, there is some positive integer n_0 such that $x_n\in U$ whenever $n > n_0$. In this case we say that $<x_n>$ is eventually in U. A sequence is called convergent if and only if there is at least one point to which it converges.

The answer to the second question (b) is only for a limited class of spaces. In fact, we have

Theorem. 1. If $<x_n>$ is a sequences of distinct points of a subset E of a top space X which converges to a point $x \in X$ then x is a limit point of the set E.

Proof. Proved earlier.

Remark. The converse of this is not true even in a Hausdorff space.

Theorem. 2. If f is a continuous mapping of the topology space X into a topology space X* and $<x_n>$ is a sequence of points of X which converges to the point $x \in X$, then the sequence $<f(x_n)>$ converges to the point $f(x) \in X^*$.

Proof. Proved earlier.

Remark : The converse of this th is also not true even in a Hausdorff space. i.e. a mapping f for which $x_n\to x$ implies $f(x_n)\to f(x)$ may not be cont. However, the converse of the above theorems hold in the case of first axiom space and first countable space.

Theorem. 3. If X is first countable pace and $E \subset X$, Then $x \in \bar{E}$ iff there is a sequence $<x_n>$ contained in E which converges to x.

Proof. Proved in Chapter IV.

Theorem 4. Let f be a mapping of the first axiom space X into topological space X*, then f is continuous at $x\in X$ iff for every sequence $<x_n>$ of pts in X converging to x we have the sequence $<f(x_n)>$ converging to the point $f(x) \in X^*$.

Proof. Proved in the Chapter IV.

Remark. The failure of the converses of the theorems 1 and 2 show that the notion of limit for sequence of points is not completely satisfactory, even in the case of Hausdorff spaces. But these converses, as shown above in Theorem 3 and 4, may be proved if we take the first axiom space. Another way to obtain these results is to generalize the notion of sequence. This leads to the idea of Moore–smith sequence, Moore – Smith family, generalized sequence or a net. Still another approach is to consider the notion of a filter which we shall discuss later on.

Nets

Definition. A set $\wedge$ is a directed set iff there is a relation $\leq$ on $\wedge$ satisfying

(a) $\lambda \leq \lambda$ for each $\lambda \in \wedge$

(b) $\lambda_1 \leq \lambda_2$ and $\lambda_2 \leq \lambda_3 \Rightarrow \lambda_1 \leq \lambda_3$.

(c) If $\lambda_1, \lambda_2 \in \wedge$, then there is some $\lambda_3 \in \wedge$ with $\lambda_1 \leq \lambda_3$ and $\lambda_2 \leq \lambda_3$.

The relation $\leq$ is sometimes referred to as a direction on $\wedge$ or is said to direct $\wedge$.

Remark. The first two properties (a) and (b) are well known requirement for an order relation. But the lack of antisymmetry shows that **direction need not be a partial order**. The concept of a net, which generalizes the notion of a sequence can now be introduced, using an arbitrary directed set to replace the integers.

Definition. A net in a set X is a function $P : \Lambda \to X$. where Λ is some directed set. The point $P(\lambda)$ is usually denoted by x_λ and we often speak of the net $<x_\lambda>_{\lambda \in \wedge}$ or the net (x_λ) if this can cause no confusion.

Example. 1. Every sequence is a net because $(N, \leq)$ where $\leq$ is the usual ordering on N is a directed set.

Example. 2. The neighbourhood system Λ_x of a point x in a topological Space X is a directed set. Here for $U, V \in \Lambda_x$, we define $U \leq V$ to mean $U \supset V$. (Note that this is in sharp contrast with our intuition that "U is smaller than V" should mean that U is contained in V). Conditions (a) and (b) hold trivally while (c) follows from the fact that the intersection of two neighbourhoods is again a neighbourhood. We define a net $P : \Lambda_x \to X$ now and denote P(U) by x_U and thus $<x_U>_{U \in \wedge x}$ is a net.

Example. 3. We give an example which is of historical significance in the definition of Riemann integrals. A partition of the unit interval [0, 1] is a finite sequence $P = \{a_0, a_1, \ldots, a_n\}$ such that $0 = a_1 < a_1 < \ldots < a_n = 1$. The interval $[a_{i-1}, a_i]$, i = 1, 2,…, n is called the ith subinterval of P. Such a partition is said to be refinement of another partition $Q = [b_0, b_1, \ldots, b_m]$ if each subinterval of P is contained in some subinterval of Q. Let us write $P \leq Q$ to mean that P is a refinement of Q. Then we see that $\leq$ directs the set of all partitions of [0, 1]. Conditions (a) and (b) of the definition are immediate. For (c) note that if P and Q are partitions, then the partition obtained by superimposing them together is a common refinement of P and Q. Let us define the Riemann net corresponding to a bounded – real valued function *f* on the unit interval [0, 1]. Let D be the set of all pairs (P, ξ) where ξ is a partition of [0, 1] say $P = \{a_0, a_1, \ldots, a_n\}$ and $\xi = \{\xi_1, \xi_2, \ldots, \xi_n\}$ is a finite sequence such that $\xi_i \in [a_{i-1}, a_i]$ for i = 1, 2,…,n. Given two elements (P, ξ) and (Q, η) of D, let us say $(P, \xi) \leq (Q, \eta)$ iff P is a refinement of Q and for each j, $\eta_j = \xi_i$ where i is so defined that the j–th subinterval of Q contained in the i-th subinterval of Q contained in the i-th subinterval of the partition P. It can be shown that $\leq$ directs D. Now define the net $\mathbf{P} : D \to R$ by

$$\mathbf{P}(P, \xi) = \sum_{i=1}^{n} f(\xi_i)\,(a_i - a_{i-1})$$

This is of course, nothing but what is called in the integral calculus as the Riemann sum of the function *f* for the partition P and the choice of points ξ_i in the ith subinterval of P and the Riemann integral as we know, is defined as the limit of such Riemann sums as the partitions gets more and more refined.

Example. 4. Let (M, ρ) be a metric space with $x_0 \in M$. Then $M - \{x_0\}$ becomes a directed set when ordered by the relation x < y iff $\rho(y, x_0) < \rho(x, x_0)$. Hence if $f : M \to N$, where N is a metric space, the restriction of *f* to $M - \{x\}$ defines a net in N.

Definition. A net $P : \Lambda \to X$ is eventually in X iff there is an element m of Λ such that if $n \in \Lambda$ and $n \leq m$ then $P(n) \in X$. This net is frequently in X iff for each m in Λ, there is n in Λ such that $n \leq m$ and $P(n) \in X$.

Definition. Let (Λ, ≤) be a directed set. A subset M of Λ is said to be Cofinal subset of Λ if every $\alpha \in \Lambda$, there exists $\not\subset \in M$ such that $\alpha \leq \not\subset$. Clearly every eventual subset is a cofinal set but converse is not true.

For example, in N any infinite subset is cofinal but not necessarily eventual.

Topology and Convergence and Nets

Definition. A net $P : \Lambda \to X$ is said to be eventually in a subset A of X iff the set $P^{-1}(A)$ is an eventual subset of A.

Definition. A net $P : \Lambda \to X$ is said to be frequently in a subset Λ of X if $P^{-1}(A)$ is a cofinal subset of Λ.

Definition. Let Λ and M be two directed sets. Then $\phi : M \to \Lambda$ is said to be an increasing cofinal function if

(a) $\phi(\mu_1) \leq \phi(\mu_2)$ whenever $\mu_1 \leq \mu_2$ (ϕ is increasing)

(b) For each $\lambda \in \wedge$, there is some $m \in M$ such that

$$\lambda \leq \phi(m) \text{ (}\phi\text{ is cofinal in }\Lambda\text{)}$$

Definition. A subnet of a net $P : \Lambda \to X$ is the composition $P_0\phi$ where $\phi : M \to \Lambda$ is an increasing cofinal function from a directed set M to ∧.

For $\mu \in M$, the point $P_0\phi(\mu)$ is often written as x_{λ_μ}, and we usually speak of the subnet (x_{λ_μ}) of (x_λ).

Definition. If (x_λ) is a net in X, a set of the form $\{x_\lambda ; \lambda \geq \lambda_0\}$ for $\lambda_0 \in \wedge$, is called a tail of (x_λ)

Definition. Let (x_λ) be a net in the space X. Then **(x_λ) is said to converge to $x \in X$**, (written as $x_\lambda \to x$) if for every neighbourhood U of x, there exists $\lambda_0 \in \wedge$ such that $\lambda \geq \lambda_0$ implies $x_\lambda \in U$.

Thus $x_\lambda \to x$ if and only if each neighbourhood of x contains a tail of (x_λ)

Sometime we say (x_λ) converges to x provided it is residually or eventually) in every neighbourhood of x.

Definition. A point $x \in X$ is said to be a cluster point of the net (x_λ) if for every neighbourhood U of x and for each $\lambda_0 \in \wedge$, there exists some $\lambda \geq \lambda_0$ such that $x_\lambda \in U$.

This is sometimes said (x_λ) has x as a cluster point if and only if (x_λ) is cofinally or frequently in each neighbourhood of x.

Example. (1) We have seen in Ex 2 for the nets that we can define a net $P : \wedge_x \to X$ on the neighbourhood system $\wedge_x$ of a point $x \in X$. We denote this net by $(x_U)_{U \in \wedge x}$. We observe that $x_U \to x$. Infact, given any neighbourhood V of x, we have $U_0 \geq V$ for some $U_0 \in \wedge_x$. Then $U \geq U_0 \geq V$ implies $U \subset U_0$ so that $x_0 \in U_0 \subset V$.

(2) We know that every sequence is a net. The two definitions of convergence of (x_n) coincide.

(3) We have seen in example 3 that

$$\mathbf{P}(P, \xi) = \sum_{i=1}^{n} f(\xi_i)\,(a_i - a_{i-1})$$

is a net. This sum is nothing but Riemann sum for the function f for the partition P and the choice of points ξ_i in the ith subinterval of the partition P. The Riemann integral is defined as the limit of this net (Riemann sum) as the partitions get more and more refined.

(4) We had seen in example 4, that the restriction of $f : M \to N$ to $M - \{x\}$, $x \in M$ was a net in N. This net converge to z_0 in N iff $\lim_{x \to x_0} f(x) = z_0$ in the elementary calculus sense.

Remark. The definition of the cluster point of a net is a generalization of the notion of a limit point of a sequence. It is obvious that if a net (x_λ) converges to x, then x is a cluster point of (x_λ). Actually a stronger result holds.

Theorem 6. A net (x_λ) has the point x as a cluster point if and only if it has a subnet which converges to x.

Proof. Let x be a cluster point of (x_λ). Let $M = \{(\lambda, U);\ \lambda \in \wedge;$ U is a neighbourhood of x containing $x_\lambda\}$ Define order $\leq$ in M as follows:

$$(\lambda_1, U_1) \leq (\lambda_2, U_2) \text{ iff } \lambda_1 \leq \lambda_2 \text{ and } U_2 \subset U_1$$

we notice that

(i) since $\lambda_1 \leq \lambda_1$ ($\wedge$ being directed and $U_1 \subset U_1$, we have

$$(\lambda_1, U_1) \leq (\lambda_1, U_1)$$

(ii) If $(\lambda_1, U_1) \leq (\lambda_2, U_2)$ and $(\lambda_2, U_2) \leq (\lambda_3, U_3)$ then $\lambda_1 \leq \lambda_2$ and $U_2 \subset U_1$ and $\lambda_1 \leq \lambda_3$ and $U_3 \subset U_2$

$\Rightarrow \lambda_1 \leq \lambda_3$ and $U_3 \subset U_1$ and thus we have

$$(\lambda_1, U_1) \leq (\lambda_3, U_3)$$

(iii) If (λ_1, U_1) and $(\lambda_2, U_2) \in M$, then since $\wedge$ is a directed set, there is a $\lambda_3 \in \wedge$ such that $\lambda_1 \leq \lambda_3$, $\lambda_2 \leq \lambda_3$.

Moreover, $U_1 \cap U_2 \subset U_1$ as well as in U_2. Then the pair $(\lambda_3, U_1 \cap U_2)$ is the pair such that $\lambda_1 \leq \lambda_3$ and $U_1 \cap U_2 \subset U_1$.

i.e. $\quad (\lambda_1, U_1) \leq (\lambda_3, U_1 \cap U_2)$

and also $\quad \lambda_2 \leq \lambda_3$ and $U_1 \cap U_2 \subset U_2$

i.e. $\quad (\lambda_2, U_2) \leq (\lambda_3, U_1 \wedge U_2)$

Thus $\leq$ is a direction on M. Define $\phi : M \to \wedge$ by $\phi(\lambda, U) = \lambda$ we have

$$\phi(\lambda_1, U_1) = \lambda_1$$

and $\quad \phi(\lambda_2, U_2) = \lambda_2$

Therefore $\lambda_1 \leq \lambda_2 \Rightarrow \phi(\lambda_1, U_1) \leq \phi(\lambda_2, U_2)$ which shows that ϕ is increasing.

Moreover if $\lambda \in \Lambda$, then there is some $(\mu, U) \in M$ such that $\lambda \leq \phi(\mu, U) = \mu$

which shows that ϕ is cofinal in Λ. Therefore T = **P** oϕ is a subnet of (x_λ) we will now show that T converges to $x \in X$. Let U_0 be a neighbourhood of x in X. Since x is a cluster point of (x_λ), the net (x_λ) is frequently in U_0. In particular fix any $\lambda_0 \in \Lambda$ such that $x_{\lambda_0} \in U_0$. Then $(\lambda_0, U_0) \in M$. Now for any $(\lambda, U) \in M$, $(\lambda, U) \geq (\lambda_0, U_0)$ implies that $U \subset U_0$, so that

$$T(\lambda, U) = Po\phi(\lambda, U) = P(\lambda) = x_\lambda \in U \subset U_0.$$

Thus we have shown that for each neighbourhood U_0 of $x \in X$, there exists $(\lambda_0, U_0) \in M$ such that

$$(\lambda, U) \geq (\lambda_0, U_0) \Rightarrow T(\lambda, U) \subset U_0$$

Hence the subnet T = Poϕ of P is convergent to the limit $x \in X$.

Conversely suppose T = Poϕ, where $\phi : M \to \wedge$ is a subset of P = (x_λ) converging to x in X. Let U be any nbd of x in X and let $\lambda_0 \in \wedge$ be given. Since T converges to x, there exists μ_U in M such that for all $\mu \in M$, $\mu \geq \mu_U$ implies $T(\mu) \in U$, i.e. $P(\phi(\mu)) = x_{\phi(\mu)} \in U$.

Since $\phi(M)$ is cofinal in $\wedge$, there is some $u_0 \in M$ such that $\lambda_0 \leq \phi(u_0)$. Choose $\mu^* \in M$ such that $\mu^* \geq \mu_0$ and $\mu^* \geq \mu_U$. Then $\phi(\mu^*) = \lambda^* \geq \lambda_0$. Since $\phi(\mu^*) \geq \phi(u_0) \geq \lambda_0$ and $x_{\varphi(\mu^*)} = x_{\lambda^*} \in U$ since $\mu^* \geq \mu_0$. Thus for any neighbourhood U of x and any $\lambda_0 \in \wedge$, there is some $\lambda^* \geq \lambda_0$ with $x_{\lambda^*} \in U$. It follows that x is a cluster point of (x_λ).

Cor. If a subnet of (x_λ) has x as a cluster point, so does (x_λ).

Proof. A subnet of a subnet of (x_λ) is a subnet of (x_λ).

We turn now to the problem of showing that nets do indeed represent the correct way of approaching convergence question in topological spaces.

Theorem 6. If $E \subset X$, then $x \in \overline{E}$ (closure of E) if and only if there is a net (x_λ) of points of E with $x_\lambda \to x$.

Proof. If $x \in \overline{E}$, then each neighbourhood U of x, meets E in at least one point x_U. We take the index set as the collection of all neighbourhoods of x ordered by reverse inclusion. Then $(x_U)_{U \in U_x}$ is a net contained in E which converges to x. To show the convergence, let V be any neighbourhood of x. We have $U_0 \subset V$ for some $U_0 \in U_x$ (since U_x represents the base of neighbourhoods). Then for all $U \geq U_0 \geq V$ we have

$$x_U \in U_0 \subset V \text{ . Hence } x_U \to x.$$

Conversely if (x_λ) is a net contained in E and $x_\lambda \to x$. Then every neighbourhood of x contains a tail of (x_λ). Thus every neighbourhood of x meets E and hence $x \in \overline{E}$.

Theorem 7. Let $f : X \to Y$. Then f is continuous if and only if whenever $x_\lambda \to x_0$ in X, then $f(x_\lambda) \to f(x_0)$ in Y.

Proof. Suppose f is continuous at x_0. Therefore given a neighbourhood V of $f(x_0)$, $f^{-1}(V)$ is a neighbourhood of x_0. Now let $x_\lambda \to x_0$, therefore $\exists\ \lambda_0$ such that $\lambda \geq \lambda_0$ implies $x_\lambda \in f^{-1}(V)$. Thus $\lambda \geq \lambda_0$ implies $f(x_\lambda) \in V$. This shows that $f(x_\lambda) \to f(x_0)$.

On the other hand, if f is a not continuous at x_0, then for some neighbourhood V of $f(x_0)$, $f(U) \not\subset V$ for any neighbourhood U of x_0. Thus for each neighbourhood U of x_0, we can pick $x_U \in U$ such that $f(x_0) \notin V$. But then (x_U) is a net in X and $x_U \to x_0$, while $f(x_U) \not\to f(x_0)$.

Theorem 8. A net $<x_\lambda>$ in a product $X = \prod_{\alpha\in\wedge} X_\alpha$ converges to x if and only if for each $\alpha\in\Lambda$, $\pi_\alpha(x_\lambda) \to \pi_\alpha(x)$ in X_α.

Proof. Let $x_\lambda \to x$ in $\prod_{\alpha\in\wedge} X_\alpha$. Since projection mapping π_α is cont. by the above theorem, $\pi_\alpha(x_\lambda) \to \pi_\alpha(x)$ in X_α.

Suppose on the other hand, that $\pi_\alpha(x_\lambda) \to \pi_\alpha(x)$ for each $\alpha\in\Lambda$ (index set). Let

$$\pi_{\alpha_1}^{-1}\,(U\alpha_1) \cap \ldots \cap \pi\alpha_n^{-1}\,(U_{\alpha n})$$

be a basic neighbourhood of x in the product space. Then for each i = 1, 2,…, n, there is a λ_i such that whenever $\lambda \geq \lambda_i$, we have $\pi_{\alpha_i}(x_\lambda) \in U_\alpha$. Thus if λ_0 is picked greater than all of $\lambda_1, \lambda_2\ldots, \lambda_n$ we have $\Pi_{\alpha_i}(x_\lambda) \in U\alpha_i$, i = 1, 2, …,n for all $\lambda \geq \lambda_0$. It follows that for $\lambda \geq \lambda_0$, $x_\lambda \in \pi_{\alpha_i}^{-1}(U\alpha_i)$ and hence $x_\lambda \to x$ in the product.

Hausdorffness and Nets

Theorem 9. A topological space is Hausdorff if and only if limits of all nets in it are unique.

Proof. Suppose X is a Hausdorff space and P : $\wedge \to X$ is a net in X. Let P converges to x and y in X. We have to show that x = y. If this is not so, then there exists open sets U and V such that $x \in U$, $y\in V$ and $U\cap V = \phi$. Since P converges to x, there exists $\lambda_1 \in\wedge$ such that for all $\lambda\in\Lambda$, $\lambda_1 \geq \lambda$, implies $P(\lambda)\in U$. Again since P converges to y, there exists $\lambda_2\in\wedge$ such that for all $\lambda\in\wedge$, $\lambda \geq \lambda_2$ implies $P(\lambda)\in V$. Now because $\wedge$ is a directed set, there exists $\lambda\in\wedge$ such that $\lambda \geq \lambda_1$ and $\lambda \geq \lambda_2$. But that $P(\lambda)\in U$, $P(\lambda) \in V$ i.e. $P(\lambda) \in U\cap V$, a contradiction since $U\cap V = \phi$, so x = y, establishing the necessity of the condition.

Conversely suppose that the limits of all nets in a space X are unique. If X is not Hausdorff, then there exist two distinct points x, y in X which do not have mutually disjoint neighbourhood in X. Let μ_x and μ_y be the neighbourhood systems in X at x and y respectively. Let $\wedge = \mu_x \times \mu_y$ and for $(U_1, V_1), (U_2, V_2) \in\wedge$, define $(U_1, V_1) \geq (U_2, V_2)$ if and only if $U_1 \subset U_2$ and $V_1 \subset V_2$. This makes $\wedge$ a directed set and we define a net P : $\wedge\to X$ as follows :

For any $U\in\mu_x$ and $V\in\mu_y$ we know that $U \cap V \neq \phi$. Define P(U, V) to be any point in $U \cap V$. We assert that the net P so defined converges to x. In fact, let G be any neighbourhood of x. Then $(G, X)\in\wedge$. Now if $(U, V) \geq (G\ X)$ in $\wedge$ then $U \subset G$ and so $P(U, V) \in U \cap V \subset U \subset G$. This proves that P converges to x. Similarly we can show that P converges to y also contradicting the hypothesis. Hence X is a T_2-space.

Remark. The proof of the converse illustrates the advantage nets have over sequences. In a sequence, the domain is always the set of positive integers, while in defining nets, we have considerable freedom in the choice of the directed set.

Compactness of Nets

Definition. A family F of subsets of a set X is said to have the finite intersection property if for any $n\in N$ and $F_1\ F_2,\ldots, F_n\in F$, the intersection $\bigcap_{i=1}^{n} F_i$ is non-empty. In particular every member of a family having finite intersection property is non-empty.

Prop. A topological space is compact if and only if every family of closed subsets of it, which has the finite intersection property, has a non-empty intersection. Now we can prove the following

Theorem 10. For a topological space X, the following statements are equivalent

(i) X is compact

(ii) Every net in X has a cluster point in X.

(iii) Every net in X has a convergent subnet in X (i.e. a subnet which converges to at least one point in X).

Proof. We have proved the equivalence of (i) and (ii) alreadby (by Theorem 5). Therefore, we only prove the equivalence of (i) and (ii). So we suppose that X is compact and let $P : \wedge \to X$ be a net in X. Suppose X has no cluster point in X. Then for each $x \in X$, there exists a neighbourhood N_x of x and an element $m_x \in \wedge$ such that for all $n \in \wedge$, $n \geq m_x$ implies $P(n) \in X - N_x$ cover X by such neighbourhoods (or more precisely, by their interiors). By compactness of X, there exist $x_1, x_2, \ldots, x_k \in X$ such that $X = \bigcup_{i=1}^{K} N_{x_i}$. Let the corresponding elements in $\wedge$ be $m_1, m_2, \ldots, m_k$. Since $\wedge$ is a directed set, there exists $n \in \wedge$ such that $n \geq m_i$ for i = 1, 2,…, k. But then $P(n) \in \bigcup_{i=1}^{K} (X - N_{x_i}) = X - \bigcup_{i=1}^{K} N_{x_i} = \phi$, a contradiction.

Hence P has at least one cluster point in X. Thus (ii) holds.

Conversely suppose that (ii) holds. Let C be a family of closed sets of X having the finite intersection property. Let **D** be the family of all finite intersections of members of C. **D** itself is closed under finite intersections and that $C \subset \mathbf{D}$. For D, E $\in$ **D**, we define $D \geq E$ to mean $D \subset E$. This makes **D**, a directed set because whenever D, E$\in$**D**, $D \cap E \in \mathbf{D}$ and $D \cap E \geq D$, $D \cap E \geq E$. Note that each member of **D** is non-empty because C is given to have the finite intersection property. So we can define a net $P ; \mathbf{D} \to X$ by P(D) = any point in D. By (ii), this net has a cluster point say x in X. We claim $x \in \bigcap_{\subset \in \mathbf{C}} C$. For if not, there exists $C \in \mathbf{C}$ such that $x \notin C$. Then $X - C$ is a neighbourhood of x (since members of **C** are closed). Also $C \in \mathbf{D}$. So by the definition of a cluster point, there exists $D \in \mathbf{D}$ such that $D \geq C$ and $P(D) \in X - C$. But then $D \subset C$, and so $X - C \subset X - D$ contradicting that $P(D) \in \mathbf{D}$. So $x \in \bigcap_{\subset \in \mathbf{C}} C$. We have thus shown that every family of closed subsets of X having finite intersection property has non-empty intersection. But this implies that X is compact.

Definition. A net $<x_\lambda>$ in a set X is an ultranet (universal net) if and only if each subset E of X, (x_λ) is either residually in E or residually in X–E.

It follows from this definition that if an ultranet is frequently in E, then it is residually in E. In particular, an ultranet in a topological space must converge to each of its cluster points.

For any directed set $\wedge$, the map $P : \wedge \to X$ defined by $P(\lambda) = x$ for all $\lambda \in \wedge$, gives an ultranet on X, called the trivial ultranet. Non-trivial ultranets can be proved to exist (relying on the axiom of choice) but none has ever been explicitly constructed. Most facts about ultranets are best developed using filters at ultrafilters.

Theorem 11. If $<x_\lambda>$ is an ultranet in X and $f : X \to Y$, then $(f(x_\lambda))$ is an ultranet in Y.

Proof. If $A \subset Y$, then $f^{-1}(A) = X - f^{-1}(Y - A)$ therefore by the definition of ultranet, $<x_\lambda>$ is eventually in either $f^{-1}(A)$ or $f^{-1}(Y - A)$ from which it follows that $(<f(x_\lambda(>))$ is eventually in either A or Y−A. Thus $(f(x_\lambda))$ is an ultranet.

Theorem 12. For a top. Space X, the following are equivalent.

(a) X is compact.
(b) Every net in X has a cluster point in X.
(c) Every net in X has a convergent subnet in X.
(d) Each ultranet in x converges.

Proof. We have already proved the equivalence of (a), (b) and (c). Moreover each ultranet is a net and by (b) it has a cluster point and hence converges to that point. Thus (b) ⇒(d). Similarly we can prove that (d) ⇒ (b) thus these four statements will be equivalent.

Filters and Their Convergence

We now introduce a second way of describing convergence in a topological space. The result is the theory of filter convergence.

Definition. A filter **H** on a set S is a non-empty collection of subsets of S such that.

(a) $\phi \notin \mathbf{H}$
(b) If $F_1, F_2 \in \mathbf{H}$, then $F_1 \cap F_2 \in \mathbf{H}$
(c) If $F \in \mathbf{H}$ and $F' \supset F$, then $F' \in \mathbf{H}$.

In view of the result "If a family **B** of subsets of a set X is closed under finite intersection, then **B** has finite intersection property". It follows that the conditions (a) and (b) imply that a filter has the finite intersection property. Condition (c) says that a filter is closed under the operation of taking supersets of its members. It implies in particular that the set S always belongs to every filter on it.

Definition. Let **H** be a filter on a set S. Then a subfamily **B** of H is said to be filter base for **H** if for any $F \in \mathbf{H}$, there exists $G \in \mathbf{B}$ such that $G \subset F$. Thus if **B** is a base for a filter **H**, then every members of **H** is a superset of some members of **B**.

On the other hand if $B \in \mathbf{B}$, then $B \in \mathbf{H}$ and so nay subset of B is in **H** by condition (c) of the definition of a filter. Thus if **B** is a base for a filter **H,** then **H** consists precisely of all supersets of members of **B**, i.e.

$$\mathbf{H} = \{F \subset S\ ;\ F \supset B,\ B \in \mathbf{B}\}$$

Definition. If $\mathbf{H}_1$ and $\mathbf{H}_2$ are filters on the same set x, we say **H**, is finer than $\mathbf{H}_2$ (or $\mathbf{H}_2$ is coarser than $\mathbf{H}_1$) if and only if $\mathbf{H}_1 \supset \mathbf{H}_2$.

Definition. Two filters $\mathbf{H}_1$ and $\mathbf{H}_2$ are said to be comparable if and only if $\mathbf{H}_1$ is finer than $\mathbf{H}_2$ or $\mathbf{H}_2$ is finer than $\mathbf{H}_1$.

Definition. A filter **H** on a set X is said to be fixed if and only if $\cap\, \mathbf{H} \neq \phi$, and fine if $\cap\, \mathbf{H} = \phi$.

Remark. The set of all filters on X is directed by the relation ≤ defined by setting $\mathbf{H}_1 \leq \mathbf{H}_2$ iff $\mathbf{H}_2$ is finer than $\mathbf{H}_1$.

Examples. (1) If X is any non-empty set, then the singleton family {X} is a filter on X. Moreover {X} is the coarset filter on X. In other words {X} is the smallest element of the ordered set of all filters on X. But there is no largest filter on X if X consists of more of more than one element as we will see later on. In fact among the families satisfying axioms (b) and (c), the power set **P**(X) is largest and it is excluded by axiom (a)

(2) Let X be any non-empty set and $x_0 \in X$. Then the family

$$\mathbf{H} = \{F \,;\, x_0 \in F\}$$

is a filter on X. To show it, we observe that **H** is non-empty since $\{x_0\} \in \mathbf{H}$. Moreover

(i) Since $x_0 \in F$ for all $F \in \mathbf{H}$, it follows that no member of **H** is empty and so $\phi \notin \mathbf{H}$.

(ii) Let $F \in \mathbf{H}$ $H \in$ **H,** then $x_0 \in F$ and $x_0 \in H$. Thus $x_0 \in F \cap \mathbf{H} \Rightarrow$ FAH $\in \mathbf{H}$.

(iii) Let $F \in \mathbf{H}$ and $H \supset F$. Then $x_0 \in F \Rightarrow x_0 \in H \Rightarrow H \in \mathbf{H}$.

Hence **H** is a filter.

(3) Let F_0 be a non-empty subset of a set X. Then the collection

$$\mathbf{H} = \{F \subset X,\ F \supset F_0\}$$

of all supersets of F_0 (in X) is a filter on X. Such a filter is known as **Atomic filter**. The set F_0 being called **Atom of the filter**. In this case, F_0 is the intersection of all members of the filter. The **filter base** for this filter **H** is the collection consisting of the single set F_0.

(4) Let X be any infinite set. Then the collection

$$\mathbf{H} = \{F \,;\, X - F \text{ is finite}\}.$$

of all cofinite subsets of X is a filter on X (Note that this filter is not atomic). Such a filter is called the **cofinite filter**. We prove it as follows

Since $X - X = \phi$, is finite. $X \in \mathbf{H}$ and hence **H** is non-empty. Further

(i) Since X is infinite and X–F is finite, it follows that F is an infinite set and hence no member of **H** is empty. Thus $\phi \notin \mathbf{H}$**.**

(ii) If $F, H \in \mathbf{H}$, then X–F and X–H are both finite. Now

$$X - (F \cap H) = (X - F) \cup (X - H) \quad \text{(Demorgan's Law)}$$

Since (X–F) and (X–H) are finite, it follows that $X - (F \cap H)$ is finite $\Rightarrow F \cap H \in \mathbf{H}$.

(iii) Let $F \in \mathbf{H}$, and $H \supset F$. Since X–F is finite $\Rightarrow$ X –H is finite. Hence $H \in \mathbf{H}$. Thus **H** is a filter.

(5) Suppose (X, T) is a topological space. Then for any $x \in X$, the neighbourhood system U_x at x is a filter. It is called the **T-neighbourhood filter** at x. Any neighbourhood base at x is a filter base for T-neighbourhood filter.

(6) Let N be the set of non-negative integers. Then the collection

$$\mathbf{H} = \{F \,;\, N - F \text{ is finite}\}$$

Is a filter on N. This is known as **Frechet Filter.**

(7) Let $P = \wedge \rightarrow X$ be a net. For each $\lambda_0 \in \wedge$, let

$$B_{\lambda_0} = \{P(\lambda) \,;\, \lambda \in \wedge \,;\, \lambda \geq \lambda_0\}$$

let $$\mathbf{H} = \{F \subset X \,;\, F \supset B_{\lambda_0} \text{ for some } \lambda_0 \in \wedge\}.$$

In other words, **H** is the collection of all supersets of the sets of the form B_{λ_0} ; $\lambda_0 \in \wedge$. Using the fact that $\wedge$ is a directed set, it can be shown that **H** is a filter on X. Obviously it depends on the net and is called the filter associated with the net P.

Theorem 13. Let $\{\mathbf{H}_\lambda\}$ be any non-empty family of filters o a non-empty set X (which must therefore be non-empty). Then the set

$$\mathbf{H} = \cap \mathbf{H}_\lambda$$

is also a filter on X.

Proof. We have

(i) Since $\phi \notin \mathbf{H}_\lambda$ for any $\lambda \in \wedge$, if follows that $\phi \notin \mathbf{H}$.

(ii) Let F, H$\in$**H**. Then $F \in \mathbf{H}_\lambda$, $H \in \mathbf{H}_\lambda$ for all $\lambda \in \wedge$. Since each $\mathbf{H}_\lambda$ is a filter it follows that $F \cap H \in \mathbf{H}_\lambda$ for all $\lambda \in \wedge$. Hence

$$F \cap H \in \mathbf{H}.$$

(iii) Let $F \in \mathbf{H}$, $H \supset F$, then $F \in \mathbf{H}_\lambda$ for all $\lambda \in \wedge$. Since each $\mathbf{H}_\lambda$ is a filter and $F \in \mathbf{H}_\lambda$, it follows that $H \in \mathbf{H}_\lambda$ for all $\lambda \in \wedge$. Hence $H \in \mathbf{H}$. Thus the theorem is proved.

The filter $\mathbf{H} = \cap \mathbf{H}_\lambda$ constructed above is called the intersection of the family of filters $\mathbf{H}_\lambda$ and is obviously the greatest lower bound of the set of the $\mathbf{H}_\lambda$ in the ordered set of all filters on X.

Theorem 14. Let A be any non-void family of subsets of a set X. Then there exists a filter on X containing **A** if and only if **A** has finite intersection property.

Proof. Suppose first that A has finite intersection property. We have to show that there exists a filter on X containing A. Let **B** = {B ; B is the intersection of a finite subfamily of A} .

Since A has finite intersection property (f. i. p) no member of **B** is empty. Hence $\phi \notin \mathbf{B}$. Now set

$$\mathbf{H} = \{F \; ; \; F \text{ contains a member of } \mathbf{B}\}$$

Evidently $\mathbf{H} \supset A$. We claim that **H** is a filter on X. In this direction, we have

(1) Since $\phi \notin \mathbf{B}$ and every member of **H** is a superset of some members of **B**, it follows that $\phi \notin \mathbf{H}$.

(2) Let $F \in \mathbf{H}$ and $H \supset F$. Since F contains a member of **B**, H must also contain that member. Hence $H \in \mathbf{H}$.

(3) Let F, H$\in$**H**. Then $F \supset B$, $H \supset C$, where B and C are members of **B**. Since B and C are finite intersections of members of **A**. $B \cap C$ must also be a finite intersection of members of **A**. Hence $B \cap C \in \mathbf{B}$. Also $F \supset B$, $H \supset C \Rightarrow F \cap H \supset B \cap C$. Thus $F \cap H$ contains a member of **B** and so $F \cap H \in \mathbf{H}$.

Thus we have shown that if **A** has f. i. p, then $\exists$ a filter **H** containing **A**.

Conversely suppose that **H** is a filter on X containing **A**. Then **H** also contains the collection **B** of finite intersection of members of **A**. hence a necessary condition for the existence of such a filter **H** is that $\phi \notin \mathbf{B}$, i.e. **A** must have f. i. p.

Cor. The filter **H** obtained above is the coarset filter which contains **A** i.e. the smallest element of the ordered family of all filters on X containing **A**.

In fact, if **H′** is any filter containing **A**, then **H′** must contain all finite intersections of members of **A** and their supersets and hence $\mathbf{H}' \supset \mathbf{H}$.

Definition. The filter **H** defined in the above theorem is said to be generated by **A** and **A** is said to be a subbase of **H**.

Observe that for **A** to be a subbase, it is necessary and sufficient that *f* has f. i. p.

Remark. The above theorem suggests the following method of constructing a filter.

Take any family **A** of subsets of X with f. i. p. Then obtain the family **B** of all finite intersections of **A**. The family **H** of all those subsets of X which contain a member of **B** is a filter on X e.g. X = {a, b, c, d} and let **A** = {{a, b}, {b, c}}, then **A** has f. i. p. Here

$$\mathbf{B} = \{\{b\}, \{a, b\}, \{b, c\}, X\}$$

Note that X is a member of **B** since it is the intersection of the void subfamily of **A**. We now form the family **H** by taking all the superset of member of **B**. Thus

$$\mathbf{H} = \{\{b\}, \{a, b\}, \{b, c\}, \{b, d\}, \{a, b, c\}, \{b, c, d\}, \{a, b, d\}, X\}$$

which is a filter on X.

Theorem 15. Let **B** be a family of non-empty subsets of a set X. Then there exists a filter on X having **B** as a base iff **B** has the property that for any $B_1, B_2 \in \mathbf{B}$, there exists $B_3 \in \mathbf{B}$ such that $B_1 \cap B_2 \supset B_3$.

Proof. Suppose there exists a filter **H** on X having **B** as a base. Then $\mathbf{B} \subset \mathbf{H}$ and $\phi \notin \mathbf{H}$; hence $\phi \notin \mathbf{B}$, Also let $B_1, B_2 \in \mathbf{B}$. Then $B_1, B_2 \in \mathbf{H}$ (since $\mathbf{B} \subset \mathbf{H}$) and so $B_1 \cap B_2 \in \mathbf{H}$ as **H** is closed under finite intersections. So by the definition of a base, there exists $B_3 \in \mathbf{B}$ such that $B_1 \cap B_2 \supset B_3$. This proves the necessity of the condition.

Conversely suppose **B** satisfies the given condition. We then construct a filter from **B** as follows: Let **H** be the family of all supersets of members of **B**. Then condition (C) in the definition of a filter automatically holds for **H**. The empty set can not be a superset of any set. Hence it follows that $\phi \notin \mathbf{H}$ as $\phi \notin \mathbf{B}$. It only remains to show that **H** is closed under finite intersections. For this it suffices to show that the intersection of any two members of **H** is again in **H** for then one can apply induction. So suppose $A_1, A_2 \in \mathbf{H}$. Then there exists $B_1, B_2 \in \mathbf{B}$ such that $B_1 \subset A_1$ and $B_2 \subset A_2$. We are given that there exists $B_3 \in \mathbf{B}$ such that $B_3 \subset B_1 \cap B_2$. But then $A_1 \cap A_2$ is a superset of $B_3 \in \mathbf{B}$ and so $A_1 \cap A_2 \in \mathbf{H}$. Thus **H** is a filter on X and **B** is a base for it by its very construction.

Cor. Any family which does not contain the empty set and which is closed under finite intersections is a base for a unique filter.

Proof. The condition in the above theorem is trivially satisfied for such a family.

It is natural to enquire if there is a corresponding notion of subbases for filters. The answer is in the affirmative.

Definition. Let **H** be a filter on a set X. Then a subfamily δ of **H** is said to be a subbase for **H** if the family of all intersections of members of δ is a base for **H**. We also say δ generates **H**.

Obviously every base is a subbase. It is easy to characterize those families that can generate filters.

Theorem 16. Let δ be a family of subsets of a set X. Then $\exists$ a filter on x having S as a subbase if and only if S has the finite intersection property.

Proof. If there exists a filter **H** on X containing S, then **H** has f. i. p. and so does every subfamily of **H**. Thus the condition is necessary.

Conversely suppose S has a finite intersection property. Let **B** be the family of all finite intersections of members of S. Then $\Phi \notin \mathbf{B}$ and **B** is closed under finite intersections. So by the Cor. above, **B** is a base for the filter **H** on X and thus S is a subbase for **H**.

So far the treatment was purely set theoretic, without mention of any topology on the set X in question. Suppose now that a topology T on X be given. We then define convergence and cluster points of filters with respect to T as follows.

Definition. A filter **H** on a topological space X is said to converge to x (written **H**$\to$x) iff every neighbourhood of x belongs to **H** iff $\mu_x \subset$ **H** i.e. iff **H** is finer than the neighbourhood filter at x). Also we say that x is a cluster point of **H** cor **H** clusters at x) iff every neighbourhood of x intersects every member of **H** i.e. iff

$$x \in \cap \{ \overline{F} ; F \in \mathbf{H}\}$$

Remark. It is clear that if **H**$\to$x, then **H** clusters at x because given $U \in \mu_x$, F$\in$**H**, both U and F are in **H** and so $U \cap F \neq \phi$.

Remark. It will be convenient to have the notions of convergence and clustering variable for filter bases, they generalize easily and obviously.

A filter base **B** converges to x iff each $U \in \mu_x$ contains some B$\in$**B** (if and only if the filter generated by **B** converges to x). Similarly **B** clusters at x iff each $U \in \mu_x$ meets each B$\in$**B** (iff the filter generated by **B** clusters at x)

Example. (1) Trivial examples of convergent filters are the neighbourhood filters.

(2) If $\mathbf{H}_1$ and $\mathbf{H}_2$ are filters on X with $\mathbf{H}_1 \subset \mathbf{H}_2$ then whenever $\mathbf{H}_1$ converges to some point in X, so does $\mathbf{H}_2$. It is probably for this reason that in such a case $\mathbf{H}_2$ is said to be a **subfilter** of $\mathbf{H}_1$ even though as subset of P(X), $\mathbf{H}_2$ is a super set (and not a subset) of $\mathbf{H}_1$.

(3) The Frechet filter on R has no cluster points.

Theorem 17. **H** has x as a cluster point iff there is a filter **G** finer than **H** which converges to A filter **H** has x as a cluster point iff some subfilter of **H** converges to x.

Proof. Suppose **H** has x as a cluster point. Therefore every neighbourhood of x intersects every member of **H** i.e. $F \cap U \neq \phi$ for every F$\in$**H** and every $U \in \mu_x$. Thus the family

$$C = \{F \cap U ; F \in \mathbf{H} \text{ and } U \in \mu_x\}$$

generates and is a base for a filter defined by

$$\mathbf{G} = \{G ; G \supset F \cap U ; F \in \mathbf{H}, U \in \mu_x\}$$

In fact (i) $\phi \notin$ **G** because the empty set can not be a superset of any set.

(ii) Suppose $G_1, G_2 \in$ **G**. Therefore $\exists$ $F_1, F_2 \in$ **H** and $U_1, U_2 \in \mu_x$ such that

$$G_1 \supset F_1 \cap U_1 \text{ and } G_2 \supset F_2 \cap U_2$$

which implies

$$G_1 \cap G_2 \supset (F_1 \cap U_1) \cap (F_2 \cap U_2)$$
$$= (F_1 \cap F_2) \cap (U_1 \cap U_2)$$

Since $F_1 \cap F_2 \in$ **H** (property of a filter) and $U_1 \cap U_2 \in \mu_x$ (property of neighbourhood system) it follows that $G_1 \cap G_2 \in$ **G** .

(iii) If $G_1 \in$ **G** and $G_2 \supset G_1$, then we have $G_1 \supset F \cap U$, $F_1 \in$**H** and $U \in \mu_x$. i.e. $G_2 \supset G_1 \supset F \cap C$, F$\in$**H** and $U \in \mu_x$

Thus **G** is a filter and C is a base for it from its very construction. Since X is a neighbourhood of x_1 therefore $X \in \mu_x$ and we have $G \supset X \cap F = F$, F$\in$**H** for all F. Thus **G** is finer than **H**. Since **H** converges to x, every neighbourhood of x belongs to **H** and therefore to **G**. Thus **G** converges to x.

Conversely suppose that **H** $\subset$ **G**$\to$x, then each neighbourhood of x belongs to **G**. Also each F$\in$**H** $\Rightarrow$ F$\in$**G**. Thus each neighbourhood of x intersect every member of **H**. Hence x is a cluster point of **H**.

Now we shall show that filter convergence is adequate to the task of describing topological concepts.

Theorem 18. If $E \subset X$, then $x \in \overline{E}$ if and only if there is a filter **H** such that $E \in \mathbf{H}$ and $\mathbf{H} \to x$.

Proof. Let $x \in \overline{E}$, then every neighbourhood U of x has a non-empty intersection with E. Then the collection

$$C = \{U \cap E ; U \in \mu_x\}$$

is a filter base for the filter

$$\mathbf{H} = \{G ; G \supset U \cap E ; U \in \mu_x\}$$

Clearly $E \in \mathbf{H}$ because $E \supset U \cap E$. This filter **H** converges to x because each neighbourhood U of x belongs to **H**.

Conversely, if $E \in \mathbf{H} \to x$, then x is a cluster point of **H** and hence $x \in \overline{E}$.

Theorem 19. Let X, Y be sets, $f : X \to Y$ a function and **H**, a filter on X. Then the family

$$C = \{f(A) ; A \in \mathbf{H}\}$$

is a base for a filter on Y.

Proof. Since by Theorem 15, "let **B** be a family of non-empty subsets of a set X. Then there exists a filter on X having **B** as a base iff **B** has the property that for any $B_1, B_2 \in \mathbf{B}$, there exists $B_3 \in \mathbf{B}$ such that $B_1 \cap B_2 \supset B_3$". Therefore we show that C satisfies then condition of the above stated Theorem. Evidently $\phi \notin C$. Also let $B_1, B_2 \in C$, Then $\exists$ $A_1, A_2 \in \mathbf{H}$ such that $f(A_1) = B_1$ and $f(A_2) = B_2$. Then $A_1 \cap A_2 \in \mathbf{H}$ (property of a filter) and $B_1 \cap B_2$ contains $f(A_1 \cap A_2)$ which is a member of C. Therefore by the above Theorem C is a base for a filter on Y.

Definition. Let **H** be a filter on X and $f : X \to Y$ be a function. Then the filter, denoted by $f(\mathbf{H})$ on Y having

$$C = \{f(A) ; A \in \mathbf{H}\}$$

as its base is called the image filter of **H** under *f*. We are now in a position to characterize continuity in terms of convergence of filters.

Theorem 20. Let X, Y be top. Spaces, $x \in X$ and $f : X \to Y$ a function. Then *f* is continuous at x iff whenever a filter **H** converges to x in X, the image filter $f(\mathbf{H})$ converges to $f(x)$ in Y.

Proof. Assume first that *f* is continuous at x and **H** is a filter which converges to x in X. We have to show that $f(\mathbf{H})$ converges to $f(x)$ in Y. Let V be a given neighbourhood of $f(x)$ in Y. Since *f* is cont. at x, $f^{-1}(V)$ is a neighbourhood of x in X. But **H** converges to x in X. Therefore every neighbourhood and in particular $f^{-1}(V) \in \mathbf{H}$. So $f(f^{-1}(V)) \in f(\mathbf{H})$. But V contains $f(f^{-1}(V))$ and so (being a superset of member of $f(\mathbf{H})$), $V \in f(\mathbf{H})$. Thus each neighbourhood V of $f(x)$ belongs to $f(\mathbf{H})$. Hence the filter $f(\mathbf{H})$ converges to $f(x)$.

Conversely suppose that $\mathbf{H} \to x$ in X implies $f(\mathbf{H}) \to f(x)$ in Y. Let **H** be the filter of all neighbourhoods of x in X. Then each neighbourhood V of $f(x)$ belongs to $f(\mathbf{H})$. Therefore for some neighbourhood U of x, we have $f(U) \subset V$. Then for each neighbourhood V of $f(x)$, there is a neighbourhood U of x such that $f(U) \subset V$. Hence *f* is continuous.

Theorem 21. Let $X = \prod_{\lambda} X_\lambda$ be the topological product of an indexed family of spaces $\{X_\lambda ; \lambda \in \wedge\}$. Let **H** be a filter on X and let $x \in X$. Then **H** converges to x in X if and only if for each λ, the filter $\pi_\lambda(\mathbf{H})$ converges to $\pi_\lambda(x)$ in X_λ.

Proof. Suppose first that **H**→x in X. Since $\prod_\lambda$ is continuous, it follows that π_λ(**H**)→ π_λ(x) . Conversely suppose that π_λ (**H**)→ π_λ(x) in X_λ. We have to show that **H** converges to x in X. Let U be a neighbourhood of x in X. Then U contains a basic open set V containing x. Let V = $\prod_\lambda V_\lambda$, where each $\mathbf{V}_\lambda$ is an open set in X_λ and $V_\lambda = X_\lambda$ for all $\lambda\in\wedge$ except for $\lambda = \lambda_1, \lambda_2,\ldots,\lambda_n$ (say). Now π_{λ_k} (**H**) converges to π_{λ_k} (x) for all k = 1, 2, …,n. So $V_{\wedge k} \in \pi_{\lambda_k}$ (**H**) and hence there exists $F_k\in$**H** such that $V_{\lambda_k} \supset \pi_{\lambda_k}(F_k)$ for k = 1, 2,…,n. Note that $\pi^{-1}_{\lambda_k}(V_{\lambda_k}) \supset \pi^{-1}_{\lambda_k}(\pi_{\lambda_k}(F_k)) \supset F_k$ for k = 1, 2,…,n. So U⊃V = $\bigcap_{K=1}^{n} \pi^{-1}_{\lambda_k}(V_{\lambda_k}) \supset \bigcap_{K=1}^{n} F_k$. But intersection $\bigcap_{K=1}^{\infty} F_k$ is in **H** since **H** is closed under finite intersections and therefore U being a superset of $\bigcap_{K=1}^{n} F_k$ is also in **H**. Thus each neighbourhood of x belongs to **H** and therefore **H**→x.

Second proof for the converse. Suppose that π_α(**H**)→ π_α(x) for each α. Let $\bigcap_{K=1}^{n} \Pi^{-1}_{\alpha_k}(U_k)$ be a basic neighbourhood of x in $\prod_\alpha X_\alpha$. Then U_k is a neighbourhood of $\pi_{\alpha_k}(x)$ for each k. Since π_{α_k} (**H**)→ π_{α_k} (x) each neighbourhood of π_{α_k} (x) belongs to π_{α_k} **(H)**. Thus $U_k\in \Pi_{\alpha_k}$ (**H**) for each k. Hence there is some $F_k\in$**H** (by the definition of the base of the filter π_{α_k} (**H**)) such that $\pi_{\alpha_k}(F_k) \supset U_k$. Then $\bigcap_{K=1}^{n} F_k \in$**H** and $\bigcap_{K=1}^{n} F_k \supset \bigcap_{K=1}^{n} \Pi^{-1}_{\alpha_k}(U_k)$ and so (by the property of the filter **H**) being a superset of the members of **H**

$$\bigcap_{K=1}^{n} \pi^{-1}_{\alpha_k}(U_k) \in \mathbf{H}.$$

Thus we have shown that each neighbourhood of x belongs to **H**. Hence **H**→x.

Ultrafilters and Compactness

Definition. A filter **H** on a set X is said to be ultrafilter if it is a maximal element in the collection of all filters on X, partially ordered by inclusion. Thus **H** is an ultrafilter iff it is not property contained in any filter on X(i.e. if there is no filter strictly finer than **H**). For example, all atomic filters whose atoms are singleton sets are maximal.

Theorem 22. Every filter is contained in an ultrafilter.

Proof. Let **H** be a filter on a set X and G be the collection of all filters on X containing **H**. Then **H**∈G and so G is non-empty. Partially order G by inclusion. Let $\{G_i ; i\in I\}$ be a non-empty linearly ordered set (chain) in G.

Let $$S = \bigcup_{i=1} G_i$$

We claim that S is a filter on X. Clearly $\phi\notin S$ because $\phi\notin G_i$ for all $i\in I$. Moreover let A, B,∈S, then there exist i, j∈I such that $A\in G_i$ and $B\in G_j$. Since the collection $\{G_i\}$ is a chain under inclusion, it follows that either $G_i \subset G_j$ or $G_j \subset G_i$. In the first case A, $B\in G_j$, and so $A\cap B\in G_i$. In either case $A\cap B\in S$. Finally suppose that C∈S and D ⊃ C. We have to show that D∈S. Now $C\in G_i$ for some i∈I. So $D\in G_i$ as G_i is a filter. But then D∈S. Thus we have shown that S is a filter. Obviously S contains **H** as each G_i does. So S∈G and by its construction, it is an upper bound for the chain $\{G_i ; i \in I\}$. We have thus shown that every chain in G has an upper bound in G. So by Zorn's Lemma, G contains a maximal element **I**. We claim that **I** is an ultrafilter i.e. **I** is also maximal in the set of all filters on X. In fact, suppose K is a filter on X such that $\mathbf{I} \subset K$. Then $\mathbf{H} \subset K$ (since $\mathbf{H} \subset \mathbf{I}$) and so $K \in G$. But **I** is maximal in G. So **H** = K . Thus **I** is an ultrafilter containing **H**.

Theorem 23. For a filter **H** on a set X, the following statements are equivalent.

(1) **H** is an ultrafilter

(2) For any $A \subset X$, either $A \in \mathbf{H}$ or $X - A \in \mathbf{H}$.

(3) For any A, $B \subset X$, $A \cap B \in \mathbf{H}$ iff either $A \in \mathbf{H}$ or $B \in \mathbf{H}$.

Proof. First we show that (1) $\Leftrightarrow$ (2). Assume **H** is an ultrafilter on X and A is a subset of X. If $A \notin \mathbf{H}$, then A contains no member of **H** or equivalently every member of **H** intersects X−A. Thus the family $\mathbf{H} \cup \{X-A\}$ has the finite intersection property and so generates a filter **G** containing **H**. Since **H** is maximal, no filter on X properly contain **H**. hence $\mathbf{G} = \mathbf{H} \Rightarrow X-A \in \mathbf{H}$ and so (2) holds.

Conversely suppose (2) holds. If **H** is not an ultrafilter, then $\exists$ a filter **G** which properly contains **H**. Then $\exists\ A \in \mathbf{G} - \mathbf{H}$. Since $A \notin \mathbf{H}$, $X-A \in \mathbf{H}$ by (2). Hence $A \in \mathbf{G}-\mathbf{H}$. Since $A \notin \mathbf{H}$, $X-A \in \mathbf{H}$ by (2). Hence $X-A \in \mathbf{G}$. So **G** contains A as well as X−A which implies $A \cap (X-A) \in \mathbf{G}$ (by definition of a filter) i.e. $\phi \in \mathbf{G}$ which is not in a filter. Thus we have a contradiction to the intersection property of a filter. Thus **H** is an ultrafilter. (2) $\Leftrightarrow$ (3). In view of the fact that every filter contains the set X, (2) follows from (3) by taking B = X−A.

Conversely assume (2) holds. Let A, $B \subset X$. Since $A \cup B$ is a superset of A as well as B, Therefore $A \in \mathbf{H}$ or $B \in \mathbf{H}$ implies $A \cup B \in \mathbf{H}$ from the very definition of a filter. On the other suppose $A \cup B \in \mathbf{H}$ but neither A nor $B \in \mathbf{H}$. Then by (2), $X-A \in \mathbf{H}$ and $X-B \in \mathbf{H}$ and so $(X-A) \cap (X-B) \in \mathbf{H}$. But $(X-A) \cap (X-B) = X - (A \cup B)$. Thus we have shown that if neither A nor B belongs to **H**, then both $A \cup B$ and its complement belong to **H** which is a contradiction to the intersection property of a filter. Hence A, $B \subset X$, $A \cup B \in \mathbf{H}$ iff $A \in \mathbf{H}$ or $B \in \mathbf{H}$. So (3) holds. Thus (2) $\Leftrightarrow$ (3).

Theorem 24. An ultrafilter converges to a point if and only if that point is a cluster point of it.

Proof. The direct implication is true for any filter because if a filter **H** converges to x Then x is also a cluster point of **H**.

Conversely suppose X is a space and $x \in X$ is a cluster point of an ultrafilter **H** on X. If **H** does not converge to x, then there exists a neighbourhood N of x such that $N \notin \mathbf{H}$. By the characterization of an ultrafilter (**H** is an ultrafilter if for any $A \subset x$ either $A \in \mathbf{H}$ or $X-A \in \mathbf{H}$), we have then that $X-N \in \mathbf{H}$. But since x is a cluster point of **H**, every neighbourhood of x intersects every member of **H**, whereas $N \cap (X-N) = \phi$. This contradiction proves that **H** converges to x in X.

Theorem 25. If f maps X onto Y and **H** is an ultrafilter on X, then $\overline{f(\mathbf{H})}$ is an ultrafilter on Y.

Proof. We know that $f(\mathbf{H})$ is a filter on Y having the sets $f(F)$; $F \in \mathbf{H}$ as a base. We want to show that $f(\mathbf{H})$ is an ultrafilter on Y whenever **H** is a filter on X. To this end, let G be any subset of Y. Since **H** is a filter on X we have $X \in \mathbf{H}$. Also

$$X = f^{-1}(G) \cup f^{-1}(Y-G)$$

Thus **H** is an ultrafilter and $f^{-1}(G) \cup f^{-1}(X-G) \in \mathbf{H}$.

$$[f(X) = Y,\ X = f^{-1}(Y)]$$

Therefore by characterization of an ultrafilter either $f^{-1}(G)\in\mathbf{H}$ or $f^{-1}(Y-G)\in\mathbf{H}$. If $f^{-1}(G)\in\mathbf{H}$, then $G\in f(\mathbf{H})$ or $f^{-1}(Y-G)\in\mathbf{H}$, then $Y-G\in f(\mathbf{H})$. Thus we have shown that for any $G \subset Y$ either $G\in f(\mathbf{H})$ or $Y-G\in f(\mathbf{H})$. Hence again by a characterization of ultrafilter, $f(\mathbf{H})$ is an ultrafilter.

Theorem 26. If **H** is an **ultrafilter** on a set X, than **H** is an ultrafilter base on every superset Y of X.

Proof. Let **B** be the class of all subsets G of Y such that G contains a member of **H**. Then **B** is a filter. Now let H be any subset of Y. Then either $X\cap H$ or $X\cap(Y-H)$ is in **H**. If $X\cap H$ is in **H**, then **H** is in **B**. If $X\cap(Y-H)\in\mathbf{H}$, then $Y-H\in\mathbf{B}$. Hence **B** is an ultrafilter. Thus **H** is an ultrafilter. Thus **H** is an ultrafilter base on Y.

Canonical way of converging nets to filters and vice-versa

Example 7. . As we have seen that any net $P : \Lambda\to X$ determines a filter having the family of sets of the form $\{P(\lambda) ; \lambda\in\wedge, \lambda \geq \lambda_0 \text{ for } \lambda_0\in\wedge\}$ as a base. This filter is known as filter associated with the net P (or filter generated by $P(\lambda)$) It may of course happen that two distinct nets determine the same filter.

Conversely given a filter **H** on X, There is a net associated with it as follows : Let $\wedge = \{(x, F) \in X\times\mathbf{H}; x \in F\}$ for $(x, F), (y, G) \in \wedge$. Define $(x, F) \geq (y, G)$ if $F \subset G$.

It is easily seen that $\geq$ directs $\wedge$ because **H** is closed under finite intersection. Now define $P : \wedge\to X$ by $P(x, F) = x$. Then P is a net in X. It is called the net associated with the filter **H** (or the net based on **H**)

Limits and cluster points are preserved in switching over from nets to filters and vice-versa. To show it we have the following two theorems.

Theorem 27 . Let $P : \wedge\to X$ be a net and **H** the filter associated with it. Let $x\in X$. Then P converges to x as a net iff **H** converges to x as a filter. Also x is a cluster point of the net P iff x is a cluster point of the filter **H**.

Proof. Suppose P converges to x. Let U be a neighbourhood of x in X. Then $\exists\ \lambda_0\in\wedge$ such that $B_{\lambda_0} = \{P(\lambda); \lambda\in\wedge; \lambda \geq \lambda_0\}$. But this means $U\in\mathbf{H}$ by the definition of **H**. So every neighbourhood of x belongs to **H**. Hence **H** converges to x.

Conversely suppose that **H** converges to x. Let U be as open neighbourhood of x. Then $U\in\mathbf{H}$. Recalling how **H** was generated, there exists $\lambda_0\in\wedge$ such that $B_{\lambda_0}\subset U$ where B_{λ_0} is defined as above. This means that $P(\lambda)\in U$ for all $\lambda\in\wedge ; \lambda \geq \lambda_0$. Thus P converges to x in X.

Suppose now that the net P clusters at x. Therefore for each neighbourhood U of x and for each $\lambda_0\in\wedge$, there is some $\lambda \geq \lambda_0$ such that $P(\lambda)\in U$. But the sets $\{P(\lambda) ; \lambda\in\wedge ; \lambda \geq \lambda_0\}$ form a base for **H**. Let $F\in\mathbf{H}$. Then

$$F \supset \{P(\lambda); \lambda\in\wedge; \lambda \geq \lambda_0\}$$

i.e. $$P(\lambda)\in F. \text{ Thus } P(\lambda)\in F\cap U.$$

Hence every neighbourhood of x intersects every member of **H**. Hence x is a cluster point of **H**.

Conversely suppose that x is a cluster point of **H** i.e. every neighbourhood U of x intersects every member F of **H**. Thus $F \cap U \neq \phi$. But $F \supset\{P(\lambda); \lambda \in\wedge ; \lambda \geq \lambda_0\}$ for some λ_0 Being a superset of $\{P(\lambda); \lambda \in\wedge ; \lambda \geq \lambda_0\}$ let belongs to **H**. Therefore $P(\lambda)\in U$ for $\lambda \geq \lambda_0$.

Theorem 28. Let **H** be a filter on a space X and P be the associated net in X. Let $x \in X$. Then **H** converges to x as a filter iff P converges to x as a net. Moreover x is a cluster point of the filter **H** iff it is a cluster point of the net **P**.

Proof. Suppose $\mathbf{H} \to x$. If U is a neighbourhood of x, then $U \in \mathbf{H}$. Pick $p \in U$. Then $(p, U) \in \wedge_{\mathbf{H}}$ and if $(q, F) \geq (p, U)$; Then $q \in F \subset U$. Thus for each neighbourhood U of x, there exists $(p, U) \in \wedge_{\mathbf{H}}$ such that $(q, F) \geq (p, U)$ implies $P(q, F) = q \in U$. Hence the net P converges to x.

Conversely, suppose that the net based on **H** converges to x. Let U be a neighbourhood of x. Then for some $(p_0, F_0) \in \wedge_{\mathbf{H}}$ we have $(p, F) \geq (p_0, F_0)$ implies $P(p, F) = p \in U$. But then $F_0 \subset U$, Otherwise there is some $q \in F_0 - U$ and then $(q, F_0) \geq (p_0, F_0)$ but $q \notin U$. Hence being a super set of the members (F_0) of **H**, belongs to **H**. Hence every neighbourhood U of x belongs to **H**. Hence $\mathbf{H} \to x$.

Now we come to the result concerning cluster points.

Suppose first **H** has x as a cluster point. Recall that the associated net $P : \wedge \to X$ is defined by taking $\wedge = \{(y, F) ; F \in \mathbf{H}, y \in F\}$ and putting $P(y, F) = y$. Let an open neighbourhood U of x and an element (y, F) of $\wedge$ be given. Then $F \cap U \neq \phi$ by definition of the cluster point of a filter. Let $z \in F \cap U$. Then $(z, F) \in \wedge$, $(z, F) \geq (y, F)$ and $(P(z, F) = z \in U$.

Thus to each neighbourhood U of x and $(y, F) \in \wedge$, there is $(z, F) \in \wedge$ such that $(z, F) \geq (y, F)$ implies $P(z, F) \in U$. Hence x is a cluster point of P.

Conversely suppose that x is a cluster point of P. Let U be any open neighbourhood of x and let $F \in \mathbf{H}$. We have to show that $F \cap U \neq \phi$ Let z be any point of F. Then $(z, F) \in \wedge$. Since x is a clustr point of P, there exists $(y, G) \in \wedge$ such that $(y, G) \geq (z, F)$ and $P(y, G) \in U$. But then $y \in G$, $G \subset F$ and $y \in U$ ($\because$ $P(y, G) = y$) showing that $y \in F \cap U$ and so $F \cap U \neq \phi$. Hence every neighbourhood of x intersects every member of **H** i.e. x is a cluster point of **H**.

Theorem 29. A top. Space is Hausdorff iff no filter can converge to more than one point in it.

Proof. Suppose X is a Hausdorff space and a filter **H** converges to x as well as y. This means $\mu_x \subset \mathbf{H}$ and $\mu_y \subset \mathbf{H}$. Now if $x \neq y$, then there exist $U \in \mu_x$ and $V \in \mu_y$ such that $U \cap V \neq \phi$ which contradicts the fact that **H** has the finite intersection property. So x = y. Thus limits of convergent filters in X are unique.

Conversely, suppose that no filter in X has more than one limit in X. If X is not Hausdorff, there exists $x, y \in X$, $x \neq y$ such that every neighbourhood of x intersect every neighbourhood of y. From this it follows that the family $\mu_x \cap \mu_y$ has finite intersection property. Therefore $\exists$ a filter **H** on X containing $\mu_x \cap \mu_y$. Evidently **H** converges both to x and y contradicting the hypothesis. So X is Hausdorff.

Theorem 30. For a topological space X, the following statements are equivalent.

(1) X is compact
(2) Every filter on X has a cluster point in X.
(3) Every filter on x has a convergent subfilter.

Proof. (ii) $\Leftrightarrow$ (iii) has been shown in the defn of subfilter.

Moreover, if a filter **H** has a cluster point x in X, Then the net based on **H** has also cluster point x in X. Therefore (i) $\Leftrightarrow$ (ii) follows from a result already proved for nets.

Theorem 31. A. top. space is compact iff every ultrafilter in it is convergent.

Proof. If a space is compact. Then every filter in it has a cluster point. In particular every ultrafilter has a cluster point and hence is convergent by a result (proved already) i.e. An ultrafilter converges to a point iff that point is a cluster point of it.

Conversely suppose X is a space with the property that every ultrafilter on it is convergent. It is sufficient to show that every filter on X has a cluster point because then the result will follow from above Theorem. Suppose **H** is a filter on X. Therefore ∃ an ultrafilter **G** containing **H**. By hypothesis, y converges to a point, say x on X. Then x is a cluster point of G. So every neighbourhood of x meets every member of **G** and in particular every member of **H** since **H** ⊂ **G**. So x is also a cluster point of **H**. Thus every filter on X has a cluster point in X. Hence X is compact.

Now we are in a position to prove the following theorem concerning the characterization of compact sets.

Theorem 32. For a top. space X, the following are equivalent

(a) X is compact

(b) Each family C of closed sets in X with the finite intersection property has non-empty intersection.

(c) Each filter in X has a cluster point.

(d) Each net in X has a cluster point.

(e) Each ultranet in X converges.

(f) Each ultrafilter in X converges.

Proof. (a) ⇒ (b) let X be compact. Suppose on the contrary that $\{F_\lambda\}$ is a family of closed sets in X having empty intersection i.e.

$$\bigcap_\lambda F_\lambda = \phi$$

$$\Rightarrow \qquad \left(\bigcap_\lambda F_\lambda\right)^C = \phi^C$$

$$\Rightarrow \qquad \bigcup_\lambda F_\lambda^C = X$$

Thus $\{F_\lambda{}^C\}$ is a converging of X. Then by compactness of X, there must be a finite subcovering of X i.e.

$$X = \bigcup_{i=1}^{n} F_i{}^C$$

But then $\phi = X^C = \left(\bigcup_{i=1}^{n} F_i{}^C\right)^C = \bigcap_{i=1}^{n} F_i$

so that the family can not have the finite intersection property. This contradiction proves that (b) holds.

(b) ⇒ (c). If **H** is a filter on X, Then $\{\ \overline{F}\ ;\ F \in$ **H**$\}$ is a family of closed sets with the finite intersection property, so (b) implies that there is a point x in $\cap\ \{\ \overline{F}\ ;\ F \in$ **H**$\}$ i.e. each neighbourhood of x intersects every members of **H**. Hence x is a cluster point of the filter **H**.

(c) ⇒ (d). Suppose that the filter **H** has x as a cluster point. We know that net $P : \wedge \to X$ based on **H** is defined by taking

$\wedge = \{(y, F)\ ;\ F \in$ **H** $;\ y \in F\}$ and putting

$P(y, F) = y$. Let an open neighbourhood U of x and an element (y, F) of $\wedge$ be given. Then $F \cap U \neq \phi$ because x is a cluster point iff every neighbourhood U of x meets every member F of **H**.

Let $z \in F \cap U$. Then $(z, F) \in \wedge$;
$(z, F) \geq (y, F)$ and $P(z, F) = z \in U$.

Thus to each neighbourhood U of x and $(y, F) \in \wedge$ there is $(z, F) \in \wedge$ such that $(z, F) \geq (y, F)$ implies $P(z, F) \in U$. Hence x is a cluster point of the net P.

(d) $\Rightarrow$ (c). If an ultranet has a cluster point then it converges to that point.

(e) $\Rightarrow$ (f). Let **H** be an ultrafilter on X. Then the neighbourhood on **H** is an ultranet which converges by (e). But the limits are preserved in satiating over form ultranets to ultrafilters. Hence **H** is convergent.

(f) $\Rightarrow$ (a) Suppose μ is an open cover of X with no finite subcover. Then $X - (U_1 \cup U_2 \ldots, \cup U_n) \neq \phi$ for each finite collection $\{U_1, \ldots, U_n\}$ from μ the sets of the form $X-(U_1 \cup U_2 \cup \ldots \cup U_n)$ Then form a filter base (since the intersection of two such sets has again the same form) generating a filter **H**. Now since each filter is contained in an ultrafilter. It follows that **H** is contained in some ultrafilter **H***. But **H*** converges to x by (*f*). Now $x \in U$ for some $U \in \mu$. Since U is a neighbourhood of x, by the defn of convergence ultrafilter, we have $U \in$ **H***. By the construction, $X - U \in$ **H** $\subset$ **H***. Since it is impossible for both U and X−U to belong to an ultrafilter, we have a contradiction. Thus μ must have a finite subcover and hence X is compact.

The Stone-Cech Compactification

Definition. Let (X, T) be a topological space. A compact space X* is said to be a compactification of X if X is homeomorphic to a dense subset of X*.

One important application of the Tychonoff product theorem is the construction of a Hausdorff compactification of any Tichnov space.

Theorem 33. (Stone-Cech Compactification Theorem)

Every Tichonov space X is homeomorphic to a dense subset $\hat{X}$ of a compact Hausdorff space βX which has the property that for every bounded, continuous, real valued mapping f defined on $\hat{X}$. There is a continuous extension to βX, that is a continuous mapping f^{β} of βX into **R** such that

$$f^{\beta} / \hat{X} = f.$$

Proof. Let $\{f_{\lambda}\}$ be the collection of all real-valued bounded continuous functions defined on X . Since these functions are bounded, we may let I_{λ} be a closed interval of real number containing the range of f_{λ}. Then by Tychonoff Product Theorem $\Pi_{\lambda} I_{\lambda}$ is compact and also it is Hausdorff since the product space, each of whose coordinate space is Hausdroff is Hausdorff. Define a mapping h of X into $\Pi_{\lambda} I_{\lambda}$ by setting

$$h(x) = < f_{\lambda}(x) >$$

for every $x \in X$. We will denote by h(X), which is the range of X, by $\hat{X}$ and its closure C(h(X)) in $\Pi_{\lambda} I_{\lambda}$ by βX. Since βX is a closed subset of a compact Hausdorff space, we have immediately that $\hat{X}$ is a dense subset of the compact Hausdorff space βX. We assert that h is a homeomorphism between X and $\hat{X}$.

The mapping h is clearly continuous $\pi_{\lambda} o h = f_{\lambda}$ which is continuous. If x and y are two distinct points of X, then there must exist some index λ such that $f_{\lambda}(x) = 0 \neq 1 = f_{\lambda}(y)$ sine X is a Tichonov space. From this it follows that $h(x) \neq h(y)$ since their λ-th coordinates differ. Finally, let us suppose that G is an open subset of X and we will show that h(a) is open in $\hat{X}$. Fix any point $x \in G$, and since X is a Tichonov space, there must be an index λ such that $f_\lambda(x) = 0$ and $f_\lambda(X - G) = \{1\}$. Clearly $\hat{X} \cap \pi_\lambda ((-\infty, 1))$ is an open subset of $\hat{X}$ containing h(x).

Furthermore, if $X \in \hat{X} \cap \pi_\lambda^{-1} ((-\infty, 1))$, then $h^{-1}(X) \subseteq G$, so that $X \in h(G)$. Thus $\hat{X} \cap \pi_\lambda^{-1} ((-\infty, 1)) \subseteq h(G)$, so that h(G) is open in $\hat{X}$.

Lastly, suppose f is a bounded, continuous, real valued mapping defined on $\hat{X}$. We must have $foh = f_{\lambda_0}$ for some index λ_0 . Let f^β be the mapping of βX into I_{λ_0} defined by setting $f^\beta (< x_\lambda >) = \pi_{\lambda_0} (< x_\lambda >) = \pi_{\lambda_0}$.

Since the projections are continuous, f^β is certainly continuous. Now $X = < f_\lambda(x) >$, then $f^\beta (X) = f_{\lambda_0}(x)$, $[f^\beta (X) = f^\beta(< f_\lambda(x) >) = \pi_{\lambda_0} (f_\lambda(x) = f_{\lambda_0}(x)]$

while $f(x) = f(< f_\lambda(x) >) = f(h(x)) = (foh)(x) = f_{\lambda_0}(x)$,

so that $f^\beta / \hat{X} = f$.

9

THE FUNDAMENTAL GROUP AND COVERING SPACES

The basic strategy of the entire subject of algebraic topology is to find methods of reducing topological problems to questions of pure algebra.

We will define a group called the fundamental group of X for any topological space X by a very simple and intuitive procedure involving the use of closed paths in X. From the definition, it will be clear that the group is a topology invariant of X i.e. if two spaces are homeomorphic, their fundamental groups are isomorphic. This gives us the possibility of proving that two spaces are not homeomorphic by proving that their fundamental groups are non-isomorphic. For example, this method is sufficient to distinguish between the various compact surfaces and in many other cases.

Not only does the fundamental group give information about spaces, but it is also useful in studying continuous maps, As we shall see, any continuous map from a space X into a space Y induces a homeomorphism of the fundamental group of X into that of Y. Certain topological properties of the continuous map will be reflected in the properties of this induced homomorphism. Thus we can prove facts about certain continuous map by studying the induced homomorphism of fundamental groups.

Thus by using the fundamental group, topological problems about space and continuous maps can sometimes be reduced to purely algebraic problems about group and homomorphisms.

Definition. A path or arc in a topological space X is a continuous map f of some closed interval into X. we generally take the closed interval [0, 1]. The images $f(0)$ and $f(1)$ of the endpoints of the interval [0, 1] are called the end points of the path or arc and the path is said to join its end points. One of the end points is called the initial point and the other is called the terminating point or the final point.

Two paths f_1 and f_2 are by definition different unless

$$f_1(x) = f_2(x) \text{ for every } x \in (0, 1).$$

Definition. The set of points in X on which (0, 1) is mapped by f is the track and is denoted by $|f|$.

Definition. A space X is called arc wise connected or path wise connected if any two points of X can be joined by an arc. An arcwise connected space is connected, but the converse statement is not true.

Definition. If the initial point of the path f_2 is the final path f_1, then $f_1 + f_2$ is the path f_0 where

$$f_0(x) = f_1(2x) \text{ for } 0 \leq x \leq \frac{1}{2}$$

$$= f_2(2x-1) \text{ for } \frac{1}{2} \leq x \leq 1.$$

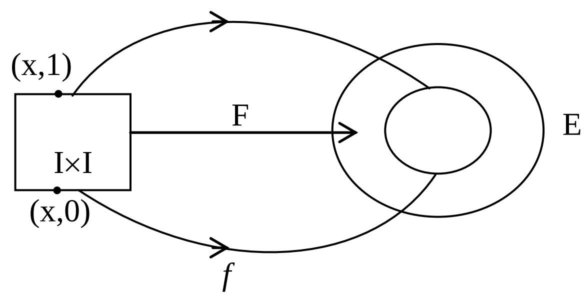
Definition. A path whose end points coincide is called a loop. Thus a path f is a loop if $f(0) = f(1)$

Homotopy of Paths

Definition. Let f and g be two paths on a topological space E. Then f and g are said to be homotopic if there exists a continuous mapping F : $I^2 \to E$ called homotopy between f and g such that

$$F(x, 0) = f(x) \text{ for } x \in I$$

$$F(x, 1) = g(x)$$



We now generalize the above definition simply by replacing the first factor in the product $I^2 = I \times I$ by any topological space A.

Definition. Let A and E be any two topological spaces and let f g be two continuous mappings of A into E. Then f and g will be said to be homotopic if there is a continuous mapping F of $A \times I$ into E such that

$$F(x, 0) = f(x)$$

$$F(x, 1) = g(x) \quad \forall\ x \in A$$

Clearly this definition reduces to the former definition if A is replaced by I.

Definition. Let f and g be two paths on a topological space E joining the points x and y i.e. f and g are continuous mapping of I into E such that $f(0) = g(0) = x$ and $f(1) = g(1) = y$ (or we can say that f and g have the same initial and terminal point) Then f and g are said to be homotopic with the fixed end points x and y if there is a continuous mapping F : $I^2 \to E$ such that

$$\left.\begin{array}{l} F(a,0) = f(a) \\ F(a,1) = g(a) \end{array}\right] \forall a \in I$$

$$\left.\begin{array}{l} F(0,b) = x \\ F(1,b) = y \end{array}\right] \forall b \in I$$

The special case of the deformation of a closed loop is covered by taking x = y explicitly.

Definition. Let f and g be two closed paths on a topological space E both beginning and ending at a point x of E. The f and g are said to be homotopic with respect to the fixed base point x if there is a continuous mapping F : $I^2 \to E$ such that

$$\left.\begin{array}{l} F(a,0) = f(a) \\ F(a,1) = g(a) \end{array}\right] \forall a \in I$$

and $$F(0, b) = F(1, b) = x \ \forall\ b \in I$$

Definition. A closed path f on a topological space E beginning and ending at x will be said to be shrinkable to x or to be homotopic to a constant w.r.t to the base point x if f is homotopic to the mapping e : I→E defined by e(a) = x for all a∈I with respect to the base point x.

Thus shrinking a path f to be point x amounts to constructing a continuous mapping F of I^2 into E whose restriction to the lower side (b= 0) gives the path f, while the other three sides are carried into x.

Definition. A topological space E will be said to be simply connected w.r. to the base point x if every closed path in E beginning and ending at x is shrinkable to x.

Theorem 1. The relation of homotopy w.r.t to x of paths based on x is an equivalence relation on the set of all paths in E based on x.

Proof. We must show that

(i) $f \simeq f$ for every path f based on x (reflexivity)

(ii) $f \simeq f$ implies $g \simeq f$ for every pair of paths based on x (symmetry)

(iii) If f, g, h are paths based on x such that $f \simeq g$ and $g \simeq h$, then $f \simeq h$ (transitivity).

(i) To prove that $f \simeq f$, define the mapping F : I^2→E by setting

$$F(s, t) = f(s).$$

Then F is clearly a continuous mapping of I^2 into E. Setting t = 0 and t = 1, we obtain

$$F(s, 0) = f(s)$$
$$F(s, 1) = f(s)$$

and setting s = 0 and s = 1, we obtain

$$F(0, t) = f(0) = x \text{ (initial point)}$$
$$F(1, t) = f(1) = x \text{ (Final point)}$$

Hence $f \simeq f$.

(ii) Let f and g be given such that $f \simeq g$. Then there is a continuous mapping F : I^2–E such that

$$F(s, 0) = f(s),\ F(s, 1) = g(s)$$
$$F(0, t) = F'(1, t) = x.$$

We define the mapping

$$F^* : I^2 \to E$$

by setting F*(s, t) = F(s, 1–t)

$$F^*(s, 0) = F(s, 1) = g(s)$$
$$F^*(s, 1) = F(s, 0) = f(s)$$
$$F^*(0, t) = F(0, 1-t) = x$$
$$F^*(1, t) = F(1, 1-t) = x$$

which proves that $g \simeq f$.

(iii) Let f, g, h be given such that $f \simeq g$ and $g \simeq h$. Therefore there exists continuous mapping F′ and F″ of I^2 into E satisfying the following :

$$\left.\begin{aligned} F'(s, 0) &= f(s) && (1)\\ F'(s, 1) &= g(s) && (2)\\ F''(s, 0) &= g(s) && (3)\\ F''(s, 1) &= h(s) && (4)\end{aligned}\right\} \text{ for } s\in I$$

$$F'(0, t) = F'(1, t) = F''(0, t) = F''(1, t) = x \text{ for } t\in I \quad (5)$$

Define F : I^2→E such that

$$F(s, t) = F'(s, 2t) \text{ if } 0 \le t \le \frac{1}{2}$$

$$F(s, t) = F''(s, 2t-1) \text{ if } \frac{1}{2} \leq t \leq 1$$

The first thing to check is that F is infact properly defined as mapping, for both parts of its definition apply when $t = \frac{1}{2}$ and they might contradict one another. Such a contradiction does not however arise for by (2) and (3),

$F'(s, 1) = F''(s, 0) = g(s)$ and so $F\left(s, \frac{1}{2}\right) = g(s)$ by both parts of the definition

The next thing to verify is that F is continuous. It is clear that if $t \neq \frac{1}{2}$, F is continuous which follows directly from the continuity of F′ or F″ according as $t < \frac{1}{2}$ or $t > \frac{1}{2}$. Suppose now that U is a nbd in E of $F\left(s, \frac{1}{2}\right)$ for some $s \in I$. Then the continuity of F′ implies that there is a number ξ' such that if $|s_1 - s| < \xi'$ and $|2t_1 - | < 2\xi'$, $F'(s_1, 2t_1)$ will lie in the nbd U of $F'(s, 1) = F\left(s, \frac{1}{2}\right)$. Similarly the continuity of F″ implies that there is a number ξ'' such that if $|s_1 - s| < \xi''$ and $|(2t_1-1) - 0| < 2\xi''$ then $F''(s_1, 2t_1-1)$ will lie in the nbd U of $F''(s, 0) = F\left(s, \frac{1}{2}\right)$. The inequalities imposed on s_1 and t_1 in the last two statements are all satisfied if $|s_1-s| < \xi$ and $\left|t_1 - \frac{1}{2}\right| < \xi$, where $\xi = \min(\xi', \xi'')$. Thus these two statements combine to show that if $|s_1-s|<\xi$ and $\left|t_1 - \frac{1}{2}\right|<\xi$, then $F(s_1, t_1) \in U$. The continuity of F is thus proved for points at which $t = \frac{1}{2}$. F has thus been shown to be a continuous mapping of I^2 into E. It remains to show that it gives the required homotopy of f and h. For t = 0,

$$F(s, 0) = F'(s, 0) = f(s) \qquad \text{by (1)}$$

and for t = 1

$$F(s, 1) = F''(s, 1) = h(s) \qquad \text{by (4)}$$

Also $\quad F(0, t) = F'(0, 2t) = x \qquad$ by (5)

where $0 \leq t \leq \frac{1}{2}$

$$F(0, t) = F''(0, 2t-1) \text{ when } \frac{1}{2} \leq t \leq 1.$$
$$= x \qquad \text{by (5)}$$

Similarly

$$F(1, t) = F'(1, 2t) \text{ if } 0 \leq t \leq \frac{1}{2}$$
$$= x \qquad \text{by (5)}$$
$$F(1, t) = F''(1, 2t-1) \text{ if } \frac{1}{2} \leq t \leq 1$$
$$= x \qquad \text{by (5).}$$

and so all the verifications are complete and $f \simeq h$.

Remark. Since homotopy w.r to a base point x is an equivalence relation, all the paths based on x are divided into equivalence classes.

Definition. The equivalence classes of paths based on $x \in E$ corresponding to the relation of homotopy with respect to the base point x will be called homotopy classes of paths on E w.r. to the base point x.

Definition. Let f and g be two closed paths on a topological space E based on a point x. Then the symbol fg, called the product of f and g is the mapping of I into E such that

$$(fg)(s) = f(2s) \qquad \text{for } 0 \le s \le \frac{1}{2}$$

$$(fg)(s) = g(2s-1) \qquad \text{for } \frac{1}{2} \le s \le 1.$$

Remark :- We claim that f g so defined is a path.

In $0 \le s < \frac{1}{2}$, f g is continuous because of continuity of f. In $\frac{1}{2} < s \le 1$, fg is continuous because of continuity of g. So $s = \frac{1}{2}$ is the only point which needs attention. Let $s = \frac{1}{2}$ and let U be a nbd in E of f(g)(s). Then due to the continuity of f at $s = \frac{1}{2}$, there exists δ_1 such that if

$$|2s' - 2s| < 2\delta_1$$

i.e. $$|s' - s| < \delta_1$$

$\Rightarrow$ $$|s - \frac{1}{2}| < \delta_1 \qquad \left(s = \frac{1}{2}\right)$$

then $$f(2s') \in U$$

Similarly due to the continuity of g at $s = \frac{1}{2}$ we have if

$$|2s' - 1 - (2s-1)| < 2\delta_2$$

i.e. $$|s' - \frac{1}{2}| < \delta_2 \qquad \left(\text{putting } s = \frac{1}{2}\right)$$

then $$g(2s'-1) \in U.$$

let $\delta = \min(\delta_1, \delta_2)$. Then if

$$|s' - \frac{1}{2}| < \delta$$

then $f(2s')$ and $g(2s'-1)$ both belong to U and therefore fg is continuous at $\frac{1}{2}$. Thus $fg : I \to E$ is a continuous mapping.

Moreover $$fg(0) = f(0) = x$$
$$fg(1) = g(1) = x$$

So the product of two paths based on x is again a path based on x.

We shall now show that the homotopy class of fg depends only on those of f and g.

Theorem 2. If f, f', g, g' are paths in a topological space E based on a point x and if

$$f \simeq f', \; g \simeq g', \text{ then } fg \simeq f'g'.$$

Proof. Since $f \simeq f'$ and $g \simeq g'$, there are continuous mapping F and G of I^2 into E such that

$$F(s, 0) = f(s) \qquad (1)$$

$$F(s, 1) = f'(s) \qquad (2)$$

$$G(s, 0) = g(s) \qquad (3)$$
$$G(s, 1) = g'(s) \qquad (4)$$
$$F(0, t) = F(1, t) = G(0, t) = G(1, t) = x \qquad (5)$$

Define $H : I^2 \to E$ by setting

$$H(s, t) = F(2s, t),\ 0 \le s \le \frac{1}{2}$$
$$H(s, t) = G(2s-1, t)\ \frac{1}{2} \le s \le 1.$$

we observe that this definition is not self contradictory for $s = \frac{1}{2}$, because it follows from the first part of the definition that

$$H(\frac{1}{2}, t) = F(1, t) = x \text{ by (5)}$$

and by the second part of the definition that

$$H(\frac{1}{2}, t) = G(0, t) = x \text{ by (5)}$$

Next the continuity of H must be shown. Only the points with $s = \frac{1}{2}$ require attention. Let U be a nbd of $H(\frac{1}{2}, t)$ for some fixed t. Since F is continuous, there exists $\delta_1 > 0$ such that if

$$|2s' - 2s| < \delta_1 \text{ and}$$
$$|t' - t| < \delta_1$$

i.e. if $|s' - s| < \delta_1$ and $|t' - t| < \delta_1$, then

$$F(2s', t') \in U$$

Similarly G is continuous and so $\exists\ \delta_2$ such that if

$$|(2s'-1) - (2s-1)| < 2\delta_2$$

and $$|t' - t| < \delta_2$$

i.e. if $|s' - s| < \delta_2$ and $|t' - t| < \delta_2$, then

$$G(2s' - 1, t') \in U$$

Let $\delta = \min(\delta_1, \delta_2)$, then if

$$|s' - \frac{1}{2}| < \text{ and } |t' - t| < \delta$$

Then we have

$$F(2s', t') \in U \text{ and } g(2s' - 1, t') \in U.$$

Hence H(s, t) is continuous.
Moreover

$$H(s, 0) = F(2s, 0) = f(2s), \text{ for } 0 \le s \le \frac{1}{2}$$
$$H(s, 0) = G(2s-1, 0) = g(2s-1) \text{ for } \frac{1}{2} \le s \le 1.$$

If this is compared with the definition of *f*g, we see that H(s, 0) = *f*g(s).
Similarly

$$H(s, 1) = F(2s, 1) = f'(2s), \text{ for } 0 \le s \le \frac{1}{2}$$

$$H(s, 1) = G(2s-1, 1) = g'(2s-1) \text{ for } \frac{1}{2} \le s \le 1.$$

Comparing this with the definition of $f'\,g'$, we see that

$$H(s, 1) = (f'\,g')(s)$$

Also
$$\left.\begin{array}{l} H(0, t) = F(0, t) = x \\ H(1, t) = G(1, t) = x \end{array}\right\} \quad \text{by (5)}$$

Thus all the necessary conditions on H have been verified and it has been proved as required that $fg \simeq f'\,g'$.

Definition. Let $\bar{f}$ and $\bar{g}$ be two homotopy classes of paths based on x w.r. to the base point x and let f be a path belonging to $\bar{f}$, g a path belonging to $\bar{g}$. The product $\bar{f}\ \bar{g}$ of $\bar{f}$ and $\bar{g}$ is defined to be the homotopy class to which the path fg belongs.

By the above theorem, this defines a homotopy class depending on the classes $\bar{f}$ and $\bar{g}$ and not on the representatives f and g. For if f' and g′ are two more paths belonging to $\bar{f}$ and $\bar{g}$ respectively $fg \simeq f'g'$ and so fg and $f'g'$ belong to the same homotopy class.

Theorem 3. Let E be a topological space and x is a point of E. Then the homotopy classes w.r.t the base point x of paths based on x are the elements of a group having the product just defined as group operation

Proof. The proof of this theorem consists of three parts (i) to show that the product operator between homotopy classes is associative (ii) to prove the existence of an identity element (iii) to prove that each homotopy class has an inverse

(i) let f, g, h be three paths based on x and $\bar{f}$, $\bar{g}$, $\bar{h}$ their homotopy classes. We have to show that $(\bar{f}\ \bar{g})\ \bar{h} = \bar{f}(\bar{g}\ \bar{h})$.

It is sufficient to show that

$$(fg)h \simeq f(gh)$$

By definition, fg is a continuous mapping of I into E such that

$$(fg)(s) = f(2s) \quad 0 \le s \le \frac{1}{2}$$

$$(fg)(s) = g(2s-1) \quad \frac{1}{2} \le s \le 1.$$

Applying the definition again to the product of fg and h, it turns out that

$$((fg)h)(s) = fg(2s) \quad 0 \le s \le \frac{1}{2}$$

$$((fg)h)(s) = h(2s-1) \quad \frac{1}{2} \le s \le 1$$

Combining these statements, it follows that (fg)h is a continuous mapping of I into E such that

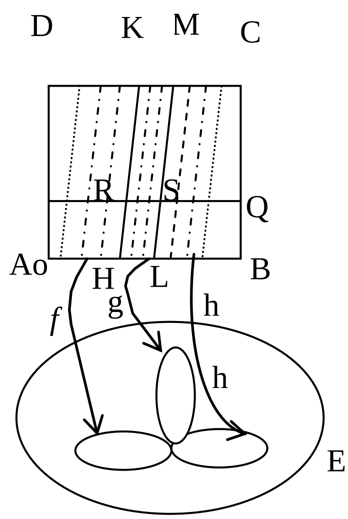
$$\left.\begin{array}{l} ((fg)h)(s) = f(4s), \quad 0 \le s \le \frac{1}{4} \\ ((fg)h)(s) = g(4s-1), \ \frac{1}{4} \le s \le \frac{1}{2} \\ ((fg)h)(s) = h(2s-1), \ \frac{1}{2} \le s \le 1 \end{array}\right\} \quad (1)$$

Similarly f(gh) is a continuous mapping of I into E such that

$$\left.\begin{aligned}(f(gh))(s) &= f(2s) \quad 0 \le s \le \frac{1}{2} \\ (f(gh))(s) &= g(4s-2) \quad \frac{1}{2} \le s \le \frac{3}{4} \\ (f(gh))(s) &= h\,(4s-3) \quad \frac{3}{4} \le s \le 1\end{aligned}\right\} \qquad (2)$$

Now if equations (1) are examined, it will be noticed that I is divided into three parts of lengths $\frac{1}{4}, \frac{1}{4}, \frac{1}{2}$ and (fg) h is constructed by applying f to the first of these subintervals, g to the second and h to the third, the appropriate change of scale being made in each case. For example the first interval is of length $\frac{1}{4}$, and (f g)h is identical with f with the scale increased by 4; the scale increase is expressed by taking 4s as the argument of f. Similarly the scale is increased by 4 in the second interval, 4s–1 being a variable going from 0 to 1 as s goes from $\frac{1}{4}$ to $\frac{1}{2}$, while in the third interval, which is of length, $\frac{1}{2}$, the scale is increased by 2. f(gh) is constructed in an exactly similar manner, except that the subdivisions are in this case of length $\frac{1}{2}, \frac{1}{4}, \frac{1}{4}$.

The idea now is to make a continuous transition from (fg)h to f(gh) by changing the three subintervals of length $\frac{1}{4}, \frac{1}{4}, \frac{1}{2}$ into those of lengths $\frac{1}{2}, \frac{1}{4}, \frac{1}{4}$ simply by stretching the first, compressing the third and sliding the second sideways as indicated for example by the arrows in the figure.



Now ABCD in the Fig can be taken as I^2. PQ representing stage t of the transition from the subdivision $\frac{1}{4}, \frac{1}{4}, \frac{1}{2}$ to $\frac{1}{2}, \frac{1}{4}, \frac{1}{4}$ is divided into intervals of lengths $\frac{1}{4}(1+t)$, $\frac{1}{4}$, $\frac{1}{4}(2-t)$. Just as (fg)h and f(gh) are defined on AB and DC, a mapping of PQ into E will be constructed by applying f, g, h to the three intervals PR, PS, SQ with suitable changes of scale. The arguments of f, g, h being chosen so that they vary from 0 to 1 as s varies along the corresponding subinterval. This mapping which will be called F(s, t) will be equal to $f\left(\frac{4s}{1+t}\right)$ on PR, to g(4s–t–1) on RS and to $h\left(\frac{4s-t-2}{2-t}\right)$ on SQ. Thus we consider the function F : I×I→X defined by

$$F(s, t) = \begin{cases} f\left(\frac{4s}{1+t}\right) & 0 \le s \le \frac{t+1}{4} \\ g(4s-t-1) & \frac{t+1}{4} \le s \le \frac{t+2}{4} \\ h\left(1-\frac{4(1-s)}{2-t}\right) & \frac{t+2}{4} \le s \le 1 \end{cases}$$

Then F is continuous. The continuity follows at once from the continuity of *f*, g and h. Also

$$F(s, 0) = [(f\,g)h](s)$$
$$F(s, 1) = [f(g\,h)](s)$$

that is F coincides with (*f*g)h on the lower side of I^2 and with *f*(gh) on the upper side. Moreover

$$F(0, t) = F(1, t) = x \text{ for all t.}$$

Hence F is the required homotopy of (*f*g)h and *f*(gh).

(ii) We shall use the symbol e to denote a constant mapping of I into E defined by

$$e(s) = x \ \forall\ s \in I.$$

The second part of the theorem will be proved by showing that the homotopy class $\bar{e}$ of the constant mapping e acts as identity. That is to say it will be shown that for any path *f* based on x,

$$ef \simeq f\,e \simeq f$$

By definition

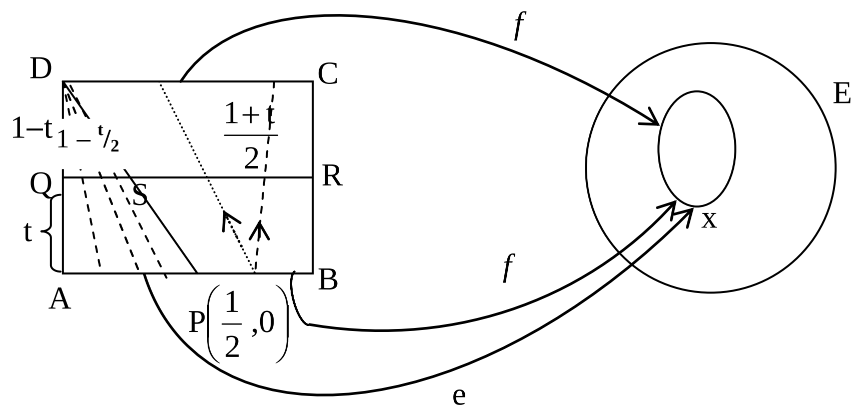
$$(ef)\,(s) = e(2s),\ 0 \le s \le \frac{1}{2}$$
$$= f(2s-1),\ \frac{1}{2} \le s \le 1.$$

that is

$$(ef)s = e(2s) = x \quad 0 \le s \le \frac{1}{2}$$
$$= f(2s-1),\ \frac{1}{2} \le s \le 1.$$

Thus I is divided in half, the first half being mapped on x, while *f* is applied to the second half with the scale doubled.

The idea now is to shrink steadily the part mapped on x while extending the part to which *f* is mapped. In the Fig below, ABCD is I^2, AB s bisected at P and PD is joined. The horizontal line QR at height t is divided into intervals of lengths $\frac{1}{2}(1-t)$ at $\frac{1}{2}(1+t)$, and as t varies from 0 to 1, it is clear that these lengths change continuously from $\frac{1}{2}$, $\frac{1}{2}$ (i.e. t = 0 in $\frac{1}{2}$ (1–t) and $\frac{1}{2}$(1+t)) to 0, 1(t = 1 in $\left(\frac{1-t}{2} \text{ and } \frac{1+t}{2}\right)$)



$$\frac{DQ}{QS} = \frac{DA}{AP} \Rightarrow \frac{DQ}{DA} = \frac{QS}{AP}$$

$$\Rightarrow \quad \frac{1-t}{1} = \frac{QS}{\frac{1}{2}}$$

$$\Rightarrow \quad QS = \frac{1-t}{2}$$

$$\Rightarrow \quad SR = 1 - \frac{1-t}{2} = \frac{2-1+t}{2} = \frac{1+t}{2}$$

(since QR = 1)

The continuous transition from e*f* applied to AB into *f* applied to DC should be possible by making QR, at stage t into E′ in such a way that QS is mapped on x and *f* is applied to SR with the appropriate change of scale.

Define F : I×I →E by

$$F(s, t) = e\left(\frac{2s}{1-t}\right) \qquad 0 \le s \le \frac{1-t}{2}$$

$$= x \qquad 0 \le s \le \frac{1-t}{2} \text{ s by definition } e(s) = x$$

$$F(s, t) = f\left(\frac{2s+t-1}{t+1}\right), \quad \frac{1-t}{2} \le s \le 1$$

Thus $F(s, 0) = e(2s) = x$ for $0 \le s \le \frac{1}{2}$

$F(s, 0) = f(2s-1)$ for $\frac{1}{2} \le s \le 1$

∴ $F(s, 0) = (e\,f)(s)$

and $F(s, 1) = f(s)$ for $0 \le s \le 1$

Moreover $F(0, t) = e(0) = x$

$F(1, t) = f(1) = x$

F(s, t) defined in this way is clearly a mapping of I^2 into E agreeing with e*f* on AB and with *f* on DC and also carrying the vertical sides AD, BC into x. It remains to show that F is continuous in the pair of variables s, t. For any point (s, t) not on the line PD, the continuity is obvious, for if (s, t) is in the triangle APD, then F carries it into x and also carries a nbd of (s, t), namely the

whole triangle APD, APD into x, while if (s, t) is in PBCD but not on PD, the continuity of F at (s, t) follows from that of *f* and the fact that *f*((2s +t −1)/(t+1)) is continuous in s and t. Finally consider point (s, t) on PD. F carries such a point into x. Now it is not hard to see that if (s′, t′) is sufficiently near to the line PD and to the right of it, then (2s′ + t′−1)/(t′+1) can be made as small as one please. It follows thus from the continuity of *f* and the definition of F that points sufficiently near to (s, t) and to the right of PD will be carried by F into a pre assigned nbd of x, all point the left of PD are carried into x itself and so the continuity of F at a point of PD is removed.
F thus satisfies all the necessary conditions to show that e*f* $\simeq$ *f*. The proof that *f*e $\simeq$ *f* is carried out in exactly the same manner.

(iii) It remains to prove that every homotopy class has an inverse that is to say, if *f* is any given path based on x, it must be shown that there is a path g based on x such that *f*g $\simeq$ e and g*f* $\simeq$ e. A path fulfilling this condition is obtained by taking *f* in reverse. Explicitly this means defining a mapping g of I into E by setting g(s) = *f*(1−s).

We shall define F : I × I →E such that it coincides with *f*g at the bottom i.e. for t = 0 and with e at the top i.e. for t = 1 and carries the vertical sides of I × I into x. Since

$$fg(s) = f(2s),\ 0 \le s \le \frac{1}{2}$$

$$= g(2s-1),\ \frac{1}{2} \le s \le 1$$

that is

$$fg(s) = d(2s),\ 0 \le s \le \frac{1}{2}$$

and

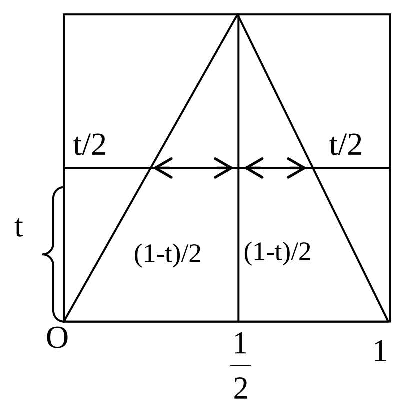
$$fg(s) = f(1-(2s-1)),\ \frac{1}{2} \le s \le 1$$

i.e.

$$= f(2-2s)\ \frac{1}{2} = s \le 1$$

we define

$$F(s,t) = \begin{cases} x & ,0 \le s \le \frac{t}{2} \\ f(2s-1), & \frac{t}{2} \le s \le \frac{1}{2} \\ f(2-t-2s), & \frac{1}{2} \le s \le 1-\frac{t}{2} \\ x, & 1-\frac{t}{2} \le s \le 1 \end{cases}$$



Then clearly

$$F(s,0) = \begin{cases} f(2s) & 0 \le s \le \frac{1}{2} \\ f(2-2s) & \frac{1}{2} \le s \le 1 \end{cases}$$

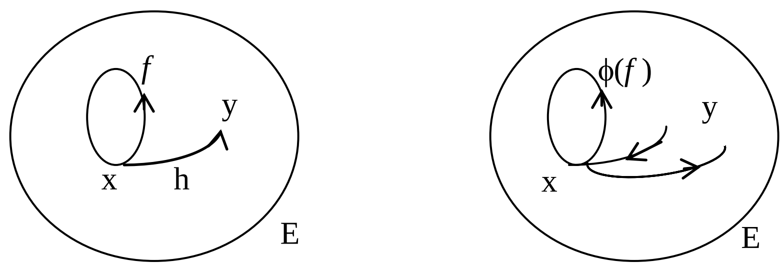
$$F(s,1) = \begin{cases} x, & 0 \le s \le \frac{1}{2} \\ x, & \frac{1}{2} \le s \le 1 \end{cases}$$

Fundamental Group

The fundamental group was introduced by French mathematician Henri Poincare (1854-1912) around 1900.

Definition. In a topological space E, the homotopy classes of closed paths w.r.t. to a base point x form a group. This group will be called the fundamental group of E relative to the base point x and will be denoted by $\pi(E, x)$

Now we will compare the groups $\pi(E, x)$ and $\pi(E, y)$ in the case where x and y can be joined by a path in the topological space E. It is clear in this case that a given path based on x leads by a simple construction to a path based on y. For if f is a given path based



on x and h is a path from x to y, then a path $\phi(f)$ based on y is obtained by going along h in reverse (i.e. from y to x) then round f and finally back to y along h. It will now be shown that this correspondence between paths based on x and those based on y leads to an isomorphism between the corresponding groups of homotopy classes.

Theorem 4. If E is a topological space and x and y are tow points of E which can be joined by a path in E, then $\pi(E, x)$ and $\pi(E, y)$ are isomorphic.

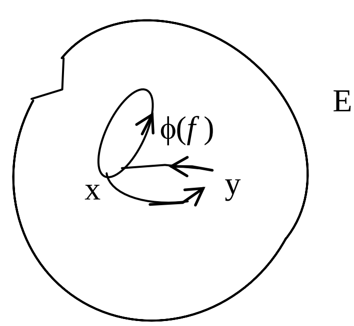
Proof. Let h be a path in E from x to y, that is to say, a continuous mapping of I into E such that $h(0) = x$ and $h(1) = y$. Let f be any path based on x that is a continuous mapping of I into E such that $f(0) = f(1) = x$. Define a path $\phi(f)$ based on y as follows.

$$\phi(f)(s) = h(1-3s), \quad 0 \le s \le \frac{1}{3}$$

$$\phi(f)(s) = f(3s-1), \quad \frac{1}{3} \le s \le \frac{2}{3}$$

$$\phi(f)(s) = h(3s-2), \quad \frac{2}{3} \le s \le 1$$

It is easy to see that $\phi^3(f)$ is a continuous mapping of I into E and carries 0 and 1 into y.



One would expect that a continuous deformation of f would lead to a corresponding deformation of $\phi(f)$; that is if f and g are homotopic w.r.t. to the base point x, one would expect $\phi(f)$ and $\phi(g)$ to be homotopic w.r.t. to the base point y. To check this suppose that homogopy of f and g w.r.t. to x is given by a continuous mapping $F : I^2 \to E$ such that

$$F(s, 0) = f(s)$$

$$F(s, 1) = g(s)$$
$$F(0, t) = F(1, t) = x \ \forall \ s, t$$

Define G by setting

$$G(S, t) = \begin{cases} h(1-3s), & 0 \le s \le \frac{1}{3} \text{ and } \forall t \\ F(3s-1, t), & \frac{1}{3} \le s \le \frac{2}{3} \text{ and } \forall t \\ h(3s-2), & \frac{2}{3} \le s \le 1 \text{ and all } t \end{cases}$$

It can be shown that G is a continuous mapping of I^2 into E. Moreover we observe

$$G(s, 0) = \begin{cases} h(1-3s), & 0 \le s \le \frac{1}{3} \text{ and all } t \\ F(3s-1, 0) = f(3s-1), & \frac{1}{3} \le s \le \frac{2}{3} \\ h(3s-2), & \frac{2}{3} \le s \le 1 \end{cases}$$

$$G(s, 1) = \begin{cases} h(1-3s), & 0 \le s \le \frac{1}{3} \\ F(3s-1, 1) = g(3s-1), & \frac{1}{3} \le s \le \frac{2}{3} \\ h(3s-2), & \frac{2}{3} \le s \le 1 \end{cases}$$

$$G(0, t) = h(1) = y$$
$$G(1, t) = h(1) = y$$

Therefore G is a homotopy between $\phi(f)$ and $\phi(g)$. It follows therefore that all paths in a given homotopy class w.r.t the base point x are mapped by ϕ into the same homotopy class w.r.t y. If $\bar{f}$ is a homotopy class w.r.t x and f is a path in this class then the homotopy class w.r.t y of $\phi(f)$ will be denoted by $\Phi(f)$. Φ is thus a well defined mapping of π (E, x) into π(E, y)

In exactly the same way a mapping $\bar{\psi}$ of π(E, y) into π(E, x) can be constructed. To do this let g be a given path based on y and define $\psi(g)$ as a path based on x, the definition being similar to that of $\phi(f)$ above.

$$\psi(g)(s) = \begin{cases} h(3s), & 0 \le s \le \frac{1}{3} \\ g(3s-1), & \frac{1}{3} \le s \le \frac{2}{3} \\ g(3-3s), & \frac{2}{3} \le s \le 1 \end{cases}$$

Having done this $\bar{\psi}$ is defined as the mapping of π(E, y) into π(E, x) which maps the homotopy class of g on that of $\psi(g)$.

Now we will show that the mapping Φ and $\bar{\psi}$ are inverse to one another, this will show that both mappings are one-one and onto. In order to do this it will be sufficient to prove that if f is a given path based on x, then $\psi(\phi(f))$ is homotopic to f with respect to the base point x and that if g is a path based on y then $\phi(\psi(g))$ is homotopic to g with respect to the base point y.

The full definition of $\psi(\phi(f))$ is as follows:

$$\psi(\phi(f))(s) = \begin{cases} h(3s), & 0 \le s \le \frac{1}{3} \\ h(4-9s), & \frac{1}{3} \le s \le \frac{4}{9} \\ f(9s-4), & \frac{4}{9} \le s \le \frac{5}{9} \\ h(9s-5), & \frac{5}{9} \le s \le \frac{2}{3} \\ h(3-3s), & \frac{2}{3} \le s \le 1 \end{cases}$$

It will be noticed that this definition amounts to dividing I into five subintervals to each of which h or h reversed or f is applied with the appropriate change of scale. Experience with this sort of situation in the last theorem should suggest that the deformation of $\psi(\phi(f))$ into f will be carried out by expanding the middle one of these subintervals to which f is applied, so that it fills the whole of I, while the remaining subintervals are compressed into the endpoints of I. And consideration similar to those followed in the last theorem suggest that the required homotopy will be given by a mapping $F : I^2 \to E$ defined as follows.

$$F(s, t) = \begin{cases} h(3s), & 0 \le s \le \frac{1-t}{3} \\ h(4-4t-9s), & \frac{1-t}{3} \le s \le \frac{4(1-t)}{9} \\ f\left(9s/(8t+1) - \frac{4-4t}{8t+1}\right), & \frac{4(1-t)}{9} \le s \le \frac{5+4t}{9} \\ h(9s-5-4t), & \frac{5+4t}{9} \le s \le \frac{t+2}{3} \\ h(3-3s), & \frac{t+2}{3} \le s \le 1 \end{cases}$$

It can now be proved that this mapping F gives the required homotopy of $\psi(\phi(f))$ and f w.r.t the base point x, the proof that g and $\phi(\psi(g))$ are homotopic with respect to the base point y is carried out in a similar manner.

Having now shown that the mapping $\Phi : \pi(E, x) \to \pi(E, y)$ is one-one and onto it remains to show that for any two paths f and g based on x, $\phi(f\,g)$ is homotopic to $\phi(f)\,\phi(g)$ w.r. the base point y. This will show that Φ is an isomorphism, as required. If one carries out reasoning similar to that used in the previous theorem, and the earlier part of the proof of this theorem, one will be led to the consideration of a mapping $F : I^2 \to E$ defined as follows :

$$F(s,t)=\begin{cases} h(1-6s/(2-t)), & 0\le s\le \frac{1}{6}(2-t) \\ f(6s+t-2), & \frac{2-t}{6}\le s\le \frac{3-t}{6} \\ h(6s+t-3), & \frac{3-t}{6}\le s\le \frac{1}{2} \\ h(t+3-6s), & \frac{1}{2}\le s\le \frac{t+3}{6} \\ g(6s-3-t), & \frac{t+3}{6}\le s\le \frac{t+4}{6} \\ h\left(\frac{(6s-t-4)}{(2-t)}\right) & \frac{t+4}{6}\le s\le 1 \end{cases}$$

$$F(s,0)=\begin{cases} h(1-3s), & 0\le s\le \frac{1}{3} \\ f(6s-2), & \frac{1}{3}\le s\le \frac{1}{2} \\ h(6s-3), & \frac{1}{2}\le s\le \frac{1}{2} \\ h(3-6s), & \frac{1}{2}\le s\le \frac{1}{2} \\ g(6s-3), & \frac{1}{2}\le s\le \frac{2}{3} \\ h(3s-2), & \frac{2}{3}\le s\le 1 \end{cases}$$

$$F(s,1)=\begin{cases} h(1-6s), & 0\le s\le \frac{1}{6} \\ f(6s-1), & \frac{1}{6}\le s\le \frac{1}{3} \\ h(6s-2), & \frac{1}{3}\le s\le \frac{1}{2} \\ h(4-6s), & \frac{1}{2}\le s\le \frac{2}{3} \\ g(6s-4), & \frac{2}{3}\le s\le \frac{5}{6} \\ h(6s-5), & \frac{5}{6}\le s\le 1 \end{cases}$$

$$(f\,g)(s)=\begin{cases} f(2s), & 0\le s\le \frac{1}{2} \\ g(2s-1), & \frac{1}{2}\le s\le 1 \end{cases}$$

Therefore

$$(\phi(f\,g))(s) = \begin{cases} h(1-3s), & 0 \le s \le \frac{1}{3} \\ fg(3s-1), & \frac{1}{3} \le s \le \frac{2}{3} \\ h(3s-2), & \frac{2}{3} \le s \le 1 \end{cases}$$

$$= \begin{cases} h(1-3s), & 0 \le s \le \frac{1}{3} \\ f(6s-2), & \frac{1}{3} \le s \le \frac{1}{2} \\ h(6s-3), & \frac{1}{2} \le s \le \frac{1}{2} \\ h(3-6s), & \frac{1}{2} \le s \le \frac{1}{2} \\ g(6s-3), & \frac{1}{2} \le s \le \frac{2}{3} \\ h(3s-2) & \frac{2}{3} \le s \le 1 \end{cases} = F(s,0)$$

Moreover

$$F(0, t) = h(1) = y$$

$$F(1, t) = h\left(\frac{6-t-4}{2-t}\right) = h(1) = y$$

$$(\phi(f)\phi(g))(s) = [\phi(f)]\,(2s),\ 0 \le s \le \frac{1}{2}$$

$$[\phi(g)]\,(2s-1)\ \frac{1}{2} \le s \le 1$$

Now

$$\phi(f)(s) = \begin{cases} h(1-3s), & 0 \le s \le \frac{1}{3} \\ f(3s\text{-}1), & \frac{1}{3} \le s \le \frac{2}{3} \\ h(3s-2), & \frac{2}{3} \le s \le 1 \end{cases}$$

Therefore

$$[\phi(f)\phi(g)](s) = \begin{cases} = h(1-6s), & 0 \le s \le \frac{1}{6} \\ = f(6s-1), & \frac{1}{6} \le s \le \frac{1}{3} \\ = h(6s-2), & \frac{1}{3} \le s \le \frac{1}{2} \\ = h(4-6s), & \frac{1}{2} \le s \le \frac{2}{3} \\ = g(6s-4), & \frac{2}{3} \le s \le \frac{5}{6} \\ = h(6s-5), & \frac{5}{6} \le s \le 1 \end{cases}$$

It can be now shown that F gives the required homotopy w.r.t base point y of $\phi(f\ g)$ and $\phi(f)\phi(g)$. This completes the proof of the theorem.

Cor. If E is an arcwise connected space, the group $\pi(E, x)$ is independent of the base point x.

Proof. In this case any pair of points x and y of E can be joined by a path in E, and so $\pi(E, x)$ and $\pi(E, y)$ are isomorphic for any x and y in E. Hence $\pi(E, x)$ is the same as $\pi(E, y)$ from the point of view of group theory.

Remark. It follows from what has been said that, if E is an arcwise connected space, one can speak without ambiguity of the fundamental group of E without mentioning any base point, the understanding being that this group is $\pi(E, x)$ for any arbitrary x in E. The fundamental group of an arcwise connected space will be denoted by $\pi(E)$.

Now it will be shown that the fundamental group of an arcwise connected space is topologically invariant. This means that if E and E′ are two homemorphic arcwise connected spaces then $\pi(E)$ and $\pi(E')$ are isomorphic.

Theorem 5. Let E and E′ be homeomorphic arcwise connected spaces, then $\pi(E)$ and $\pi(E')$ are isomorphic.

Proof. Let f be the homeomorphism of E onto E′ and let g be the inverse of f. If $x \in E$, then we write $y = f(x)$. If h is any path in E based on x, then the mapping $foh\ I \to E'$ is a path in E′ based on y, for

$$foh(0) = f(h(0)) = f(x) = y$$
$$foh(1) = f(h(1)) = f(x) = y \quad [\, f : E \to E',\ h : I \to E\ h(0) = h(1) = x\ foh : I \to E]$$

Suppose that k is a second path in E based on x and that h is homotopic to k, so there exists a continuous mapping $F : I^2 \to E$ such that

$$\left.\begin{array}{l} F(s,0) = h(s) \\ F(s,1) = k(s) \end{array}\right\} s \in I$$

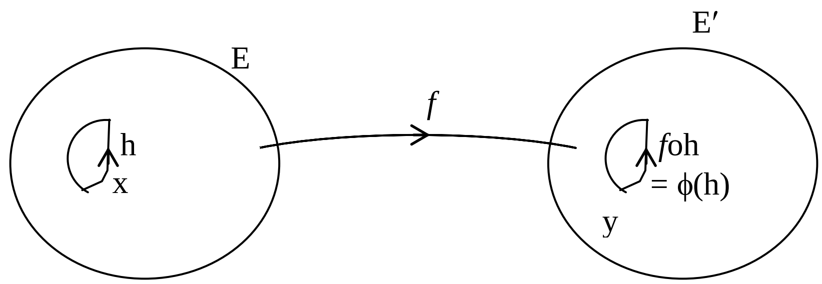
$$F(0, t) = F(1, t) = x,\ t \in I$$

Then the mapping $foF : I^2 \to E'$ is such that

$$(foF)\ (s, 0) = f(F(s, 0)) = f(h(s)) = foh(s)$$

$$(foF)\ (s, 1) = f(F(s, 1)) = f(k(s)) = fok(s)$$
$$(foF)(0, t) = f(F(0, t)) = f(x) = y$$
$$(foF)\ (1,t) = f(F(1, t)) = f(x) = y.$$

Thus whenever h and k based on x are homotopic, foh and fok based on y are homotopic this shows that if $\bar{h}$ is a given homotopy class on E w.r. the base point x, then for any representative path h of $\bar{h}$, foh always lies in the same homotopy class $\phi(\bar{h})$ on E′, w.r.t the point y. Thus we have constructed a mapping $\phi : \pi(E) \to \pi(E')$



We claim that ϕ is an isomorphism between the two groups. We have seen that $f$oh and $f$ok are paths on E′. Then the product of these two paths in E′ is defined by

$$\left.\begin{aligned}[(f\text{oh})(f\text{ok})](s) &= (f\text{oh})(2s), \quad 0 \le s \le \frac{1}{2} \\ [(f\text{oh})(f\text{ok})](s) &= (f\text{ok})\,(2s-1), \quad \frac{1}{2} \le s \le 1\end{aligned}\right\} \qquad (1)$$

Also

$$(hk)(s) = h(2s), \quad 0 \le s \le \frac{1}{2}$$

$$(hk)(s) = k(2s-1), \quad \frac{1}{2} \le s \le 1$$

and therefore

$$\left.\begin{aligned}[f\text{o}(hk)](s) &= (f\text{oh})(2s), \quad 0 \le s \le \frac{1}{2} \\ [f\text{o}(hk)](s) &= (f\text{ok})\,(2s-1), \quad \frac{1}{2} \le s \le 1\end{aligned}\right\}$$

From (1) and (2) it follows that

$$(f\text{oh})\,(f\text{ok}) = f\text{o}(hk)$$

which proves that ϕ is homomorphism.

Since any path h′ on E′ based on y can be obtained by comparing f with $f$oh′ and so any homotopy class in E′ w.r.t y can be obtained as an image under ϕ. Hence ϕ is onto mapping.

Finally suppose that $\phi(\bar{h})$ is the identity of π(E′). If h is a path in the class $\bar{h}$, this means that $f$oh is homotopic to the constant mapping in E′ w.r.t y. that is to say, there is a continuous mapping F* : $I^2 \to E'$ such that

$$F^*(s, 0) = (f\text{oh})(s)$$
$$F^*(s, 1) = e(s) = y$$
$$F^*(0, t) = F^*(1, t) = y$$

Then goF* : $I^2 \to E$, where g is the inverse of f is the mapping such that

$$(\text{goF}^*)\,(s, 0) = \text{goF}^*(s, 0) = g(f\text{oh})(s)$$
$$= [(\text{go}f)\text{oh}](s) = h(s)$$
$$(\text{goF}^*)\,(s, 1) = g(y) = g(f(x)) = x$$
$$(\text{goF}^*)\,(0, t) = (\text{goF}^*)\,(1, t) = x$$

that is to say, h is homotopic to the constant mapping with respect to x. Thus $\phi(\bar{h})$ is the identity of π(E′) $\Rightarrow$ $\bar{h}$ is the identity of π(E), which proves that the mapping is one to one also. Hence ϕ is an isomorphism.

Definition. A closed path f on a topological space E beginning and ending at x will be said to be shrinkable to x or to be homotopic to a constant w.r.t to the base point x if f is homotopic to the mapping e : I→E defined by e(s) = x for all s∈I, w.r.t the base point x.

Definition. A topological space E will be said to be simply connected w.r.t base point x if every closed path in E beginning and ending at x is shrinkable to x.

It follows therefore that a space E is simply connected w.r. the base point x iff $\pi(E, x)$ reduces to identity element only.

Theorem 6. Every closed path beginning and ending at a point x on a circular disc is homotopic to a constant w.r.t base point x.

Proof. Set up polar coordinates in the plane of the disc taking x as a pole then if $f : I \to E$ is the given closed path, $f(s)$ will have polar coordinates $(r(s), \theta(s))$. Define F(s, t) as the point with polar coordinates $((1-t)r(s), \theta(s))$. Since $(r(s), \theta(s))$ in E, all points $(r, \theta(s))$ with $r \le r(s)$ are on E and so $F(s, t) \in E$. F is thus a mapping of I^2 into E and it can be seen that F is continuous. Also setting t = 0, F(s, 0) is the point $(r(s), \theta(s)) = f(s)$ while the radial coordinate r of F(s, 1) is zero.

The Fundamental Group of the Circle

Our goal in this section is to show that the fundamental group of the circle is isomorphic to the group of integers. In order to do this we need several preliminary results. We begin with the definition of a function p that maps R onto S^1. Define $p : R \to S^1$ by $p(x) = (\cos 2\pi x, \sin 2\pi x)$. One can think of p as a function that wraps R around S^1. In particular, note that for each integer n, p is a one-to-one map of [n, n + 1] onto S^1. Furthermore, if U is the open subset of S^1 indicated in Figure 8.7, then $p^{-1}(U)$ is the union of a pairwise disjoint collection of open intervals.

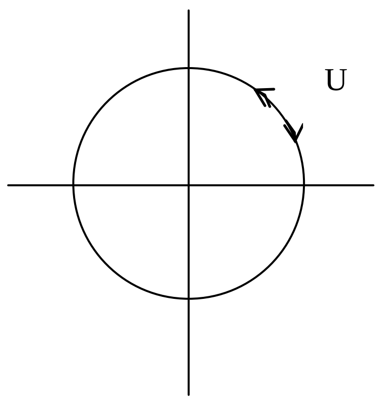


Figure 8.7

To be specific, let $U = \{(x, y) \in S^1 : x \;\; 0 \text{ and } y > 0\}$. Then if $x \in p^{-1}(U)$, cos 2πx and sin 2πx are both positive, so $p^{-1}(U) = \bigcup_{n \in Z} (n, n + \frac{1}{4})$. Moreover, for each $n \in Z$, $|_{[n, n+\frac{1}{4}]}$ is a one-to-one function from the closed interval $(n, n + \frac{1}{4}]$ onto $\bar{U}$. Thus, since $[n, n + \frac{1}{4}]$ is compact, $p|_{[n, n+\frac{1}{4}]}$ is a homemorphism of $[n, n + \frac{1}{4}]$ onto $\bar{U}$. Therefore $p_{(n, n+\frac{1}{4})}$ is a homeomorphism of $(n, n + \frac{1}{4})$ onto U.

Throughout the remainder of this section, p denotes the function defined above.

The following result is known as the Covering Path Property.

Theorem 7. Let $\alpha : I \to S^1$ be a path and let $x_0 \in R$ such that $p(x_0) = \alpha(0)$. Then there is a unique path $\beta : I \to R$ such that $\beta(0) = x_0$ and $p \circ \beta = \alpha$.

Proof. Since α is continuous, for each $t \in I$ there is a connected neighborhood U_t of t such that $\alpha(U_t)$ is a proper subset of S^1. Since I is compact, the open cover $\{U_t : t \in I\}$ of I has a finite subcover $\{U_1, U_2, \ldots, U_m\}$. If $U_i = I$ for some i, then $\alpha(I)$ is a connected proper subset of S^1. If $U_i \ne I$ for any i, choose U_i so that $0 \in U_i$. There exists $t' \in I$ such that $t' \ne 0$, $[0, t'] \subseteq U_i$, and $t' \in U_i \cap$

U_j for some $j \neq i$. If $U_i \cup U_j = I$, $\alpha([0, t'])$ and $\alpha([t', 1])$ are connected proper subsets of S^1. If $U_i \cup U_j \neq I$, choose $t'' \in I$ such that $t'' > t'$, $[t', t''] \subseteq U_j$ and $t'' \in U_j \cap U_k$ for some k ($i \neq k \neq j$). Since $\{U_1, U_2,\ldots,U_m\}$ is a finite cover of I, after a finite number of steps, we will obtain $t_0, t_1, t_2,\ldots, t_n$ such that $0 = t_0$, $t_n = 1$, $t_{i-1} < t_i$ and $\alpha([t_{i-1}, t_i])$ is a connected proper subset of S^1 for each i = 1, 2,…, n.

For each i = 0, 1, 2,…, n let P_i be the statement : There is a unique continuous function $\beta_i : [0, t_i] \to R$ such that $\beta_i(0) = x_0$ and $po\beta_i = \alpha|_{[0,t_i]}$. In order to prove the theorem, it is sufficient to show that P_i is true for each i. It is clear that P_0 is true. Suppose $0 < i \leq n$ and P_{i-1} is true. Let V_{i-1} denote the component of $p^{-1}(\alpha[t_{i-1}, t_i]))$ that contains $\beta_{i-1}(t_{i-1})$, and let $p_{i-1} = p|_{V_{i-1}}$. Then $p_{i-1} : V_{i-1}\to\alpha([t_{i-1}, t_i])$ is a homemorphism. Define $\beta_i : [0, t_i] \to R$ by

$$\beta_i(t) = \begin{cases}\beta_{i-1}(t), & \text{if } 0 \leq t \leq t_{i-1}\\ (p_{i-1})^{-1}(\alpha(t)), & \text{if } t_{i-1} \leq t \leq t_i.\end{cases}$$

Then β_i is continuous, $\beta_i(0) = x_0$, and $po\beta_i =|_{[0,t_i]}$. We must show that β_i is unique. Suppose $\beta_i' : [0, t_i] \to R$ is a continuous function such that $\beta_i'(0) = x_0$ and $po\beta_i' =|_{[0,t_i]}$. If $0 \leq t \leq t_{i-1}$, then $\beta_i'(t) = \beta_i(t)$ since P_{i-1} is true. Suppose $t_{i-1} < t \leq t_i$. Since$(po\beta_i')(t) = \alpha(t) \in\alpha([t_{i-1}, t_i])$, $\beta_i'(t) \in p^{-1}(\alpha([t_{i-1}, t_i]))$. Therefore $\beta_i'(t) \in V_{i-1}$ since β_i' is continuous and V_{i-1} is a component of $p^{-1}(\alpha([t_{i-1}, t_i]))$. Therefore $\beta_i(t), \beta_i'(t) \in V_{i-1}$ and $(p_{i-1} \circ \beta_i)(t) = (p_{i-1} \circ \beta_i')(t)$. Hence $\beta_i(t) = \beta_i'(t)$ since p_{i-1} is a homeomorphism.

Example. Let $\alpha : I\to S^1$ be a homemorphism that maps I onto $\overline{\{(x,y)\in S^1 : x > 0 \text{ and } y > 0\}}$ and has the property that $\alpha(0) = (1, 0)$ and $\alpha(1) = (0, 1)$, and let $x_0 = 2$. Then the function $\beta : I \to R$ given by Theorem 6 is a homemorphism of I onto [2,2.25], which has the property that $\beta(0) = 2$ and $\beta(1) = 2.25$.

The following result is known as the Covering Homotopy Property.

Theorem 8. Let (X, *T*) be a topological space, let $f : X\to R$ be a continuous function, and let $H : X \times I\to S^1$ be a continuous function such that $H(x, 0) =(p\, o f)(x)$ for each $x\in X$. Then there is a continuous function $F : X \times I\to R$ such that $F(x, 0) = f(x)$ and $(p\, oF)(x, t) = H(x, t)$ for each $(x, t) \in X \times I$.

Proof. For each $x\in X$, define $\alpha_x : I\to S^1$ by $\alpha_x(t) = H(x, t)$. Now $(p\, of)(x) = \alpha_x(0)$, and hence, by last Theorem (6) there is a unique path $\beta_x : I\to R$ such that $\beta_x(0) = f(x)$ and $po\beta_x = \alpha_x$. Define $F : X \times I \to R$ by $F(x, t) = \beta_x(t)$. Then $(poF)(x, t) = (po\beta_x)(t) = H(x, t)$ and $F(x, 0) = \beta_x(0) = f(x)$.

We must show that F is continuous. Let $x_0\in X$. For each $t \in I$, there is a neighborhood M_t of x_0 and a connected neighborhood N_t or t such that $H(M_t \times N_t)$ is contained in a proper connected subset of S^1. Since I is compact, the open cover $\{N_t : t\in I\}$ of I has a finite subcover $\{N_{t_1}, N_{t_2},..., N_{t_n}\}$. Let $m = \bigcap_{i=1}^{n} M_{t_i}$. Then M is a neighborhood of x_0, and, for each i = 1,2,…m, $H(M \times N_{t_i})$ is contained a proper connected subset of S^1. Thus there exist $t_0, t_1, \ldots, t_m$ such that $t_0 = 0$, $t_m = 1$, and, for each j = 1, 2,…,m, $t_{j-1} < t_j$ and $H(M \times[t_{j-1}, t_j])$ is contained in a proper connected subset of S^1.

For each j = 0, 1,…,m, let P_j be the statement : There is a unique continuous function $G_j : M \times [0, t_j] \to R$ such that $G_j(x, 0) = f(x)$ and $(p \circ G_j)(x, t) = H(x, t)$. We want to show that P_j is true for each j. It is clear that P_0 is true. Suppose $0 < j \leq m$ and P_{j-1} is true. Let U_{j-1} be a connected proper subset of S^1 that contains $H(M \times I_{j-1})$, let V_{j-1} denote the component of $p^{-1}(U_{j-1})$ that contains $G_{j-1}(M \times \{t_{j-1}\})$, and let $p_{j-1} = p|_{V_{j-1}}$. Then $p_{j-1} : V_{j-1} \to U_{j-1}$ is a homemorphism. Define $G_j : M \times [0, t_j] \to R$ by

$$G_j(x, t) = \begin{cases} G_{j-1}(x,t), & \text{if } 0 \le t \le t_{j-1} \\ (p_{j-1})^{-1}(H(x,t)), & \text{if } t_{j-1} \le t \le t_j \end{cases}$$

Then G_j is continuous, $G_j(x, 0) = f(x)$, and $(p \circ G_j)(x, t) = H(x, t)$. We must show that G_j is unique. Suppose $G'_j : M \times [0, t_j] \to R$ is a continuous function such that $G'_j(x, 0) = f(x)$ and $(p \circ G'_j)(x, t) = H(x, t)$. If $0 \le t \le t_{j-1}$, then $G'_j(x, t) = G_j(x, t)$ because P_{j-1} is true. Suppose $t_{j-1} < t \le t_j$. Since $(p \circ G_j)(x, t) = H(x, t) \in U_{j-1}$, $G'_j(x, t) \in p^{-1}(U_{j-1})$. Therefore, since G'_j is continuous and V_{j-1} is a component of $p^{-1}(U_{j-1})$, $G'_j(x, t) \in V_{j-1}$. Hence $G_j(x, t)$ and $G'_j(x, t)$ are members of V_{j-1} and $(p_{j-1} \circ G_j)(x, t) = (p_{j-1} \circ G'_j)(x, t)$. Thus, since p_{j-1} is a homemorphism, $G_j(x, t) = G'_j(x, t)$. We have proved that there is a unique continuous function $G : M \times I \to R$ such that $G(x, 0) = f(x)$ and $(p \circ G)(x, t) = H(x, t)$. Therefore $G = F|_{(M \times I)}$. Since M is a neighborhood of x_0 in X, F is continuous at (x_0, t) for each $t \in I$. Since x_0 is an arbitrary member of X, F is continuous. This proof also shows that F is unique.

It α is a loop in S^1 at (1, 0), then $p(0) = \alpha(0)$, and hence, by Theorem 6 there is a unique path $\beta = I \to R^1$ such that $\beta(0) = 0$ and $p \circ \beta = \alpha$. Since $(p \circ \beta)(1) = \alpha(1) = (1, 0)$, $\beta(1) \in p^{-1}(1, 0)$, and hence $\beta(1)$ is an integer. The integer $\beta(1)$ is called the degree of the loop α, and we write $\deg(\alpha) = \beta(1)$.

Example. Define $\alpha : I \to S^1$ by $\alpha(x) = (\cos 4\pi x, -\sin \pi x)$. Then α "wraps" I around S^1 twice in a clockwise direction. The unique path $\beta : I \to R$ such that $\beta(0) = 0$ and $p \circ \beta = \alpha$ given by Theorem 6 is defined by $\beta(x) = -2x$ for each $x \in I$. Therefore $\deg(\alpha) = \beta(1) = -2$.

Theorem 9. Let α_1 and α_2 be loops in S^1 at (1, 0) such that $\alpha_1 \simeq_p \alpha_2$. Then $\deg(\alpha_1) = \deg(\alpha_2)$.

Proof. Since $\alpha_1 \simeq_p \alpha_2$, there is a continuous function $H : I \times I \to S^1$ such that $H(x, 0) = \alpha_1(x)$ and $H(x, 1) = \alpha_2(x)$ for each $x \in I$ and $H(0, t) = H(1, t) = (1, 0)$ for each $t \in I$. By Theorem 6 there is a unique path $\beta_1 : I \to R$ such that $\beta_1(0) = 0$ and $p \circ \beta_1 = \alpha_1$. Thus, by Theorem 8, there is a unique continuous function $F : I \times I \to R$ such that $(p \circ F)(x, t) = H(x, t)$ for each $(x, t) \in I \times I$ and $F(x, 0) = \beta_1(x)$ for each $x \in I$. Define $\gamma : I \to R$ by $\gamma(t) = F(0, t)$ for each $t \in I$. Then γ is continuous, and $(p \circ \gamma)(t) = (p \circ F)(0, t) = H(0, t) = (1, 0)$ for each $t \in I$. Therefore $\gamma(I) \subseteq p^{-1}(1, 0)$, and, since $\gamma(I)$ is connected and $p^{-1}(1, 0)$ is a discrete subspace of R, γ is a constant function. Thus $F(0, t) = \gamma(t) = \gamma(0) = F(0, 0) = \beta_1(0) = 0$ for each $t \in I$. Define $\beta_2 : I \to R$ by $\beta_2(x) = F(x, 1)$ for each $x \in I$. Then $\beta_2(0) = F(0, 1) = 0$ and $(p \circ \beta_2)(x) = (p \circ F)(x, 1) = H(x, 1) = \alpha_2(x)$ for each $x \in I$. By definition, $\deg(\alpha_1) = \beta_1(1)$ and $\deg(\alpha_2) = \beta_2(1)$. Now define a path $\delta : I \to R$ by $\delta(t) = F(1, t)$ for each $t \in I$. Again $(p \circ \delta)(t) = (p \circ F)(1, t) = H(1, t) = (1, 0)$ for each $t \in I$, and hence $\delta(I) \subseteq p^{-1}(1, 0)$. Therefore δ is a constant function, and hence $F(1, t) = \delta(t) = \delta(0) = F(1, 0) = \beta_1(0) = 0$ for each $t \in I$. Therefore $\beta_2(1) = F(1, 1) = F(1, 0) = \beta_1(1)$, and hence $\deg(\alpha_1) = \deg(\alpha_2)$.

Remark :- Now we are ready to prove that the fundamental group of the circle is isomorphic to the group of integers. Since S^1 is pathwise connected, the fundamental group of S^1 is independent of the base point.

Theorem 10. $\pi_1(S^1, (1,0))$ is isomorphic to the group of integers.

Proof. Define $\phi : \pi_1(S^1, (1, 0)) \to Z$ by $\phi([\alpha]) = \deg(\alpha)$. By Theorem 9 ϕ is well-defined. Let $[\alpha_1]$, $[\alpha_2] \in \pi_1 (S^1,(1, 0))$. By Theorem 7 there are unique paths β_1, β_2, $I\to R$ such that $\beta_1(0) = \beta_2(0) = 0$, $p \circ \beta_1 = \alpha_1$, and $p \circ \beta_2 = \alpha_2$. By definition, $\deg(\alpha_1) = \beta_1(1)$ and $\deg(\alpha_2) = \beta_2(1)$. Define $\delta : I\to R$ by

$$\delta(x) = \begin{cases} \beta_1(2x), & \text{if } 0 \le x \le \frac{1}{2} \\ \beta_1(1) + \beta_2(2x-1), & \text{if } \frac{1}{2} \le x \le 1. \end{cases}$$

Since $\beta_2(0) = 0$, δ is continuous. Now $\delta(0) = \beta_1(0) = 0$, and, since

$$p[(\beta_1(1) + \beta_2(2x-1)) = p(\beta_2(2x-1)),$$

$$(p \circ \delta)(x) = \begin{cases} (p\circ\beta_1)(2x), & \text{if } 0 \le x \le \frac{1}{2} \\ (p\circ\beta_2)(2x-1), & \text{if } \frac{1}{2} \le x \le 1 \end{cases}$$

$$= \begin{cases} \alpha_1(2x), & \text{if } 0 \le x \le \frac{1}{2} \\ \alpha_2(2x-1), & \text{if } \frac{1}{2} \le x \le 1 \end{cases} = (\alpha_1 * \alpha_2)(x).$$

Therefore $\phi([\alpha_1] \circ [\alpha_2]) = \phi([\alpha_1 * \alpha_2]) = \deg(\alpha_1 * \alpha_2) = \delta(1) = \beta_1(1) + \beta_2(1) = \deg(\alpha_1) + \deg(\alpha_2) = \phi([\alpha_1]) + \phi([\alpha_2])$, and so ϕ is a homomorphism.

Now we show that ϕ maps $\pi_1 (S^1, (1, 0))$ onto Z. Let $z \in Z$, and define a path $\alpha : I\to R$ by $\alpha(t) = zt$ for each $t \in I$. Then $\alpha(0) = 0$ and $\alpha(1) = z$, so $p\circ\alpha : I\to S^1$ is a loop in S^1 at $(1, 0)$. Therefore $[p \circ \alpha]\in\pi_1 (S^1, (1, 0))$, and, by definition, $\deg(p \circ \alpha) = \alpha(1) = z$. Therefore$\phi([p \circ \alpha]) = \deg (p \circ \alpha) = z$.

Finally, we show that ϕ is one-to-one. Let $[\alpha_1]$, $[\alpha_2] \in\pi_1 (S^1, (1, 0))$ such that $\phi([\alpha_1]) = \phi([\alpha_2])$. Then $\deg(\alpha_1) = \deg(\alpha_2)$. By Theorem 7 there are unique paths $\beta_1, \beta_2 : I\to R$ such that $\beta_1(0) = \beta_2(0) = 0$, $p \circ \beta_1 = \alpha_1$, and $p \circ \beta_2 = \alpha_2$. By definition, $\deg(\alpha_1) = \beta_1(1)$ and $\deg(\alpha_2) = \beta_2(1)$. So $\beta_1(1) = \beta_2(1)$. Define $F : I \times I \to R$ by $F(x, t) = (1-t)\beta_1(x) + t\beta_2(x)$ for each $(x, t) \in I \times I$. Then $F(x, 0) = \beta_1(x)$ and $F(x, 1) = \beta_2(x)$ for each $x\in I$ and $F(0, t) = 0$ and $F(1, t) = \beta_1(1) = \beta_2(1)$ for each $t\in I$. Therefore $p \circ F : I \times I\to S^1$ is a continuous function such that $(p \circ F)(x, 0) = (p \circ \beta_1)(x) = \alpha_1(x)$ and $(p \circ F)(x, 1) = (p \circ \beta_2)(x) = \alpha_2(x)$ for each $x\in I$ and $(p \circ F)(0, t) = p(0) = (1, 0)$ and $(p \circ F)(1, t) = (p \circ \beta_1)(1) = \alpha_1(1) = (1, 0)$ for each $t\in I$. Thus $\alpha_1 \simeq_p \alpha_2$, so $[\alpha_1] = [\alpha_2]$.

Covering Spaces

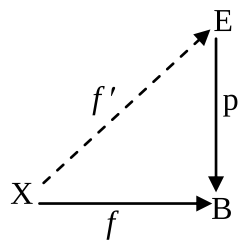
In this section we generalize Theorem 7 and 8 by replacing the function $p:R\to S^1$ defined by $p(x) = (\cos 2\pi x, \sin 2\pi x)$ with a function called a “covering map” from an arbitrary topological space (E, T) into another topological space (B, u).

Definition. Let (E, T) and (B, u) be topological spaces, and let $p : E\to B$ be a continuous surjection. A n open subset U of B is evenly covered by p if $p^{-1}(U)$ can be written as the union of a pairwise disjoint collection $\{V_\alpha : \alpha\in\wedge\}$ of open sets such that for each $\alpha \in\wedge$, $p|_{V_\alpha}$ is a homemorphism of V_α onto U. Each V_α is called a slice of $p^{-1}(U)$.

Definition. Let (E, T) and B(, u) be topological spaces, and let $p: E\to B$ be a continuous surjection. If each member of B has a neighborhood that is evenly covered by p, then p is **covering map** and E is a **covering space** of B.

Note that the function $p : R\to S^1$ in Section 8.3 is covering map. Covering spaces were introduced by Poincare in 1883.

Definition. Let (E, *T*), (B, *u*), and (X, *v*) be topological spaces, let $p : E \to B$ be a covering map, and let $f : X \to B$ be a continuous function. A lifting of f is a continuous function $f' : X \to E$ such that $p \circ f' = f$(see Figure below).



Notice that Theorem 7 provides a lifting of a path in S^1, where p is the specific function discussed in last section rather than an arbitrary covering map.

Theorem 11. Let (E, *T*) and (B, *u*) be topological spaces, let $p : E \to B$ be a covering map, let $e_0 \in E$, let $b_0 = p(e_0)$, and let $\sigma : I \to B$ be a path such that $\sigma(0) = b_0$. Then there exists a unique lifting $\sigma' : I \to E$ of f such that $\sigma'(0) = e_0$.

Proof. Let $\{U_\alpha : \alpha \in \wedge\}$ be an open cover of B such that for each $\alpha \in \wedge$, U_α is evenly covered by p. So, there exists $t_0, t_1, \ldots, t_n$ such that $0 = t_0 < t_1 < \ldots < t_n = 1$ and for each $i = 1, 2, \ldots, n$, $\sigma([t_{i-1}, t_i]) \subseteq U_\alpha$ for some $\alpha \in \wedge$. We define the lifting σ' inductively.

Define $\sigma'(0) = e_0$, and assume that $\sigma'(t)$ has been defined for all $t \in [0, t_{i-1}]$, where $1 \le i \le n$. There exists $\alpha \in \wedge$ such that $\sigma([t_{i-1}, t_i]) \subseteq U_\alpha$. Let $v = \{V_{\alpha\beta} : \beta \in \Gamma\}$ be the collection of slices of $p^{-1}(U_\alpha)$. Now $\sigma'(t_{i-1})$ is a member of exactly one member $V_{\alpha\gamma}$ of v . For $t \in [t_{i-1}, t_i]$, define $\sigma'(t) = (p|_{v_{\alpha\gamma}})^{-1}(\sigma(t))$. Since $p|_{v_{\alpha\gamma}} : V_{\alpha\gamma} \to U_\alpha$ is a homemorphism, σ' is continuous on $[t_{i-1}, t_i]$ and hence on $[0, t_i]$. Therefore, by induction, we can define σ' on I.

It follows immediately from the definition of σ' that $p \circ \sigma' = \sigma$.

The uniqueness of σ' is also proved inductively. Suppose σ'' is another lifting of σ such that $\sigma''(0)e_0$, and assume that $\sigma''(t) = \sigma''(t)$ for all $t \in [0, t_{i-1}]$, where $1 \le i \le n$. Let U_α be a member of the open cover of B such that $\sigma([t_{i-1}, t_i]) \subseteq U_\alpha$, and let $V_{\alpha\gamma}$ be the member of *V* chosen in the definition of σ'. Since σ'' is a lifting of σ, $\sigma''([t_{i-1}, t_i]) \subseteq p^{-1}(U_\alpha) = \bigcup_{\beta \in \Gamma} V_{\alpha\beta}$. Since *V* is a collection of pairwise disjoint open sets and $\sigma''([t_{i-1}, t_i])$ is connected, $\sigma''([t_{i-1}, t_i])$ is a subset of one member of *V*. Since $\sigma''(t_{i-1}) = \sigma'(t_{i-1}) \in V_{\alpha\gamma}$ $\sigma''([t_{i-1}, t_i]) \subseteq V_{\alpha\gamma}$. Therefore, for each $t \in [t_{i-1}, t_i]$, $\sigma''(t) \in V_{\alpha\gamma} \cap p^{-1}(\sigma(t))$. But $V_{\alpha\gamma} \cap p^{-1}(\sigma(t)) = \{\sigma'(t)\}$, and so $\sigma''(t) = \sigma'(t)$. Therefore $\sigma''(t) = \sigma'(t)$ for all $t \in [0, t_i]$. By induction, $\sigma'' = \sigma'$.

Theorem 12. Let (E, *T*) and (B, *u*) be topological spaces, let $p : E \to B$ be a covering map, let $e_0 \in E$, let $b_0 = p(e_0)$, and let $F : I \times I \to B$ be a continuous function such that $F(0, 0) = b_0$. Then there exists lifting $F' : I \times I \to E$ of F such that $F'(0, 0) = e_0$. Moreover, if F is a path homotopy then F' is also.

Proof. Define $F'(0, 0) = e_0$. By Theorem 11, there exists a unique lifting $F' : \{0\} \times I \to E$ of $F|_{\{0\}\times I}$ such that $F'(0, 0) = e_0$ and a unique lifting $F : I \times \{0\} \to E$ of $F|_{I\times\{0\}}$ such that $F(0, 0) = e_0$. Therefore, we assume that a lifting F' of F is defined on $(\{0\} \times I) \cup (I \times \{0\})$ and we extend it to $I \times I$.

Let $\{U_\alpha : \alpha \in \wedge\}$ be an open cover of B such that for each $\alpha \in \wedge$, U_α is evenly covered by p. So, there exists $s_0, s_1, \ldots, s_m$ and $t_0, t_1, \ldots, t_n$ such that $0 = s_0 < s_1 < \ldots < s_m = 1$, $0 = t_0 < t_1 < \ldots < t_n = 1$, and for each i = 1, 2,…,m 1, 2, …, m and j = 1, 2,…,n, let $U_i \times J_j = [s_{i-1}, s_i] \times [t_{j-1}, t_j]$. We define F′ Inductively on the rectangles $I_i \times J_j$ in the following order :

$$I_1 \times J_1, I_2 \times J_1, \ldots, I_m \times J_1, I_1 \times J_2, I_2 \times J_2, \ldots, I_m \times J_2, I_1 \times J_3, \ldots, I_m \times J_n.$$

$I_1 \times J_n$	$I_2 \times J_n$	…	$I_m \times J_n$
⋮	⋮		⋮
$I_1 \times J_2$	$I_2 \times J_2$	…	$I_m \times J_2$
$I_1 \times J_1$	$I_2 \times J_2$	…	$I_m \times J_1$

Suppose $1 \le p \le m$, $1 \le p \le n$, and assume that F′ is defined on $C = (\{0\} \times I) \cup (I\times\{0\} \cup \bigcup_{i=1}^{m} \bigcup_{j=1}^{q-1} (I_i \times J_j) \cup \bigcup_{i=1}^{p-1} (I_i \times J_q)$. We define F′ on $I_p \times J_q$.

There exists $\alpha \in \wedge$ such that $F(I_p \times J_q) \subseteq U_\alpha$. Let $V = \{V_{\alpha\beta} : \beta \in \wedge\}$ be the collection of slices of $p^{-1}(U_\alpha)$. Now F′ is already defined on $D = C \cap (I_p \times J_q)$. Since D is connected, F′(D) is connected. Since V is a collection of pairwise disjoint open sets, F′(D) is a subset of one member $V_{\alpha\gamma}$ of V. Now $p|_{v_{\alpha\gamma}}$ is a homemorphism of $V_{\alpha\gamma}$ onto U_α. Since F′ is a lifting of $F|_C$, $((p|_{v_{\alpha\gamma}}) \circ F')(x) = F(x)$ for all $x \in D$. For $x \in I_p \times I_q$, define $F'(x) = (p|_{v_{\alpha\gamma}})^{-1} (F(x))$. Then F′ is a lifting of $F|_{(C \cup (I_q \times J_q))}$. Therefore, by induction we can define F′ on $I \times I$.

Now suppose F is a path homotopy. Then $F(0, t) = b_0$ for all t∈I(. Since F′ is a lifting of F, $F'(0, t) \in p^{-1}(b_0)$ for all t∈I. Since $\{0\} \times I$ is connected, F′ is continuous, and $p^{-1}(b_0)$ has the discrete topology as a subspace of E, $F'\{0\} \times I)$ is connected and hence it must be a single point. Likewise, there exist $b_1 \in B$ such that $F(1, t) = b_1$ for all t∈I, so $F'(1, t) \in p^{-1}(b_1)$ for all t∈I, and hence $F'(\{1\} \times I)$ must be a single point. Therefore F′ is a path homotopy.

Theorem 13. Let (E, *T*) and (E, *u*) be topological spaces, let $p : E \to B$ be a covering map, let $e_0 \in E$, let $b_0 = p(e_0)$, let $b_1 \in B$, let α and β be paths in B from b_0 to b_1 that are path homotopic, and let α′ and β′ be liftings of α and β respectively such that $\alpha'(0) = \beta'(0) = e_0$. Then $\alpha'(1) = \beta'(1)$ and $\alpha' \simeq_p \beta'$.

Proof. Let $F : I \times I \to B$ be a continuous function such that $F(x, 0) = \alpha(x)$ and $F(x, 1) = \beta(x)$ for all x∈I and $F(0, t) = b_0$ and $F(1, t) = b_1$ for all t∈I. By Theorem 12 there exists a lifting $F' : I \times I \to E$ of F such that $F'(0, t) = e_0$ for all $t \in I$ and $F'(\{1\} \times I)$ is a set consisting of a single point, say e_1. The continuous function $F'|_{(I\times\{0\})}$ is a lifting of $F|_{(I\times\{0\})}$ such that $F'(0, 0) = e_0$. Since the lifting of paths is unique $F'(x, 0) = \alpha'(x)$ for all x∈I. Likewise, $F'|_{(I\times\{1\})}$ is a lifting of $F|_{(I\times\{1\})}$ such that $F'(0, 1) = e_0$. Again by Theorem 11 $F'(x, 1) = \beta'(x)$ for all x∈I. Therefore $\alpha'(1) = \beta'(1) = e_1$ and $\alpha' \simeq_p \beta'$.

The Fundamental Theorem of Algebra

Definition. Let (X, T) and (Y, U) be topological spaces and let $h : X \to Y$ be a continuous function. Then h is inessential if it is homotopic to a constant map and h is essential if if it is not inessential.

Result 1. Let (X,T) be a topological space and let $h : S^1 \to X$ be a continuous function. Then the following are equivalent.

(1) h is inessential

(2) h can be extended to a continuous function

Result 2. Let (Y, U) be a topological space and let $h : S^1 \to Y$ be an inessential function. Then h_* is the zero homomorphism.

The Fundamental Theorem of Algebra says that if $n \in N$, then every polynomial equation of degree n with complex coefficients has at least one solution in the set of complex numbers. This theorem is difficult to prove, and most proofs involve non-algebraic concepts. We give a proof that uses the concepts used in the last sections.

Theorem 14. Let $n \in N$ and let $x^n + a_{n-1} x^{n-1} +\ldots+ a_1x + a_0 = 0$ be a polynomial equation of degree n with complex coefficients. Then this equation has at least one solution in the set of complex numbers.

Proof. We consider the members of S^1 to be complex numbers and define continuous function f : $S^1 \to S^1$ by $f(z) = z^n$. Let $s_0 = (1, 0)$ and consider the induced homomorphism $f_* : \pi_1(S^1, s_0) \to \pi_1(S^1, s_0)$. Define $\sigma : I \to S^1$ by $\sigma(x) = (\cos 2\pi x, \sin 2\pi x) = e^{2\pi ix}$. Then $f_*([\sigma]) = [f \text{ o } \sigma] \in \pi_1 (S^1, (1, 0))$. Since $(f \text{ o } \sigma)(0) = (1, 0)$, the unique path $\alpha : I \to R$ given by Theorem 7, is the path defined by $g(x) = nx$. Let $p : R \to S^1$ be the standard covering map defined by $p(x) = (\cos 2\pi x, \sin 2\pi x)$. Then $p \text{ o } \alpha = f \text{ o } \sigma$, so $\deg (f \text{ o } \sigma) = n$. From the proof that $\pi_1(S^1, (1, 0))$ is isomorphic to the group of integers (Theorem 10), we see that $[f \text{ o } \sigma]$ is not the identity element of $\pi_1 (S^1, (1, 0))$. Therefore f_* is not the zero homomorphism.

First we show that we may assume that $|a_{n-1}| + |a_{n-2}| +\ldots+ |a_0| < 1$. We let c be a positive real number and substitute x = cy in the given polynomial equation to obtain the equation

$$(cy)^n + a_{n-1}(cy)^{n-1} +\ldots+ a_1(cy) + a_0 = 0$$

or

$$y^n + (a_{n-1}/c)y^{n-1} +\ldots+ (a_1/c^{n-1})y + a_0/c^n = 0.$$

Now choose c large enough so that $|a_{n-1}/c| + |a_{n-2}/c^2| +\ldots+ |a_1/c^{n-1}| + |a_0/c^n| < 1$. Then if y_0 is a solution of $y^n + (a_{n-1}/c)y^{n-1} +\ldots+ (a_1/c^{n-1})\, y + a_0/c^n = 0$, $x = cy_0$ is a solution of $x^n + a_{n-1}\, x^{n-1} +\ldots+ a_1x + a_0 = 0$. Therefore it is sufficient to show that $y^n + (a_{n-1}/c)\, y^{n-1} +\ldots+ a_1/c^{n-1})\, y + a_0/c^n = 0$ has a solution. This means that in the given polynomial equation, we may assume that $|a_{n-1}| + |a_{n-2}| +\ldots+ |a_0| < 1$.

The proof that the given polynomial equation has a solution in B^2 is by contradiction. Suppose $x^n + a_{n-1}\, x^{n-1} +\ldots+ a_1x + a_0 = 0$ has no solution in B^2. Then there is a continuous function $q : B^2 \to R^2 - \{(0, 0)\}$ defined by $q(z) = z^n + a_{n-1}\, z^{n-1} +\ldots+ a_1z + a_0$. Let $r : S^1 \to R^2 - \{(0, 0)\}$ be the restriction of q to S^1. Then $q : B^2 \to R^2 - \{(0, 0)\}$ is an extension of r. So, by result 1, r is inessential.

We arrive at a contradiction by showing that r is homotopic to a continuous function that is essential. Define $k : S^1 \to R^2 -\{(0, 0)\}$ by $k(z) = z^n$, and define $H : S^1 \times I \to R^2 - \{(0, 0)\}$ by $H(z, t) = z^n + t(a_{n-1}\, z^{n-1} + a_{n-2}\, z^{n-2} +\ldots+ a_0)$. Note $H(x, t) \neq (0, 0)$ for $(x, t) \in S^1 \times I$ because

$$\begin{aligned}|H(x, t)| &\geq |z^n| - |t(a_{n-1}\, z^{n-1} + a_{n-2}\, z^{n-2} +\ldots+ a_0| \\ &\geq 1 - t\,(|a_{n-1}\, z^{n-1}\,| + |a_{n-2}\, z^{n-2}| +\ldots+ |a_0|) \\ &= 1-t\,(|a_{n-1}| + |a_{n-2}| +\ldots+ |a_0|) > 0.\end{aligned}$$

We complete the proof by showing that k is essential. Note that $k = j \text{ o } f$, where $j : S^1 \to R^2 - \{(0, 0)\}$ is the inclusion map, so $k_* = j_* \text{ o} f_*$. Since the fundamental group of S^1 is isomorphic to the group of integers, f_* is essentially the homomorphism that takes an integer a into the product na. Furthermore j_* is an isomorphism. Therefore k_* is not the zero homomorphism. By result 2, k is essential.

10

PARACOMPACT SPACES

Paracompact spaces were first introduced by Dieudonne (one generalisation des espaces compacts, J. Math. Pures. Appl 23(1944), 65-76) as a natural generalisation of compact spaces still retaining enough structure to enjoy many of the properties of compact spaces, yet sufficiently general to include a much wider class of spaces. The notion of paracompactness gained structure with the proof by A. H. Stone, that every metric space is paracompact and the subsequent use of this result in the solutions of the general metrization problem by Bing, Nagata and Smirnov.

We need some new terminologies for coverings and collections which recently have proved to be very powerful tool for study not only of paracompact spaces but also of metric spaces.

Covering of a Space

Definition. If μ and v are covers of X (Topological space) we say that μ refines v and write μ<v iff each $U\in\mu$ is contained in some $V\in v$. Then we say that μ is a refinement of v.

Definition. (i) Any subcovering of a given covering is a refinement of that covering.

(ii) A compact space can be characterized by saying that every open cover of it has a finite cover which is its refinement.

(iii) The relation < of refinement is a pre-ordering i.e. it is reflexive, transitive but is not partially ordering as it is not antisymmetric.

Definition. If $\{C_\gamma\} < \{A_\alpha\}$ and $\{C_\gamma\} < \{B_\beta\}$ then $(C_\gamma\}$ is called common refinement of $\{A_\alpha\}$ and $\{B_\beta\}$.

Definition. If μ is a cover of X and $A \subset X$, the star of A with respect to μ is the set

$$\text{St}(A, \mu) = U(U\in\mu ; A\cap U \neq \phi\}$$

Definition. We say μ star refines v or μ is a star refinement of v written as $\mu^* < v$ iff for each $U \in \mu$, there is some $V\in v$ such that

$$\text{St}(U, \mu) \subset V.$$

Definition. μ is a barycentric refinement of v, written as $\mu \,\Delta\, v$, provided the sets St(x, μ), for $x\in X$ (topological space) refines v.

Theorem 1. A barycentric refinement w of a barycentric refinement v of μ is a star refinement of μ.

Proof. Suppose $W_0\in w$. Choose a fixed $y_0 \in W_0$. For each $W\in w$ such that $W \cap W_0 \neq \phi$ choose a z $\in W\cap W_0$.

Then

$$W \cup W_0 \subseteq U\{W_1 ; z\in W_1, W_1 \in w\}$$
$$= \text{St}(z, w) \subset \text{some } V \in v$$

because w is a barycentric refinement of v. Now since each such V contains y_0, we conclude that

$$\text{St}(W_0, w) = U\{W \in w ; W_0 \cap W \neq \phi\}$$
$$\subseteq \text{St}(y_0, v) \subset \text{some } U\in\mu,$$

Since v is a barycentric refinement of μ. Thus for each $W_0 \in w$, there is some $U \in \mu$ such that st $(W_0, w) \subset U$. Hence w is a star refinement of μ. This completes the proof.

Theorem 2. A barycentric refinement of any refinement of μ is a barycentric refinement of μ. Infact, let

$$w = \{W_\alpha ; \alpha \in A\}$$

be barycentric refinement of a family

$$v = \{V_\alpha ; \alpha \in A\}$$

where v is a refinement of μ. Let $x \in X$, then

$$St(x, w) = U\{W_\alpha ; x \in W_\alpha ; W_\alpha \in w\}$$
$$\subseteq V_\alpha \{V_\alpha \in v\}$$

since w is a barycentric refinement of v. But since v is a refinement of μ, to each $V_\alpha \in v$, there is some $U \in \mu$ such that $V_\alpha \subseteq U$. Thus

$$St(x, w) \subseteq U \text{ for some } U \in \mu.$$

$\Rightarrow$ w is a barycentric refinement of μ .

Locally Finite Covering

Definition. A collection μ of subsets of X if called **locally finite** or (neighbourhood finite) iff for every $x \in X$ has a neighbourhood meeting only finite $U \in \mu$. we call μ **point finite** if and only if each $x \in X$ belongs to only finitely many $U \in \mu$. Obviously every locally finite collection is point finite.

Remark. (1) But every point finite need not be locally finite. e. g. consider the covering of the set R.

$$\{\{x\} ; x \in R\} \text{ is a covering of R.}$$

It is a point finite covering of R but not locally finite. Since in usual topology of R, every point has neighbourhood (an open interval) which contains uncountable number of points of R. So every point has no neighbourhood which intersects finite number of members of the covering. However if we consider the discrete topology of R, then above covering is also locally finite. Thus for any space X, $\{\{x\} ; x \in X\}$ is a point finite cover, which is locally finite only under stringent conditions on X.

(2) The covering of R by the sets [n, n+1], as n ranges through all integers is point finite.

Definition. A collection μ of subsets of X is discrete if and only if each $x \in X$ has a neighbourhood meeting at most one element of μ clearly every discrete collection of sets is locally finite.

Finally, we have that any property of collection of sets in X, there is a corresponding σ-property which we illustrate with an example.

A collection v of subsets of X is called **σ–locally finite** iff $v = \bigcup_{n=1}^{\infty} V_n$ where each V_n is a locally finite collection in X, i.e. iff v is the countable union of locally finite collections family) in X.

Similarly, a collection v of subsets of X is **σ–discrete** iff v is the countable union of discrete collections in X.

If v is a σ–locally finite cover of X, the subcollections V_n which are locally finite and make up v will not usually be covers.

Theorem 3. If $\{A_\lambda ; \lambda \in \wedge\}$ is a locally finite system of sets in X, then so is

$\{\ \overline{A}_\lambda\ ;\ \lambda\in\wedge\}$

Proof. Given $p\in X$ and find an open neighbourhood U of p such that $U\cap A_\lambda = \phi$ except for finitely many λ. [since $\{A_\lambda\ ;\ \lambda\in\wedge\}$ is a locally finite system of sets]

$\Rightarrow \quad U \cap \overline{A}_\lambda = \phi$ except for some λ.

$U \cap \overline{A}_\lambda \neq \phi$ for some λ

$\Rightarrow \quad \{\ \overline{A}_\lambda;\ \lambda\in\wedge\}$ is locally finite.

Theorem 4. If $\{A_\lambda\ ;\ \lambda\in\wedge\}$ is a locally finite system of sets, then $U\ \overline{A}_\lambda = \overline{UA_\lambda}$. In particular, the union of a locally finite collection of closed sets is closed.

Proof. $U\ \overline{A}_\lambda \subset \overline{UA_\lambda}$ (Trivial) .

Conversely, suppose $p\in \overline{UA_\lambda}$. Now some neighbourhood of p meets only finitely many of the sets A_λ, say $A_{\lambda_1},\ldots,A_{\lambda_n}$. Since every neighbourhood of p meets UA_λ, every neighbourhood of p must then meet

$A_{\lambda_1}, U\ldots U\, A_{\lambda_n}$

Hence $p\in\ \overline{A_{\lambda_1}\cup A_{\lambda_2}\cup\ldots\cup A_{\lambda_n}}$ so that for some k, $p\in\ \overline{A}_{\lambda_k}$. Thus

$\overline{UA_\lambda}\subset U\overline{A}_\lambda$ proving the lemma.

Theorem 5. For each $\lambda\in\wedge$ $\{U\ \overline{A}_\lambda\ ;\ \lambda\in\wedge\}$ is closed.

Proof. Let $B = U\{\ \overline{A}_\lambda\ ;\ \lambda\in\wedge\}$ we will show $\overline{B} = B$

$\Rightarrow \quad$ B is closed.

Suppose $x\notin B$, we will show x can not be a limit point of B.

$x\notin B \Rightarrow x\notin U\ \overline{A}_\lambda \Rightarrow x\notin\ \overline{A}_\lambda\ \forall\lambda\in\wedge.$

If there is a neighbourhood U which is not disjoint for finite number of $\overline{A}_\lambda$'s.

i.e. $U\cap\ \overline{A}_{\lambda_i}\ \phi$ for $i = 1, 2,\ldots,n$. Then $X - \overline{A}_{\lambda_i}$ is open neighbourhood of x and so

$\bigcap_1^n \{X-\overline{A}_{\lambda_i}\}$ is open neighbourhood of x.

Thus $U \cap\{\bigcap_1^n (X-\overline{A}_{\lambda_i})$ is neighbourhood of x disjoint with each $\overline{A}_\lambda$ and $U\cap(\bigcap_1^n\{X-\overline{A}_{\lambda_i}\})$ is disjoint with $U\ \overline{A}_\lambda = B$. Thus x is not a limit point of B.

Paracompact Spaces

Definition. A Hausdorff space X is **paracompact** if and only if each open cover of X has an open locally finite refinement. For example A discrete topological space is paracompact since every open cover of a discrete space X has a locally finite refinement. $\{\{x\}\ ;\ x\in X\}$ is an open cover of x and refines every open covering of X and $\{\{x\}\ ;\ x\in X\}$ is locally finite since every point has a neighbourhood $\{x\}$ which intersects only one set namely $\{x\}$.

Remark. Some of the books use regular space instead of Hausdorff space in defining paracompact space. .

Michaell Theorem on Characterization of Paracompactness

Theorem 6. 1. (Michael) If X is a T_3-space, the following are equivalent

(a) X is paracompact

(b) Each open cover of X has an open σ-locally finite refinement.

(c) Each open cover has a locally finite refinement (not necessarily open)

(d) Each open cover of X has a closed locally finite refinement

Proof. (a) $\Rightarrow$ (b) Since X is paracompact, each open cover of X has an open locally finite refinement. Also, a locally finite cover is σ-locally finite. It follows therefore that each open cover of X has an open σ-locally finite refinement.

(b) $\Rightarrow$ (c) let μ be an open cover of X. By (b), there is a refinement v of μ such that $v = \bigcup_{n=1}^{\infty} v_n$, where each v_n is a locally finite collection of open sets, say

$$v_n = \{V_{n_\beta} ; \beta \in B\}. \text{ For each n,}$$

let $W_n = \bigcup_{\beta} V_{n_\beta}$. Then $\{W_1, W_2, \ldots..\}$

covers X . Define $A_1 = W_n - \bigcup_{i<n} W_i$

Then $\{A_n ; n \in N\}$ is a locally finite refinement of $\{W_n ; n \in N\}$. Now consider $\{A_n \cap V_{n_\beta} ; n \in N\}$, $\beta \in B$.

This is a locally finite refinement of v and since v is a refinement of μ. Thus each open cover of X has a locally finite refinement.

(c) $\Rightarrow$ (d). Let μ be an open cover of X. For each $x \in X$, pick some U_x in μ such that $x \in U_x$ and [as X is T_3-space] by regularity, find an open neighbourhood V_x of x such that $\overline{V}_x \subset U_x$. Now $\{V_x ; x \in X\}$ is an open cover of X and so by (c) has a locally finite refinement $\{A_\beta, \beta \in B\}$. Then $\{\overline{A}_\beta ; \beta \in B\}$ is still locally finite by the (proved earlier) and for each β, if $A_\beta \subset V_x$, then $\overline{A}_\beta \subset \overline{V}_x \subset U$ for some $U \in \mu$. It follows that $\{\overline{A}_\beta ; \beta \in B\}$ is a closed locally finite refinement of μ. [closure].

(d) $\Rightarrow$ (a). Let μ be an open cover of X, v a closed locally finite refinement. For each $x \in X$, let W_x be a neighbourhood of x meeting only finitely many $V \in v$. Now let A be a closed locally finite refinement of $\{w_x ; x \in X\}$. For each $V \in v$, let

$$V^* = X - U\{A \in \mathbf{A} ; A \cap V = \phi\}.$$

Then $\{V^*, V \in v\}$ is an open cover (The sets V^* are open by the result proved above) and is also locally finite. For consider $x \in X$, there is a neighbourhood U of x meeting only $A_1, A_2, \ldots, A_n$, say from **A**. But whenever $U \cap V^* \neq \phi$ we have $A_k \cap V^* \neq \phi$ for some $K = 1, 2, \ldots, n$ which implies $A_k \cap V \neq \phi$. Since each A_K meets only finitely many V, we must then have $U \cap V^* = \phi$ for all but finitely many of the V^*. Hence $\{V^* ; \vee \in v\}$ is locally finitely .

Now for each $V \in v$, pick $U \in \mu$ such that $V \subset U$ and form the set $U \cap V^*$ The collection of sets which results, as V ranges through v, serves as an open locally finite refinement of μ.

Corollary. Every Lindelof T_3-space X is Paracompact.

Since X is Lindelof, every open covering of X has a countable subcovery and A countable subcover is a σ-locally finite refinement. Thus each open cover of X has an open σ-locally finite refinement. Then by the above theorem, X is paracompact.

Remark. (1) A compact space can be characterized by the fact that for every open cover of X, there is a finite cover which is refinement of it.

(2) Since a finite subcover is a locally finite refinement, we have the fundamental result that a **compact Hausdorff space is Paracompact.**

(3) A discrete space is paracompact since every open cover of it is locally finite [paracompact ⇔ locally finite by the above theorem].

(4) The real line **R** is paracompact that is not compact. The fact that R is paracompact is a consequence of the theorem that every metrizable space is paracompact. But a direct proof is given as follows.

Suppose we are given an open covering μ of R. For each integer n, choose a finite number of elements of μ that cover the interval [n, n+1] and intersect each one with the open interval (n–1, n+2). Let the resulting collection of open sets be denoted by $\mathbf{B}_n$. Then the collection

$$\beta = \bigcup_{n\in Z} \beta_n$$

is a locally finite open refinement of μ that covers R.

Theorem. 7. (A. H. Stone). Every metric space is paracompact.

Proof. Let μ be an open cover of the metric space (X, d). For each n = 1, 2,3,… and $U\in\mu$, let $U_n = \{x\in U \,;\, d(x, X-U) \geq \frac{1}{2^n}\}$ Then we observe that

$$d(U_n, X - U_{n+1}) \geq \frac{1}{2^n} - \frac{1}{2^{n+1}} = \frac{1}{2^{n+1}}$$

Since by triangular inequality

$$d(x, X-U) \leq d(x, y) + d(y, X-U)$$

$$\Rightarrow \quad d(U_n, x-U) \leq d(x, y) + d(y, X-U) \quad \forall\, x\in U_n.$$

$$\Rightarrow \quad \frac{1}{2^n} \leq d(U_n, X-U) \leq d(x, y) + d(y, x-U) \quad \forall\, x\in U_n.$$

Since for all $y\in U_{n+1}$

$$d(y, X - U) \geq \frac{1}{2^{n+1}}$$

and $\forall\, y\in X - U_{n+1}$ [Reverse]

$$d(y, X-U) < \frac{1}{2^{n+1}}$$

⇒ Now from (∗)

$$\frac{1}{2^n} \leq d(x, y) + d(y, X-U)$$

$$\leq d(x\,;y) + \frac{1}{2^{n+1}}$$

$$\Rightarrow \quad \frac{1}{2^n} - \frac{1}{2^{n+1}} \leq d\,(U_n, X-U_{n+1}) \;\forall\, x\in U_n,\, y\in X-U_{n+1}$$

$$\Rightarrow \quad d(U_n, X-U_{n+1}) \geq \frac{1}{2^{n+1}}$$

Let < be a well ordering of the elements of μ. For each n = 1, 2,… and $U\in\mu$, let

$$U_n^* = U_n - U\{V_{n+1} \,;\, V\in\mu,\, V < U\}$$

For each U, $V\in\mu$, and each n = 1, 2,…, we have

$$U_n^* \subset X-V_{n+1}$$

or

$$V_n^* \subset X - U_{n+1}$$

depending on which comes first in the well-ordering. In either case,

$$d(U_n^*, V_n^*) \geq \frac{1}{2^{n+1}}$$

Since $\quad U_n^* \subset U_n,\ V_n^* \subset X - U_{n+1}$

$$\Rightarrow \quad d(U_n^*, V_n^*) \geq d(U_n, X - U_{n+1}) \geq \frac{1}{2^{n+1}}$$

Hence defining an open set $\tilde{U}_n$, for each $U \in \mu$ and $n \in N$, by

$$\tilde{U}_n = \{x \in X ; d(x, U_n^*) < \frac{1}{2^{n+3}}\}$$

Now $\quad d(U_n^*, V_n^*) \leq d(U_n^*, \tilde{U}_n) + d(\tilde{U}_n, \tilde{V}_n) + d(\tilde{V}_n, V_n^*)$

$$\Rightarrow \quad \frac{1}{2^{n+1}} \leq d(U_n^*, V_n^*) \leq \frac{1}{2^{n+3}} + \frac{1}{2^{n+3}} + d(\tilde{U}_n, \tilde{V}_n)$$

$$\Rightarrow \quad d(\tilde{U}_n, \tilde{V}_n) \geq \frac{1}{2^{n+1}} - \frac{2}{2^{n+3}} = \frac{1}{2^{n+2}}\ .$$

and so $V_n = \{\tilde{U}_n ; U \in \mu\}$ is discrete for each n [i.e. every point $x \in X$ has a neighbourhood meeting at most one element of μ]

Hence $v = UV_n$ is σ-discrete [A collection v of subsets of X is σ-discrete iff v is the countable union of discrete collections in X] and thus σ-locally finite [σ-locally finite definition] Moreover v refines μ . Thus we have proved that each open cover of X has an open σ-locally finite refinement. Hence by Michael' theorem (X, d) is paracompact.

The relationship between paracompactness and normality is given in the next th :

Paracompactness and Normality

Theorem 8. Every paracompact space is normal.

Proof. We shall show first that a **paracompact space is regular**. Suppose A is closed set in a paracompact space X and $x \notin A$. For each $y \in A$. $\exists$ open set V_y containing y such that $x \notin \bar{V}_y$. Then the sets V_y ; $y \in A$ together with the set X–A, form an open cover of X. Let w be a locally finite refinement and

$$V = U\{W \in w ; W \cap A \neq \phi\}$$

Then V is open and contains A, and $\bar{V} = U\{\bar{W} ; W \cap A \neq \phi\}$ [By a result proved earlier]

But each set W is contained in some V_y since w is refinement and hence, $\bar{W}$ is contained in $\bar{V}_y$. hence $x \notin \bar{W}$. (since $x \notin \bar{V}_y$). Thus $x \notin \bar{V}$. But $V \supseteq A$. Thus x and A are seperated by open sets in X. i.e. the space is regular. Now we will prove that the space is normal. Suppose A and B are disjoint closed sets i X . Since the space is regular, to each $y \in A$, $\exists$ an open set V_y containing y and $\bar{V}_y \cap B = \phi$. Then the sets V_y together with X–A form an open cover of X. Let w be an open locally finite refinement and

$$V = \{W \in w; W \cap A \neq \phi\}$$

Then V is open and contains A and

$$\bar{V} = U\{\bar{W} ; W \cap A \neq \phi\}.$$

But each such W is contained in some V_y (since w is refinement) and hence each $\overline{W}$ is contained in $\overline{V}_y$. Thus there is an open set V such that

$$A \subset V \text{ and } \overline{V} \cap B = \phi .$$

Thus X is normal.

Remark. Every normal space need not be paracompact e.g. space of ordinals which are less than the first uncountable rationals w.r.t. order topology is a normal space but not paracompact.

Theorem. 9. Each F_σ-set in a paracompact space is paracompact i.e. every closed subspace of a paracompact space X is paracompact.

Proof. Let $F = \bigcup_{1}^{\infty} F_n$ be an F_σ-subset of a paracompact space. X, where each F_n is closed in X. Let $\{U_\alpha ; \alpha \in A\}$ be an open covering of F and each $U_\alpha = F \cap V_\alpha$ where V_α is open in X. For each fixed n, $\{X - F_n\} U \{V_\alpha ; \alpha \in A\}$ is an open covering of X and so has an open locally finite refinement W_n. Let

$$\beta_n = \{W \cap F ; W \in w_n\}$$

Then each β_n is locally finite collection of open subsets of F and $\bigcup_{n=1}^{\infty} \beta_n$ refines $\{U_\alpha ; \alpha \in A\}$. Thus $\{U_\alpha ; \alpha \in A\}$ has an open σ-locally finite refinement. Thus by Michael theorem, F is paracompact.

Theorem. 10. (Michael). If each open set in a paracompact space X is paracompact, then every subspace is paracompact.

Proof. Given an $B \subset X$ and any open covering $\{W \cap B\}$, where each W is open in X. Then UW is an open set and so is paracompact by hypothesis. Therefore there is an open locally finite refinement {V} of the covering {W} of UW. Then $\{V \cap B\}$ is a locally finite refinement of $\{W \cap B$ and hence every covering of $\{W \cap B\}$ of B has an open locally finite refinement. Thus is paracompact.

Remark. (1) If $\overline{S}_r$ is a well-ordered set and $\overline{S_r \times S_r}$ is compact Hausdorff and is therefore paracompact. The subspace $S_r \times \overline{S}_r$ is not paracompact because it is not even normal thus an arbitrary subspace of a paracompact space need not be paracompact.

(2) The product of paracompact spaces need not be paracompact e.g. the space R (real line) is paracompact but R×R i.e. R^2 is not paracompact because it is not normal.

Theorem. 11. The product of a paracompact space with a compact T_2-space is paracompact.

Proof. Let X be a paracompact space and let Y be a compact space. Let μ be an open cover of X×Y. For fixed $x \in X$, a finite no. of elements of μ, say $U^x_{\alpha_1}, ..., U^x_{\alpha_n}$ cover $\{x\} \times Y$. [since Y is compact] and X is paracompact [every open cover of X has an open locally finite refinement] Pick an open neighbourhood V_x of x in X such that $V_x \times Y \subset \bigcup_{i=1}^{n_x} U_{\alpha_i}$. The sets V_x as x ranges through X, form an open cover of X. Let v be an open locally finite refinement. For each $V \in v$, $V \subset V_x$ for some x. Consider the set $(V \times Y) \cap U^x_{\alpha_i}$, i = 1, 2,…, n_x, formed as V ranges through v. This is a refinement of v and an open cover w of X×Y. Moreover, given $(x, y) \in X \times Y$, there is neighbourhood of x which meets only finitely many $V \in v$ and the neighbourhood U×Y of (x, y) can then meet only finitely many sets of w. Hence X×Y is paracompact.

Theorem. 12. (Michael) Paracompactness is invariant under continuous closed surjections (i.e. Let X be paracompact and p : X→Y a continuous closed surjection. Then Y is paracompact).

Proof. Let $\{U_\alpha ; \alpha\in A\}$ be any open covering of Y. Since a paracompact space is normal and normality is invariant under cont. closed sujrections, Y is normal. We know that a space Y is paracompact iff each open cover of Y has an open σ-locally finite refinement therefore to prove the theorem it is sufficient to show that $\{U_\alpha ; \alpha\in A\}$ has an open σ-locally finite refinement. We assume that A is well ordered (A partially ordered set W is called well ordered (or an ordinal) if each non-empty subset $B \subset W$ has a first element i.e. for each $B \neq 0$, $\exists$ a $b_0\in B$ satisfying $b_0 < b$ for all $b\in B$) and begin by constructing an open covering $\{V_{\alpha,n} ; (\alpha,n) \in A\times Z^+\}$ of X such that

(1) For each n, $\{\overline{V}_{\alpha,n} ; \alpha\in A\}$ is a covering of X and a precise locally finite refinement of $\{p^{-1}(U_\alpha) ; \alpha\in A\}$

(2) If $\beta > \alpha$, then $p(\overline{V}_{\beta,n+1}) \cap p(\overline{V}_{\alpha,n}) = \phi$.

Proceeding by induction, we take a precise open neighbourhood. finite refinement of $\{p^{-1}(U_\alpha)\}$ and shrink it to get $\{\overline{V}_{\alpha,1}\}$ Assuming $\{V_{\alpha, i}\}$ to be defined for all $i\leq n$, let

$$W_{\alpha,n+1} = p^{-1}(U_\alpha) - p^{-1}\, p\,(\bigcup_{\lambda<\alpha} \overline{V}_{\lambda,n})$$

Since by locally finiteness, $\bigcup_{\lambda<\alpha} \overline{V}_{\lambda,n}$ is closed and p is a closed map, each $W_{\alpha,n+1}$ is open. Further $\{W_{\alpha,n+1} ; \alpha\in A\}$ is a covering of X. Infact given $x\in X$, let α_0 be the first index for which $x\in p^{-1}(U_\alpha)$. Then $x\in W_{\alpha_0,n+1}$, since $p^{-1}\, p(\overline{V}_{\lambda,n}) \subset p^{-1}(U_\lambda)$ for each λ. Taking a precise, open locally finite refinement of $\{W_{\alpha,n+1} ; \alpha\in A\}$ shrink it to get $\{\overline{V}_{\alpha,n+1}\}$.

Clearly (1) holds i.e. Y is normal and since $\overline{V}_{\beta,n+1}$ is not in the inverse image of any $p(\overline{V}_{\alpha,n})$ for $\alpha < \beta$. Condition (2) is also satisfied.

For each n and α, let

$$H_{\alpha,n} = Y - p\left(\bigcup_{\beta\neq\alpha} \overline{V}_{\beta,n}\right) \text{which is an open set. We have}$$

(a) $H_{\alpha,n} \subset (\overline{V}_{\alpha,n}) \subset U_\alpha$ for each n and α.

$$\begin{aligned} p^{-1}(H_{\alpha,n}) &= X - p^{-1}p\left(\bigcup_{\beta\neq\alpha} \overline{V}_{\beta,n}\right) \\ &\subset X - p^{-1}\, p(X - \overline{V}_{\alpha,n}) \\ &\subset \overline{V}_{\alpha,n} \subset p^{-1}(U_\alpha) \end{aligned}$$

(b) $H_{\alpha,n} \cap H_{\beta,n} = \phi$ for each n whenever $\alpha \neq \beta$.

In fact $y\in H_{\alpha,n} \Rightarrow y\in p(\overline{V}_{\alpha,n})$ and is in no other $p(\overline{V}_{\beta,n})$

(c) $\{H_{\alpha,n} ; (\alpha, n) \in A \times Z^+\}$ is an open covering of Y.

Let $y\in Y$ be given, for each fixed n, there is because of (1), a first α_n with $y\in p(\overline{V}_{\alpha,n})$. Choosing now

$$\alpha_K = \min \{\alpha_n;\, n\in Z^+\}$$

We have $y\in(\overline{V}_{\alpha_{k,k}})$. If $\beta < \alpha_k$, then the defn. of α_K shows $y\notin p(\overline{V}_{\beta,K+1})$, if $\beta > \alpha_K$, then by (2), we find that $y\notin p(\overline{V}_{\beta,K+1})$, therefore we conclude that

$$\Rightarrow \qquad y\in Y - p\left(\bigcup_{\beta\neq\alpha_K} \overline{V}_{\beta,K+1}\right) \Rightarrow y\in H_{\alpha K,\, K+1}.$$

$\Rightarrow \quad \{H_{\alpha,n} ‘ (\alpha, n) \in A \times Z^+\}$ is a covering of Y.

To complete the proof, we need only modify $H_{\alpha,n}$ slightly to assume locally finiteness for each n. Choose a precise open locally finite refinement of

$$\{p^{-1}(H_{\alpha,n}) ; (\alpha, n) \in A \times Z^+\}$$

and shrink it to get an pen locally finite covering $\{K_{\alpha,n}\}$ satisfying $p(K_{\alpha,n}) \subset (H_{\alpha,n})$

$p(\overline{K}_{\alpha,n}) \subset H_{\alpha,n}$ For each n, let $S_n = \{y$; some neighbourhood of y intersects at most one $H_{\alpha,n}\}$
S_n is open and contains the closed set

$$\bigcup_{\alpha} p(\overline{K}_{\alpha,n}) = p(\bigcup_{\alpha} \overline{K}_{\alpha,n})$$

and so by normality of Y, we find an open set G_n with

$$\bigcup_{\alpha} p(\overline{K}_{\alpha,n}) \subset G_n \subset \overline{G}_n \subset S_n$$

Now the open covering

$$\{G_n \cap H_{\alpha,n} ; (\alpha, n) \in A \times Z^+\}$$

is σ-locally finite refinement of $\{U_\alpha\}$ with $\{G_n \cap H_{\alpha,n}; \alpha \in A\}$, n = 1, 2,… is locally finite. Thus by Michael's theorem "(If X is a T_3-space, the following are equivalent

(a) X is paracompact.

(b) each open cover of X has an open σ-locally finite refinement.

(c) each open cover has a locally finite refinement.

(d) Each open cover of X has a closed locally finite refinement]", Y is paracompat.

Thus Y is paracompact.

Theorem 13. Let X be normal and $\mu = \{u_\alpha ; \alpha \in A\}$ an locally finite open covering. Then μ has an open barycentric refinement.

Proof. Since X is normal, we can shrink μ to an open covering **B** = $\{V_\alpha ; \alpha \in A\}$ such that $\overline{V}_\alpha \subset U_\alpha$ for each α. Then **B** is locally finite. For each $x \in X$, define

$$W(x) = \cap\{U_\alpha ; x \in \overline{V}_\alpha\} \cap [\cap\{C\ \overline{V}_\beta ; x \notin \overline{V}_\beta\}]$$

We show that $w = \{W(x); x \in X\}$ is the required open covering. Being the intersection of finite no. of open sets, $\cap\{U_\alpha; x \in \overline{V}_\alpha\}$ and the last term $\cap\{C\ \overline{V}_\beta ; x \notin \overline{V}_\beta\} = \{C\ U\ \overline{V}_\beta\}$ is an open set. Therefore each W(x) is open. Next W is a covering since $x \in W(x)$ for each $x \in X$. Finally fix any $x_0 \in X$ and choose a $\overline{V}_\alpha$ containing x_0. Now for each x such that $x_0 \in W(x)$, we must have $x \in \overline{V}_\alpha$ also, otherwise $W(x) \subset C\ \overline{V}_\alpha$. Now because $x \in \overline{V}_\alpha$, we conclude that $W(x) \subset U_\alpha$. Thus $St(x, W) \subset U_\alpha$ which proves that W is an open barycentric refinement of μ. This completes the proof.

Nagata- Smirnov Metrization Theorem

Let T be a fixed, infinite cardinal no and suppose that ∧ is a fixed set of elements whose cardinality is T. The generalized Hilbert space (H^T, d_H^T) of weight T is the set H^T of all real valued mappings *f* defined on ∧ such that each mapping is different from zero on at most a countable subset of ∧ and the series $\sum_{\lambda \in \wedge} [f(\lambda)]^2$ converges, with the metric d_H^T defined by setting $d_H^T(f, g) = \sqrt{\sum_{\lambda \in \wedge} [f(\lambda) - g(\lambda)]^2}$ d_H^T is a metric for H^T.

To prove the main theorem, first we prove two lemmas.

Lemma 1. In a T_3-space with a σ-locally finite base, every open set is an F_σ-set.

(A subset A of a space X is called a F_σ set in X if it equals the union of a countable collection of closed subsets of X).

Proof. Let X be a T_3-space with a σ-locally finite base $\{B_{n,\lambda};\ n\in N,\ \lambda\in\wedge_n\}$ and suppose G is an open subset of X. By regularity, for each x∈G, there exists an open set containing x, whose closure is contained in G. Since for each fixed integer K, the collection $\{B_{K,\lambda};\ \lambda\in\wedge_K\}$ is locally finite, if we let

$$B_K = \bigcup_{x\in G} B_{K,\lambda(x)}, \text{ then}$$

$$\overline{B_K} = \bigcup_{x\in G} B_{K,\lambda(x)} \subseteq G .$$

Thus we have

$$G = \bigcup_{K\in N} \overline{B_K}$$

which is countable union of closed sets.

Lemma 2. A T_3-space with a σ-locally finite base is normal.

Proof. Let F and K be disjoint closed subsets of the T_3-spce X with σ-locally finite base $\{B_{n,\lambda};\ n\in N,\ \lambda\in\wedge_n\}$. By regularity, for each point x∈F, there exists a basic open set $B_{n(x),\lambda(x)}$ containing x whose closure is contained in X–K, and for each y∈K, there exists a basic open set $B_{n(y),\lambda(y)}$ containing y whose closure is contained in X–F. If we let

$$B_{k,F} = \bigcup_{x\in F} B_{k,\lambda(x)} \text{ and}$$

$$B_{k,K} = \bigcup_{y\in K} B_{k,\lambda(y)}, \text{ then by the local finiteness of } \{B_{k,\lambda};\ \lambda\in\wedge_k\}$$

$$\overline{B_{k,F}} = \bigcup_{x\in F} \overline{B}_{k,\lambda(x)} \subseteq X-K$$

and

$$\overline{B_{k,K}} = \bigcup_{y\in K} \overline{B_{k,\lambda(y)}} \subseteq X-F$$

Thus the sets

$$G_{n,F} = B_{n,F} - \bigcup_{k\subseteq n} \overline{B_{k,K}}$$

and

$$G_{n,K} = B_{n,K} - \bigcup_{k\leq n} \overline{B_{k,F}}$$

are open sets with the property that $G_{n,F}$ contains every point x∈F for which n(x) = n, and $G_{n,K}$ contains every point y∈K for which n(x) = n. Finally, we may let

$$G_F = \bigcup_{n\in N} G_{n,F} \text{ and } G_K = \bigcup_{n\in N} G_{n,K}$$

and obtain two disjoint open sets containing F and K respectively. Hence X is normal.

Now we state the main theorem.

Theorem 14 (Nagata-Smirnov Metrization Theorem).

A topological space is metrizable iff it is a T_3-space with a σ-locally finite base.

Proof. The necessity, of the condition follows from Stone's Theorem (Every metric space is paracompact). If we consider the open covering of a metric space X by balls $\{B(x,\frac{1}{n});x\in X\}$, we may find for each n, a locally finite open cover which refines it. The union of these covers is a σ-locally finite base for X.

Conversely, suppose that X is a T_3-space with a σ-locally finite base $\{B_{n,\lambda};\ n\in N,\ \lambda\in\wedge_n\}$. We shall prove that X is metrizable. We shall denote by $\wedge$ the collection of all pairs (n, λ) with $n\in N$ and $\lambda\in\wedge_n$, with which we have indexed the base and suppose that cardinality of $\wedge$ is T. We shall show that X is homeomorphic to a subset of H^T.

Now by the above lemmas, X is normal and every open subset is an F_σ-set. Now using the result "If E is an open F_σ-set in a normal space X, then there exists a continuous mapping $f : X\to[0, 1]$ such that $f(x) > 0$ iff $x\in E$, thus and $X\setminus E = f^{-1}(0)\}$ so for each $(n, \lambda)\in\wedge\exists$ a continuous mapping $f_{n,\lambda} : X\to[0, 1]$ such that $f_{n,\lambda}(x) > 0$ iff $x\in B_{n,\lambda}$. For each fixed integer n, the family $\{B_{n,\lambda}\ ;\ \lambda\in\wedge_n\}$ is locally finite, and so for each fixed point $x\in x$, $f_{n,\lambda}(x)\neq 0$ for at most a finite number of values of λ. Hence $1+\sum_\beta f^2_{n,\beta}(x)$ is a well defined continuous mapping of X which is never less than one. From this it follows that we may define a continuous mapping $g_{n,\lambda} : X\to[0, 1]$ by setting

$$g_{n,\lambda}(x) = f_{n,\lambda}(x)\left[1+\sum_\beta f^2_{n,\beta}(x)\right]^{-1/2}$$

Again, we see that $g_{n,\lambda}(x) > 0$ if $x\in B_{n,\lambda}$ while for a fixed integer n and fixed point $x\in X$, $g_{n,\lambda}(x)\neq 0$ for at most a finite number of values of λ. It is obvious that $\sum_\lambda g^2_{n,\lambda} < 1$

and then $\sum_\lambda [g_{n,\lambda}(x) - g_{n,\lambda}(y)]^2 < 2$ for all x , $y\in X$.

We now set $h_{n,\lambda}(x) = 2^{-\frac{n}{2}} g_{n,\lambda}(x)$

Then

$$\sum_{(n,\lambda)} h^2_{n,\lambda}(x) = \sum_n 2^{-n}\sum_\lambda g^2_{n,\lambda}(x)$$

$$= \sum_n 2^{-n} = 1$$

Thus for each $x\in X$, $h_{n,\lambda}(x)$ is a real valued mapping of $\wedge$ which is such that the mapping is different from zero on at most a countable subset of $\wedge$ and is such that the series of squares converges. From the definition of generalized Hilbert space of weight T, it follows that for each $x\in X$, we have found a point $f(x) = h_{n,\lambda}(x)$ in H^T. Thus we have defined a mapping f of X into H^T which is onto some subset of $f(X)$. We shall show that f is a homeomorphism.

If x and y are distinct points of X, then there exists a basic open set $B_{n,\lambda}$ containing x but not y since X is a T_1-space. It follows that $h_{n,\lambda}(x) > 0$, while $h_{n,\lambda}(y) = 0$ so $f(x)\neq f(y)$ and hence f is one to one.

Now suppose that $x\in X$ and $\in>0$ are given. First choose an integer $N = N(\in)$ such that $2^{-N} < \frac{\in^2}{4}$.

By the local finiteness property, there must exist an open set G containing x which has a non-empty intersection with at most a finite number of the sets $B_{n,\lambda}$ with $n\le N$. Let us denote these sets by $B_{n_i\lambda_i}$, where $N_i\le N$ for i = 1, 2,…, K . Since each function $h_{n,\lambda}$ is continuous we may find for each $\langle n, \lambda\rangle\in\wedge$, an open set $G_{n,\lambda}$ containing x such that

$$|h_{n,\lambda}(x) - h_{n,\lambda}(y)| < \frac{\in}{\sqrt{2K}}$$

for every $y\in G_{n,\lambda}$. Let us set

$$G^* = G \cap \left(\bigcap_{i=1}^{K} G_{n_i,\lambda_i} \right)$$

Which is an open set containing x. We now note that for <n, λ> ∈ ∧, but not equal to some (n_i, λ_i).

$$h_{n,\lambda}(x) = h_{n,\lambda}(y) = 0 \quad \forall\ y \in G^*$$

Thus we have for $y \in G^*$

$$\sum_{n \le N,\lambda} [h_{n,\lambda}(x) - h_{n,\lambda}(y)]^2 < K\left(\frac{\in}{\sqrt{2K}}\right)^2 = \frac{\in^2}{2}.$$

while $\displaystyle\sum_{n > N,\lambda} [h_{n,\lambda}(x) - h_{n,\lambda}(y)]^2$

$$= \sum_{n>N} 2^{-n} \sum_{\lambda} [g_{n,\lambda}(x) - g_{n,\lambda})y)]^2$$

$$\le 2 \sum_{n>N} 2^{-n} = 2(2^{-N}) < 2\left(\frac{\in^2}{4}\right) = \frac{\in^2}{2}$$

Thus we have show that

$$d_H^T(f(x), f(y)) = \sqrt{\sum_{n,\lambda} [h_{n,\lambda}(x) - h_{n,\lambda}(y)]^2} < \in$$

$\forall\ y \in G^*$ and so *f* is cont.

Finally, let G be an arbitrary open set in X and choose a point $x \in G$ We must have $x \in B_{n,\lambda} \subseteq G$ for some <n, λ> ∈ ∧. Let $h_{n,\lambda}(x)$, which is a positive real no, be denoted by δ. If *f*(y) is a point of *f*(X) such that $d_H^T(f(x), f(y)) < \delta$, then $h_{n,\lambda}(y)$ is also positive and so $y \in B_{n,\lambda} \subseteq G$. Thus

$$f^{-1}(B(f(x), \delta)) \subseteq G.$$

so that *f* is open. From this, it follows that *f* is a homeomorphism.