FLUID DYNAMICS

M.A./MSc. Mathematics (Final)

MM-504 and 505 (A_2)

Directorate of Distance Education
Maharshi Dayanand University
ROHTAK – 124 001

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Contents

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Max. Marks : 100
Time : 3 Hours

Note: Question paper will consist of three sections. Section I consisting of one question with ten parts of 2 marks each covering whole of the syllabus shall be compulsory. From Section II, 10 questions to be set selecting two questions from each unit. The candidate will be required to attempt any seven questions each of five marks. Section III, five questions to be set, one from each unit. The candidate will be required to attempt any three questions each of fifteen marks.

UNIT-I

Kinematics — Lagrangian and Eulerian methods. Equation of continuity. Boundary surface. Stream lines. Path lines and streak lines. Velocity potential. Irrotational and rotational motions. Vortex lines.

Equations of Motion—Lagrange's and Euler's equations of motion. Bernoulli's theorem. Equation of motion byflux method. Equations referred to moving axes Impulsive actions. Stream function.

UNIT-II

Irrotational motion in two-dimensions. Complex velocity potential. Sources, sinks, doublets and their images. Conformal mapping, Milne-Thomson circle theorem. Two-dimensional irrotational motion produced by motion of circular, co-axial and elliptic cylinders in an infinite mass of liquid. Kinetic energy of liquid. Theorem of Blasius. Motion of a sphere through a liquid at rest at infinity. Liquid streaming past a fixed sphere. Equation of motion of a sphere. Stoke's stream function.

UNIT-III

Vortex motion and its elementary properties. Kelvin's proof of permance. Motions due to circular and rectilinear vertices.

Wave motion in a gas. Speed of Sound. Equation of motion of a gas. Subsonic, sonic and supersonic flows of a gass. Isentropic gas flows. Flow through a nozzle. Normal and oblique shocks.

UNIT-IV

Stress components in a real fluid. Relations between rectangular components of stress. Connection between stresses and gradients of velocity. Navier-stoke's equations of motion. Plane Poiseuille and Couette flows between two parallel plates. Theory of Lubrication. Flow through tubes of uniform cross section in form of circle, annulus, ellipse and equilateral triangle under constant pressure gradient. Unsteady flow over a flat plate.

UNIT-V

Dynamical similarity. Buckingham p-theorem. Reynolds number. Prandt's boundary layer. Boundary layer equations in two-dimensions. Blasius solution. Boundary-layer thickness. Displacement thickness. Karman integral conditions. Separations of boundary layer flow.

UNIT-I

Basic Concepts and Definitions

(i) Let $\bar{q} = \hat{i}u + \hat{j}v + \hat{k}w,$ then

$$|\bar{q}| = \sqrt{u^2 + v^2 + w^2} = q$$

D.C's are given by $l = \cos\alpha = \dfrac{u}{|\bar{q}|}$, $m = \cos\beta = \dfrac{v}{|\bar{q}|}$, $n = \cos\gamma = \dfrac{w}{|\bar{q}|}$

where l, m, n, are components of a unit vector i.e. $l^2 + m^2 + n^2 = 1$

(ii) $\bar{a}.\bar{b} = ab\cos\theta$, $\bar{a} \times \bar{b} = ab\sin\theta\,\hat{n}$

(iii) $\nabla\phi = \hat{i}\dfrac{\partial\phi}{\partial x} + \hat{j}\dfrac{\partial\phi}{\partial y} + \hat{k}\dfrac{\partial\phi}{\partial z}$, where ϕ is a scalar and

$\nabla \equiv \hat{i}\dfrac{\partial}{\partial x} + \hat{j}\dfrac{\partial}{\partial y} + \hat{k}\dfrac{\partial}{\partial z}$ is a vector (operator)

(iv) div $\bar{q} = \nabla.\bar{q} = \dfrac{\partial u}{\partial x} + \dfrac{\partial v}{\partial y} + \dfrac{\partial w}{\partial z}, \bar{q} = (u, v, w)$

If $\nabla \cdot \bar{q} = 0,$ then $\bar{q}$ is said to be solenoidal vector.

(v) $d\bar{r} = \hat{i}dx + \hat{j}dy + \hat{k}dz, d\phi = \dfrac{\partial\varphi}{\partial x}dx + \dfrac{\partial\varphi}{\partial y}dy + \dfrac{\partial\varphi}{\partial z}dz$

and

$$\nabla\phi = \hat{i}\frac{\partial\phi}{\partial x} + \hat{j}\frac{\partial\phi}{\partial y} + \hat{k}\frac{\partial\phi}{\partial z},$$

Therefore,

$$d\phi = (\nabla\phi).\ d\bar{r}$$

(vi) $\text{Curl}\,\bar{q} = \nabla \times \bar{q} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ u & v & w \end{vmatrix}$

$$= \hat{i}\left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}\right) + \hat{j}\left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}\right) + \hat{k}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)$$

(vii) **(a)** Gradient of a scalar is a vector.
(b) Divergence of a scalar and curl of a scalar are meaningless.
(c) Divergence of a vector is a scalar and curl of a vector is a vector.

(viii) $\nabla\cdot\nabla\phi = \nabla^2\phi = \dfrac{\partial^2\phi}{\partial x^2}+\dfrac{\partial^2\phi}{\partial y^2}+\dfrac{\partial^2\phi}{\partial z^2}$

where ∇^2 is Laplacian operator.

(ix) Curl grad $\phi = 0$, div curl $\bar{q} = 0$

(x) Curl curl $\bar{q} = \text{grad div } \bar{q} - \nabla^2\bar{q}$

i.e. $\nabla^2\bar{q} = \text{grad}\,\text{div}\,\bar{q} - \text{curl}\,\text{curl}\,\bar{q}$

(xi) **Gauss's divergence theorem**

(a) $\int_S \bar{q}\cdot d\bar{S} = \int_V \text{div } \bar{q} \text{ dv}$

(b) $\int_S \hat{n}\times\bar{q}\,dS = \int_V \text{curl } \bar{q} \text{ dv}$

(xii) **Green's theorem**

(a) $\int_V \nabla\phi\cdot\nabla\psi dV = \int_S \phi\nabla\psi\cdot d\bar{S} - \int_V \phi\nabla^2\psi dV$

$= \int_S \psi\nabla\phi\cdot d\bar{S} - \int_V \psi\nabla^2\phi\cdot dV$

(b) $\int_V (\phi\nabla^2\psi - \psi\nabla^2\phi) dV = \int_V \left(\phi\dfrac{\partial\psi}{\partial n} - \psi\dfrac{\partial\phi}{\phi n}\right)dS$

(xiii) **Stoke's theorem** $\int_C \bar{q}\cdot d\bar{r} = \int_S \text{curl } \bar{q}\cdot d\bar{S} = \int_S \text{curl } \bar{q}\cdot\hat{n}\,dS$

(xiv) **Orthogonal curvilinear co-ordinates :**

Let there be three orthogonal families of surfaces

$$f_1(x, y, z) = \alpha,\ f_2(x, y, z) = \beta,\ f_3(x, y, z) = \gamma \qquad (1)$$

where x, y, z are Cartesian co-ordinates of a point P(x, y, z) in space. The surfaces

$$\alpha = \text{constant},\ \beta\ \ \text{constant},\ \gamma = \text{constant} \qquad (2)$$

form an orthogonal system in which every pair of surfaces is an orthogonal system. The values α, β, γ are called orthogonal curvilinear co-ordinates.
From three equations in (1), we can get

$$x = x(\alpha, \beta, \gamma),\ y = y(\alpha, \beta, \gamma),\ z = z(\alpha, \beta, \gamma)$$

The surfaces (2) are called co-ordinate surfaces.
Let $\bar{r}$ be the position vector of the point P(x, y, z)

i.e. $\bar{r} = x\,\hat{i} + y\,\hat{j} + z\,\hat{k} = \bar{r}\,(\alpha, \beta, \gamma)$

A tangent vector to the α-curve (β = constant, γ = constant) at P is $\frac{\partial \bar{r}}{\partial \alpha}$. A unit tangent vector is

$$\hat{e}_1 = \frac{\partial \bar{r}/\partial \alpha}{|\partial \bar{r}/\partial \alpha|}$$

or
$$\frac{\partial \bar{r}}{\partial \alpha} = h_1 \hat{e}_1$$

where $h_1 = \left|\frac{\partial \bar{r}}{\partial \alpha}\right| = \sqrt{\left(\frac{\partial x}{\partial \alpha}\right)^2 + \left(\frac{\partial y}{\partial \alpha}\right)^2 + \left(\frac{\partial z}{\partial \alpha}\right)^2}$

Similarly, $\hat{e}_2, \hat{e}_3$ are unit vectors along β-curve and γ-curve respectively such that

$$\frac{\partial \bar{r}}{\partial \beta} = h_2 \hat{e}_2, \frac{\partial \bar{r}}{\partial \gamma} = h_3 \hat{e}_3$$

Further,
$$d\bar{r} = \frac{\partial \bar{r}}{\partial \alpha} d\alpha + \frac{\partial \bar{r}}{\partial \beta} d\beta + \frac{\partial \bar{r}}{\partial \gamma} d\gamma$$

$$= h_1\, d\alpha\ \hat{e}_1 + h_2 d\beta\, \hat{e}_2 + h_3 d\gamma\, \hat{e}_3$$

Therefore,

$$(ds)^2 = d\bar{r}.d\bar{r} = h_1^2 d\alpha^2 + h_2^2 d\beta^2 + h_3^2 d\gamma^2$$

where $h_1\, d\alpha$, $h_2\, d\beta$, $h_3\, d\gamma$ are arc lengths along α, β and γ curves.
In orthogonal curvilinear co-ordinates, we have the following results.

(i) grad $\phi = \left(\frac{1}{h_1}\frac{\partial \phi}{\partial \alpha}, \frac{1}{h_2}\frac{\partial \phi}{\partial \beta}, \frac{1}{h_3}\frac{\partial \phi}{\partial \gamma}\right)$

(ii) If $\bar{q} = (q_1, q_2, q_3)$, then

$$\text{div } \bar{q} = \frac{1}{h_1 h_2 h_3}\left[\frac{\partial}{\partial \alpha}(h_2 h_3 q_1) + \frac{\partial}{\partial \beta}(h_3 h_1 q_2) + \frac{\partial}{\partial \gamma}(h_1 h_2 q_3)\right]$$

(iii) If curl $\bar{q} = \bar{\xi} = (\xi_1, \xi_2, \xi_3)$, then

$$\xi_1 = \frac{1}{h_2 h_3}\left[\frac{\partial}{\partial \beta}(h_3 q_3) - \frac{\partial}{\partial \gamma}(h_2 q_2)\right]$$

$$\xi_2 = \frac{1}{h_3 h_1}\left[\frac{\partial}{\partial \gamma}(h_1 q_1) - \frac{\partial}{\partial \alpha}(h_3 q_3)\right]$$

$$\xi_3 = \frac{1}{h_1h_2}\left[\frac{\partial}{\partial\alpha}(h_2q_2) - \frac{\partial}{\partial\beta}(h_1q_1)\right]$$

(iv) $$\nabla^2\phi = \frac{1}{h_1h_2h_3}\left[\frac{\partial}{\partial\alpha}\left(\frac{h_2h_3}{h_1}\frac{\partial\phi}{\partial\alpha}\right) + \frac{\partial}{\partial\beta}\left(\frac{h_3h_1}{h_2}\frac{\partial\phi}{\partial\beta}\right) + \frac{\partial}{\partial\gamma}\left(\frac{h_1h_2}{h_3}\frac{\partial\phi}{\partial\gamma}\right)\right].$$

The Cartesian co-ordinate system (x, y, z) is the simplest of all orthogonal co-ordinate systems. In many problems involving vector field theory, it is convenient to work with other two most common orthogonal co-ordinates i.e. cylindrical polar co-ordinates and spherical polar co-ordinates denoted respectively by (r, θ, z) and (r, θ, ψ). For cylindrical co-ordinates, $h_1 = 1$, $h_2 = r$, $h_3 = 1$. For spherical co-ordinates, $h_1 = 1$, $h_2 = r$, $h_3 = r \sin\theta$.

1. Fluid Dynamics

Fluid dynamics is the science treating the study of fluids in motion. By the term fluid, we mean a substance that flows i.e. which is not a solid. Fluids may be divided into two categories

(i) liquids which are incompressible i.e. their volumes do not change when the pressure changes

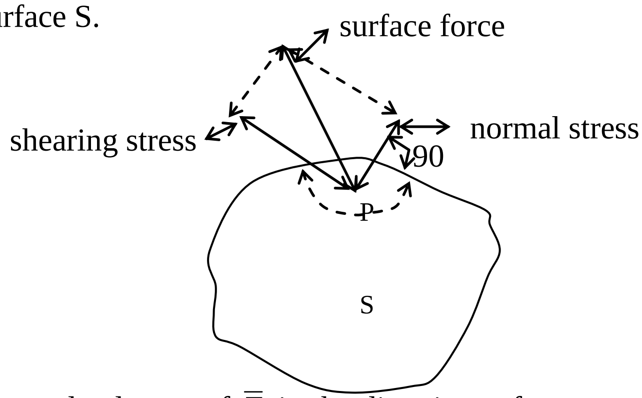
(ii) gases which are compressible i.e. they undergo change in volume whenever the pressure changes. The term hydrodynamics is often applied to the science of moving incompressible fluids. However, there is no sharp distinctions between the three states of matter i.e. solid, liquid and gases.

In **microscopic** view of fluids, matter is assumed to be composed of molecules which are in random relative motion under the action of intermolecular forces. In solids, spacing of the molecules is small, spacing persists even under strong molecular forces. In liquids, the spacing between molecules is greater even under weaker molecular forces and in gases, the gaps are even larger.

If we imagine that our microscope, with which we have observed the molecular structure of matter, has a variable focal length, we could change our observation of matter from the fine detailed microscopic viewpoint to a longer range **macroscopic** viewpoint in which we would not see the gaps between the molecules and the matter would appear to be continuously distributed. We shall take this macroscopic view of fluids in which physical quantities associated with the fluids within a given volume V are assumed to be distributed continuously and, within a sufficiently small volume δV, uniformly. This observation is known as **Continuum hypothesis**. It implies that at each point of a fluid, we can prescribe a unique velocity, a unique pressure, a unique density etc. Moreover, for a continuous or ideal fluid we can define a **fluid particle** as the fluid contained within an infinitesimal volume whose size is so small that it may be regarded as a geometrical point.

1.1. Stresses : Two types of forces act on a fluid element. One of them is **body force** and other is **surface force.** The body force is proportional to the mass of the body on which it acts while the surface force is proportional to the surface area and acts on the boundary of the body.

Suppose $\overline{F}$ is the surface force acting on an elementary surface area dS at a point P of the surface S.



Let F_1 and F_2 be resolved parts of $\overline{F}$ in the directions of tangent and normal at P. The normal force per unit area is called the **normal stress** and is also called **pressure**. The tangential force per unit area is called the **shearing stress**.

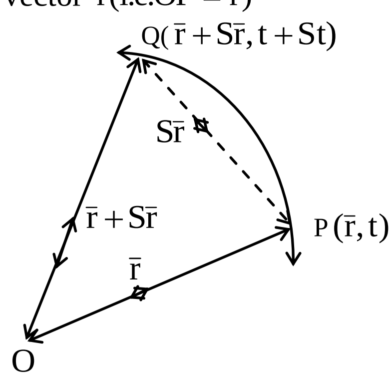
1.2. Viscosity : It is the internal friction between the particles of the fluid which offers resistance to the deformation of the fluid. The friction is in the form of tangential and shearing forces (stresses). Fluids with such property are called **viscous** or **real** fluids and those not having this property are called **inviscid or ideal or perfect fluids.**

Actually, all fluids are real, but in many cases, when the rates of variation of fluid velocity with distances are small, viscous effects may be ignored.

From the definition of body force and shearing stress, it is clear that body force per unit area at every point of surface of an ideal fluid acts along the normal to the surface at that point. Thus ideal fluid does not exert any shearing stress.

Thus, we conclude that viscosity of a fluid is that property by virtue of which it is able to offer resistance to shearing stress. It is a kind of molecular frictional resistance.

1.3. Velocity of Fluid at a Point : Suppose that at time t, a fluid particle is at the point P having position vector $\overline{r}$(i.e.$\overline{OP} = \overline{r}$)



and at time t + δt the same particle has reached at point Q having position vector $\bar{r}+\delta\bar{r}$. The particle velocity $\bar{q}$ at point P is

$$\bar{q} = \underset{\delta t \to 0}{\mathrm{Lt}} \frac{(\bar{r}+\delta\bar{r})-\bar{r}}{\delta t} = \underset{\delta t \to 0}{\mathrm{Lt}} \frac{\delta\bar{r}}{\delta t} = \frac{d\bar{r}}{dt}$$

where the limit is assumed to exist uniquely. Clearly $\bar{q}$ is in general dependent on both $\bar{r}$ and t, so we may write

$$\bar{q} = \bar{q}(\bar{r}, t) = \bar{q}(x, y, z, t),$$

$$\bar{r} = x\,\hat{i} + y\,\hat{j} + z\,\hat{k} \text{ (P has co-ordinates (x, y, z))}$$

Suppose,

$$\bar{q} = u\,\hat{i} + v\,\hat{j} + w\,\hat{k}$$

and since

$$\bar{q} = \frac{d\bar{r}}{dt} = \frac{dx}{dt}\hat{i} + \frac{dy}{dt}\hat{j} + \frac{dz}{dt}\hat{k},$$

therefore

$$u = \frac{dx}{dt}, \quad v = \frac{dy}{dt}, \quad w = \frac{dz}{dt}.$$

1.4. Remarks. (i) A point where $\bar{q} = \bar{0}$, is called a stagnation point.

(ii) When the flow is such that the velocity at each point is independent of time i.e. the flow pattern is same at each instant, then the motion is termed as steady motion, otherwise it is unsteady.

1.5. Flux across any surface : The flux i.e. the rate of flow across any surface S is defined by the integral

$$\int_S \rho(\bar{q}\cdot\hat{n})\,dS$$

where ρ is the density, $\bar{q}$ is the velocity of the fluid and $\hat{n}$ is the outward unit normal at any point of S.

Also, we define

Flux = density × normal velocity × area of the surface.

2. Eulerian and Lagrangian Methods (Local and Total range of change)

We have two methods for studying the general problem of fluid dynamics.

2.1. Eulerian Method : In this method, we fix a point in the space occupied by the fluid and observation is made of whatever changes of velocity, density pressure etc take place at that point. i.e. point is fixed and fluid particles are allowed to pass through it. If P(x, y, z) is the point under reference, then x, y, z do not depend upon the time parameter t, therefore $\dot{x}, \dot{y}, \dot{z}$ do not exist (dot denotes derivative w.r.t. time t).

Let f(x, y, z, t) be a scalar function associated with some property of the fluid (e.g. its density) i.e. f(x, y, z, t) = f($\bar{r}$, t), where $\bar{r} = x\hat{i} + y\hat{j} + z\hat{k}$ is the position vector of the point P, then

$$\frac{\partial f}{\partial t} = \underset{\delta t \to 0}{\text{Lt}} \frac{f(\bar{r}, t + \delta t) - f(\bar{r}, t)}{\delta t} \qquad (1)$$

Here, $\frac{\partial f}{\partial t}$ is called local time rate of change.

2.2. Lagrangian Method :- In this case, observations are made at each point and each instant, i.e., any particle of the fluid is selected and observation is made of its particular motion and it is pursued throughout its course.

Let a fluid particle be initially at the point (a, b, c). After lapse of time t, let the same fluid particle be at (x, y, z). It is obvious that x, y, z are functions of t. But since the particles which have initially different positions occupy different positions after the motion is allowed. Hence the co-ordinates of the final position i.e. (x, y, z) depend on (a, b c) also. Thus

$$x = f_1(a, b, c, t),\ y = f_2(a, b, c, t),\ z = f_3(a, b, c, t).$$

For this case, if f(x, y, z, t) be scalar function associated with the fluid, then

$$\frac{df}{dt} = \underset{\delta t \to 0}{\text{Lt}} \frac{f(\bar{r} + \delta\bar{r}, t + \delta t) - f(\bar{r}, t)}{\delta t} \qquad (2)$$

where $\dot{x}, \dot{y}, \dot{z}$ exist.

Here $\frac{df}{dt}$ is called an individual time rate or total rate or particle rate of change.

Now, we establish the relation between these two time rates (1) & (2).

We have

$$f = f(x, y, z, t)$$

Therefore,

$$\frac{df}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} + \frac{\partial f}{\partial z}\frac{dz}{dt} + \frac{\partial f}{\partial t}$$

$$= \left(\frac{\partial f}{\partial x}\hat{i} + \frac{\partial f}{\partial y}\hat{j} + \frac{\partial f}{\partial z}\hat{k}\right) \cdot \left(\frac{dx}{dt}\hat{i} + \frac{dy}{dt}\hat{j} + \frac{dz}{dt}\hat{k}\right) + \frac{\partial f}{\partial t}$$

$$= \nabla f \cdot \bar{q} + \frac{\partial f}{\partial t}$$

where

$$\bar{q} = \frac{dx}{dt}\hat{i} + \frac{dy}{dt}\hat{j} + \frac{dz}{dt}\hat{k} = (u, v, w)$$

Thus

$$\frac{df}{dt} = \frac{\partial f}{\partial t} + \bar{q} \cdot \nabla f \qquad (3)$$

2.3. Remarks. (i) The relation

$$\frac{df}{dt} = \frac{\partial f}{\partial t} + \bar{q}.\nabla f$$

$$\Rightarrow \qquad \frac{df}{dt} = \left(\frac{\partial}{\partial t} + \bar{q}.\nabla\right) f$$

$$\Rightarrow \qquad \frac{d}{dt} \equiv \frac{\partial}{\partial t} + \bar{q}.\nabla$$

The operator $\frac{d}{dt}\left(\text{also denoted by}\frac{D}{Dt}\right)$ is called Lagrangian operator or material derivative i.e. time rate of change in Lagrangian view. Sometimes, it is called 'differentiation following the fluid'.

(ii) Similarly, for a vector function $\bar{F}(x, y, z, t)$ associated with some property of the fluid (e.g. its velocity, acceleration), we can show that

$$\frac{d\bar{F}}{dt} = \frac{\partial \bar{F}}{\partial t} + \bar{q} \cdot \nabla \bar{F}$$

Hence the relation (3) holds for both scalar and vector functions associated with the moving fluid.

(iii) The Eulerian method is sometimes also called the flux method.

(iv) Both Lagrangian and Eulerian methods were used by Euler for studying fluid dynamics.

(v) Lagrangian method resembles very much with the dynamics of a particle

(vi) The two methods are essentially equivalent, but depending upon the problem, one has to judge whether Lagrangian method is more useful or the Eulerian.

3. Streamlines, Pathlines and Streaklines

3.1. Streamlines : It is a curve drawn in the fluid such that the direction of the tangent to it at any point coincides with the direction of the fluid velocity vector $\bar{q}$ at that point. At any time t, let $\bar{q}$ = (u, v, w) be the velocity at each point P(x, y, z) of the fluid. The direction ratios of the tangent to the curve at P(x, y, z) are $d\bar{r}$ = (dx, dy, dz) since the tangent and the velocity at P have the same direction, therefore $\bar{q} \times d\bar{r} = \bar{0}$

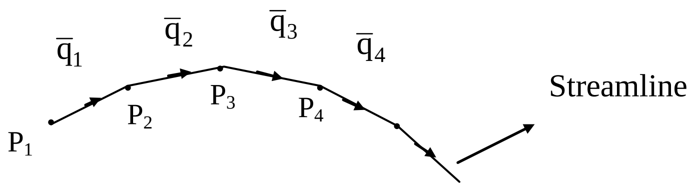
i.e. $(u\hat{i} + v\hat{j} + w\hat{k}) \times (dx\hat{i} + dy\hat{j} + dz\hat{k}) = \bar{0}$

i.e. $(vdy - w\,dy)\,\hat{i} + (wdx - udz)\hat{j} + (udy - vdx)\hat{k} = \bar{0}$

i.e. vdz – wdy = 0 = wdx – udz = udy – vdx

$$\Rightarrow \frac{dx}{u} = \frac{dy}{v} = \frac{dz}{w}$$

These are the differential equations for the streamlines.
i.e. their solution gives the streamlines.



In the figure, if $\bar{q}_1, \bar{q}_2, \bar{q}_3$,....... denote the velocities at neighbouring points P_1, P_2, P_3,…., then the small straight line segments P_1P_2, P_2P_3, P_3P_4… collectively give the approximate form of the streamlines.

3.2. Pathlines: When the fluid motion is steady so that the pattern of flow does not vary with time, the paths of the fluid particles coincide with the streamlines. But in case of unsteady motion, the flow pattern varies with time and the paths of the particles do not coincide with the streamlines. However, the streamline through any point P does touch the pathline through P. Pathlines are the curves described by the fluid particles during their motion i.e. these are the paths of the particles.
The differential equations for pathlines are

$$\frac{d\bar{r}}{dt} = \bar{q} \text{ i.e.} \frac{dx}{dt} = u, \frac{dy}{dt} = v, \frac{dz}{dt} = w \qquad (1)$$

where now (x, y, z) are the Cartesian co-ordinates of the fluid particle and not a fixed point of space. The equation of the pathline which passes through the point (x_0, y_0, z_0), which is fixed in space, at time t = 0 say, is the solution of (1) which satisfy the initial condition that $x = x_0$, $y = y_0$, $z = z_0$ when t = 0. The solution gives a set of equations of the form

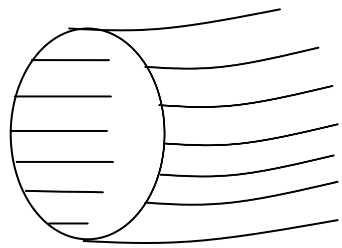
$$\left.\begin{aligned} x &= x(x_0, y_0, z_0, t) \\ y &= y(x_0, y_0, z_0, t) \\ z &= z(x_0, y_0, z_0, t) \end{aligned}\right\} \qquad (2)$$

which, as t takes all values greater than zero, will trace out the required pathline.

3.3. Remarks : (i) Streamlines give the motion of each particle at a given instant whereas pathlines give the motion of a given particle at each instant.

We can make these observations by using a suspension of aluminium dust in the liquid.

(ii) If we draw the streamlines through every point of a closed curve in the fluid, we obtain a **stream tube**. A stream tube of very small cross-section is called a **stream filament.**



(iii) The components of velocity at right angle to the streamline is always zero. This shows that there is no flow across the streamlines. Thus, if we replace the boundary of stream tube by a rigid boundary, the flow is not affected. The principle of conservation of mass then gives that the flux across any cross-section of the stream tube should be the same.

3.4. Streaklines : In addition to streamlines and pathlines, it is useful for observational purpose to define a streakline. This is the curve of all fluid particles which at some time have coincided with a particular fixed point of space. Thus, a streakline is the locus of different particles passing through a fixed point. The streakline is observed when a neutrally buoyant marker fluid is continuously injected into the flow at a fixed point of space from time $\tau = -\infty$. The marker fluid may be smoke if the main flow involves a gas such as air, or a dye such as potassium permanganate ($KMnO_4$) if the main flow involves a liquid such as water.

If the co-ordinates of a particle of marker fluid are (x, y, z) at time t and the particle coincided with the injection point (x_0, y_0, z_0) at some time τ, where $\tau \leq t$, then the time-history (streakline) of this particle is obtained by solving the equations for a pathline, subject to the initial condition that $x = x_0$, $y = y_0$, $z = z_0$ at $t = \tau$. As τ takes all possible values in the angle $-\infty \leq \tau \leq t$, the locations of all fluid particles on the streakline through (x_0, y_0, z_0) are obtained. Thus, the equation of the streakline at time t is given by

$$\left.\begin{aligned} x &= x(x_0, y_0, z_0, t, \tau) \\ y &= y(x_0, y_0, z_0, t, \tau) \\ z &= z(x_0, y_0, z_0, t, \tau) \end{aligned}\right\} (-\infty \leq \tau \leq t) \qquad (2)$$

3.5. Remark: (i) For a steady flow, streaklines also coincide with streamlines and pathlines.

(ii) Streamlines, pathlines and streaklines are termed as **flowlines** for a fluid.

4. Velocity Potential

Suppose that $\bar{q} = u\,\hat{i} + v\,\hat{j} + w\,\hat{k}$ is the velocity at any time t at each point P(x, y, z) of the fluid. Also suppose that the expression u dx + vdy + wdz is an exact differential, say – dϕ.

Then, –dϕ = udx + vdy + wdz

i.e. $-\left(\frac{\partial\phi}{\partial x}dx + \frac{\partial\phi}{\partial y}dy + \frac{\partial\phi}{\partial z}dz + \frac{\partial\phi}{\partial t}dt\right)$ = u dx + vdy + wdz where $\phi = \phi(x, y, z, t)$ is some scalar function, uniform throughout the entire field of flow.

Therefore,

$$u = \frac{-\partial\phi}{\partial x}, v = \frac{-\partial\phi}{\partial y}, w = \frac{-\partial\phi}{\partial z}, \frac{\partial\phi}{\partial t} = 0$$

But

$$\frac{\partial\phi}{\partial t} = 0 \quad \Rightarrow \phi = \phi(x, y, z)$$

Hence

$$\bar{q} = u\,\hat{i} + v\,\hat{j} + w\,\hat{k} = -\left(\frac{\partial\phi}{\partial x}\hat{i} + \frac{\partial\phi}{\partial y}\hat{j} + \frac{\partial\phi}{\partial z}\hat{k}\right) = -\nabla\phi$$

where ϕ is termed as the **velocity potential** and the flow of such type is called flow of potential kind.

In the above definition, the negative sign in $\bar{q} = -\nabla\phi$ is a convention and it ensures that flow takes place from higher to lower potentials. The level surfaces ϕ(x, y, z, t) = constant, are called **equipotentials** or **equipotential surfaces**.

4.1. Theorem : At all points of the field of flow the equipotentials (i.e. equipotential surfaces) are cut orthogonally by the streamlines.

Proof. If the fluid velocity at any time t be $\bar{q}$ = (u, v, w), then the equations of streamlines are

$$\frac{dx}{u} = \frac{dy}{v} = \frac{dz}{w} \tag{1}$$

The surfaces given by

$$\bar{q} \cdot d\bar{r} = 0 \text{ i.e. } udx + vdy + wdz = 0 \qquad (2)$$

are such that the velocity is at right angles to the tangent planes. The curves (1) and the surfaces (2) cut each other orthogonally. Suppose that the expression on the left hand side of (2) is an exact differential, say, $-d\phi$, then

$$d\phi = udx + vdy + wdz \qquad (3)$$

where ϕ is velocity potential.

The necessary and sufficient condition for the relations.

$$u = -\frac{\partial\phi}{\partial x}, v = -\frac{\partial\phi}{\partial y}, w = -\frac{\partial\phi}{\partial z}$$

i.e. $\bar{q} = -\nabla\phi$ to hold is

$$\text{curl } \bar{q} = \text{curl } (-\nabla\phi) = \bar{0} \qquad (4)$$

The solution of (2) i.e. $d\phi = 0$ is

$$\phi(x, y, z) = \text{const} \qquad (5)$$

The surfaces (5) are called equipotentials. Thus the equipotentials are cut orthogonally by the stream lines.

4.2. Note : When curl $\bar{q} = \bar{0}$, the flow is said to be irrotational or of potential kind, otherwise it is rotational. For irrotational flow, $\bar{q} = -\nabla\phi$.

4.3. Example. The velocity potential of a two dimensional flow is $\phi = c\,xy$. Find the stream lines

Solution. The stream lines are given by

$$\frac{dx}{u} = \frac{dy}{v} = \frac{dz}{w}$$

where $\bar{q} = (u, v, w)$

For an irrotational motion (i.e. motion of potential kind)

we have

$$\text{curl } \bar{q} = 0 = \text{curl } (-\nabla\phi)$$

i.e. $\bar{q} = -\nabla\phi$, where ϕ is the velocity potential.

From here,

$$(u, v, w) = -\left(\frac{\partial\phi}{\partial x}, \frac{\partial\phi}{\partial y}, \frac{\partial\phi}{\partial z}\right) = -(cy, cx, 0)$$

i.e $\quad u = -cy, v = -cx, w = 0$

Therefore, streamlines are

$$\frac{dx}{-cy} = \frac{dy}{-cx} = \frac{dz}{0}$$

i.e. $\quad x\,dx - y dy = 0, dz = 0$

i.e. $\quad x^2 - y^2 = a^2, z = K$

which are rectangular hyperbolae

4.4. Example. If the speed of fluid is everywhere the same, the streamlines are straight.

Solution. The streamlines are given by the differential equations.

$$\frac{dx}{u} = \frac{dy}{v} = \frac{dz}{w}$$

where u, v, w are constants. The solutions are

$$vx - uy = \text{constant}, vz - wy = \text{constant}$$

The intersection of these planes are necessarily straight lines. Hence the result.

4.5. Example. Find the stream lines and path lines of the particles for the two dimensional velocity field.

$$u = \frac{x}{1+t}, v = y, w = 0$$

Solution. For streamlines, the differential equations are

$$\frac{dx}{u} = \frac{dy}{v} = \frac{dz}{w}$$

Therefore,

$$(1+t)\frac{dx}{x} = \frac{dy}{y} = \frac{dz}{0}$$

Here t = constant = t_0 (at given instant), therefore the solutions are

$$(1+t_0)\log x = \log y + c_1,\ z = c_2$$

or $$\log x^{1+t_0} = \log y + \log a,\ z = c_2.$$

or $$x^{1+t_0} = ay,\ z = c_2.$$

which are the required stream lines.

For path lines, we have

$$\frac{dx}{dt} = u, \frac{dy}{dt} = v, \frac{dz}{dt} = w$$

Therefore,

$$\frac{dx}{dt} = \frac{x}{1+t}, \frac{dy}{dt} = y, \frac{dz}{dt} = 0$$

$$\Rightarrow \quad \frac{dx}{x} = \frac{dt}{1+t}, \frac{dy}{y} = dt,\ dz = 0$$

$$\Rightarrow \quad \log. x = \log(1+t) + \log a,\ \log y = t + \log b,\ z = c$$

$$\Rightarrow \quad x = a(1+t),\ y = be^t,\ z = c$$

$$\Rightarrow \quad y = be^{\frac{x-a}{a}};\ z = c$$

which are the required path lines.

4.6. Note. In case of path lines, t must be eliminated since these give the motion at each instant (i.e. independent of t).

4.7. Example. Obtain the equations of the streamlines, path lines and streaklines which pass through (l, l, 0) at t = 0 for the two dimensional flow

$$u = \frac{x}{t_0}\left(1+\frac{t}{t_0}\right), v = \frac{y}{t_0},\ w = 0.$$

where l and t_0 are constants having respectively the dimensions of length and time.

Solution. We define the dimensionless co-ordinates X, Y, Z and time T by writing

$$X = \frac{x}{l},\ Y = \frac{y}{l},\ Z = \frac{z}{l},\ T = \frac{t}{t_0}$$

such that $dX = \frac{1}{l}dx,\ dY = \frac{1}{l}dy,\ dZ = \frac{1}{l}dz,\ dT = \frac{1}{t_0}dt$

and $\quad u = \frac{Xl}{t_0}(1+T), v = \frac{Yl}{t_0},\ w = 0$

Streamlines are given by

$$\frac{dx}{u} = \frac{dy}{v} = \frac{dz}{w}$$

$$\Rightarrow \frac{t_0 l\, dX}{Xl(1+T)} = \frac{t_0 l\, dY}{Yl} = \frac{l dZ}{0}$$

$$\Rightarrow \quad \frac{dX}{X(1+T)} = \frac{dY}{Y} = \frac{dZ}{0}$$

Integrating these, we get

$$Z = \text{constant} = C_1 \text{ (say)} \qquad (1)$$

and $\log X = (1+T) \log Y + \log C_2$, where C_2 is constant

$$\Rightarrow X = C_2\, Y^{(1+T)} \qquad (2)$$

As variables X, Y, Z and T are independent and C_1 & C_2 are constants, equations (1) & (2) give the complete family of stream lines at all times $t = t_0T$. In particular, $X = 1 = Y$, $Z = 0$ and $T = 0 \Rightarrow C_1 = 0$, $C_2 = 1$ and we get stream line as $Y = X$ i.e. $y = x$ and $z = 0$.

Pathlines are given by

$$\frac{dX}{dT} = X(1+T), \frac{dY}{dT} = Y, \frac{dZ}{dT} = 0$$

Now, X, Y, Z are the dimensionless co-ordinates of a fluid particle and are functions of T.

Therefore, $\quad \frac{dX}{X} = (1+T)dT \quad \Rightarrow \log X = \left(T + \frac{T^2}{2}\right) + \log K_1$

$\Rightarrow \qquad X = K_1 \ e^{T+T^2/2}$ (3)

$$\frac{dY}{dT} = Y \quad \Rightarrow \frac{dY}{Y} = dT \ \Rightarrow \log Y = T + \log K_2$$

$\Rightarrow \qquad Y = K_2 \, e^T.$ (4)

$dZ = 0 \Rightarrow Z = \text{constant} = K_3$ (5)

These are the parametric equations of path lines. The path line through P(1, 1, 0) i.e. X = 1 = Y, Z = 0, T = 0 is obtained when $K_1 = K_2 = 1$, $K_3 = 0$

$$\Rightarrow X = e^{T+\frac{T^2}{2}}, \ Y = e^T, \ Z = 0$$

Elimination of T gives.

$$X = \ e^{T\left(1+\frac{T}{2}\right)} = \left[e^T\right]^{\left(1+\frac{T}{2}\right)} = Y^{\left(1+\frac{T}{2}\right)} \ = \ Y^{\left(1+\frac{1}{2}\log Y\right)}, Z = 0$$

The pathline which passes through X = Y = 1, Z = 0 when T = τ is given by

$$X = \exp.\left[T + \frac{1}{2}T^2 - \tau - \frac{1}{2}\tau^2\right],$$

$$Y = \exp\,(T-\tau),\ Z = 0$$

These are the parametric equations of the streaklines true for all values of T. At T = 0, the equations give

$$X = \exp.\left(-\tau - \frac{\tau^2}{2}\right),\ Y = \exp(-\tau),\ Z = 0.$$

Eliminating τ, we have.

$$-\tau = \log Y \text{ i.e. } \tau = -\log Y$$

Therefore,

$$X = \exp\ \left(-\tau(1+\tau/2)\right) = \left[e^{-\tau}\right]^{\left(1+\frac{\tau}{2}\right)} = \left(Y\right)^{(1+\tau/2)} = Y^{\left(1-\frac{\log Y}{2}\right)}, Z = 0$$

4.8. Article. To obtain the differential equations for streamlines in cylindrical and spherical co-ordinates

We know that the streamlines are obtained from the differential equations

$$\bar{q} \times d\bar{r} = \bar{0} \qquad (1)$$

where $\bar{q}$ is the velocity vector and $\bar{r}$ is the position vector of a liquid particle.

If the motion is irrotational, then

$$\bar{q} = -\nabla\phi$$

Therefore, the differential equations (1) become

$$\nabla\phi \cdot \times d\,\bar{r} = \bar{0} \qquad (2)$$

(i) In cylindrical co-ordinates (r, θ, z), we have

$$d\,\bar{r} = (dr, r\, d\theta, dz)$$

and

$$\nabla\phi = \text{grad } \phi = \left(\frac{\partial\phi}{\partial r}, \frac{1}{r}\frac{\partial\phi}{\partial\theta}, \frac{\partial\phi}{\partial z}\right)$$

Thus, the different equations (2) become

$$\left(\frac{\partial\phi}{\partial r}, \frac{1}{r}\frac{\partial\phi}{\partial\theta}, \frac{\partial\phi}{\partial z}\right) \times (dr, rd\theta, dz) = \bar{0}$$

$$\Rightarrow \frac{dr}{\partial\phi/\partial r} = \frac{rd\theta}{1/r \cdot \partial\phi/\partial\theta} = \frac{dz}{\partial\phi/\partial z}\,. \qquad (3)$$

(ii) In spherical co-ordinates (r, θ, ψ), we have

$$d\,\bar{r} = (dr, rd\theta, r \sin\theta\, d\psi)$$

and $$\nabla\phi = \text{grad } \phi = \left(\frac{\partial\phi}{\partial r}, \frac{1}{r}\frac{\partial\phi}{\partial\theta}, \frac{1}{r\sin\theta}\frac{\partial\phi}{\partial\psi}\right)$$

The differential equations (2) become.

$$\left(\frac{\partial\phi}{\partial r}, \frac{1}{r}\frac{\partial\phi}{\partial\theta}, \frac{1}{r\sin\theta}\frac{\partial\phi}{\partial\psi}\right) \times (dr, rd\theta, r \sin\theta\, d\psi) = \bar{0}$$

$$\Rightarrow \frac{dr}{\partial\phi/\partial r} = \frac{rd\theta}{\frac{1}{r}\partial\phi/\partial\theta} = \frac{r\sin\theta d\psi}{\frac{1}{r\sin\theta}\partial\phi/\partial\psi} \qquad (4)$$

Equations (3) and (4) are the required differential equations.

4.9. Example. Show that if the velocity potential of an irrotational fluid motion is $\phi = \frac{A}{r^2}\psi\cos\theta,$ where (r, θ, ψ) are the spherical polar co-ordinates of any point, the lines of flow lie on the surface $r = k \sin^2\theta$, k being a constant.

Solution. The differential equations for lines of flow (streamlines) are

$$\frac{dr}{\partial\phi/\partial r} = \frac{rd\theta}{\frac{1}{r}\partial\phi/\partial\theta} = \frac{r\sin\theta d\psi}{\frac{1}{r\sin\theta}\partial\phi/\partial\psi}$$

From first two members, we have

$$\frac{dr}{-\psi\frac{2A}{r^3}\cos\theta} = \frac{rd\theta}{\frac{1}{r}\left(-\psi\frac{A}{r^2}\sin\theta\right)}$$

$$\Rightarrow \frac{dr}{\cos\theta} = \frac{2rd\theta}{\sin\theta} \qquad \Rightarrow \frac{dr}{r} = 2\frac{\cos\theta}{\sin\theta}d\theta$$

$$\Rightarrow \log r = 2 \log \sin\theta + \log k \quad \Rightarrow r = k \sin^2\theta$$

Hence the result.

4.10. Note. In the above example, the velocity potential, in Cartesian co-ordinates, can be written as

$$\phi = A(x^2 + y^2 + z^2)^{-3/2}\, z.\, \tan^{-1}\left(\frac{y}{x}\right),$$

where

$$x = r \sin\theta \cos\psi,\ y = r \sin\theta \sin\psi,\ z = r \cos\theta$$

are spherical polar substitutions.

Also, the streamlines $r = k \sin^2\theta$ can be written as $r^3 = k\, r^2 \sin^2\theta$

$$\Rightarrow \quad (x^2 + y^2 + z^2)^{3/2} = k\,(x^2 + y^2)$$

i.e. $\qquad x^2 + y^2 + z^2 = k^{2/3}\,(x^2 + y^2)^{2/3}$

which are the streamlines in Cartesian co-ordinates.

4.11. Example. At the point in an incompressible fluid having spherical polar co-ordinates (r, θ, ψ), the velocity components are ($2M\bar{r}^3$ cosθ, $M\bar{r}^3$sinθ, 0) where M is a constant. Show that velocity is of potential kind. Find the velocity potential and the equations of streamlines.

Solution. Here $d\,\bar{r} = dr\,\hat{r} + rd\theta\,\hat{\theta} + r\sin\theta d\psi\,\hat{\psi}$

$$\bar{q} = 2M\bar{r}^3\cos\theta\,\hat{r} + M\bar{r}^3\sin\theta\,\hat{\theta}$$

Then,

$$\text{curl }\bar{q} = \frac{1}{r^2\sin\theta}\begin{vmatrix} \hat{r} & r\hat{\theta} & r\sin\theta\,\hat{\psi} \\ \partial/\partial r & \partial/\partial\theta & \partial/\partial\psi \\ 2M\bar{r}^3\cos\theta & M\bar{r}^2\sin\theta & 0 \end{vmatrix}$$

$$= \frac{1}{r^2\sin\theta}\left[\hat{r}\cdot 0 + r\hat{\theta}\cdot 0 + r\sin\theta\,\hat{\psi}(-2M\bar{r}^3\sin\theta + 2M\bar{r}^3\sin\theta)\right] = \bar{0}$$

Therefore, the flow is of potential kind.

Now, using the relation $\bar{q} = -\nabla\varphi = -\left(\frac{\partial\varphi}{\partial r}\hat{r} + \frac{1}{r}\frac{\partial\varphi}{\partial\theta}\hat{\theta} + \frac{1}{r\sin\theta}\frac{\partial\varphi}{\partial\psi}\hat{\psi}\right)$, we have

$$2M\bar{r}^3\cos\theta\,\hat{r} + M\bar{r}^3\sin\theta\,\hat{\theta} = \left(-\frac{\partial\phi}{\partial r}\hat{r} - \frac{1}{r}\frac{\partial\phi}{\partial\theta}\hat{\theta} - \frac{1}{r\sin\theta}\frac{\partial\phi}{\partial\psi}\hat{\psi}\right)$$

From here,

$$\frac{\partial\phi}{\partial r} = 2M\bar{r}^3\cos\theta, -\frac{\partial\phi}{\partial\theta} = M\bar{r}^2\sin\theta, \frac{\partial\phi}{\partial\psi} = 0$$

Therefore,

$$d\phi = \frac{\partial\phi}{\partial r}dr + \frac{\partial\phi}{\partial\theta}d\theta + \frac{\partial\phi}{\partial\psi}d\psi$$

$$= (-2M\bar{r}^3\cos\theta)dr - (M\bar{r}^2\sin\theta)d\theta$$

$$= d\,(M\bar{r}^2\cos\theta)$$

Integrating, we get

$$\phi = M\bar{r}^2\cos\theta$$

which is the required velocity potential.

The streamlines are given by

$$\frac{dr}{-\frac{\partial\phi}{\partial r}} = \frac{r d\theta}{-\frac{1}{r}\frac{\partial\phi}{\partial\theta}} = \frac{r\sin\theta d\psi}{-\frac{1}{r\sin\theta}\frac{\partial\phi}{\partial\psi}}$$

or $$\frac{dr}{2M\bar{r}^3\cos\theta} = \frac{r d\theta}{M\bar{r}^3\sin\theta} = \frac{r\sin\theta d\psi}{0}$$

From the last term, ψ = constant.

From the first two terms, we get

$$\frac{dr}{r} = \frac{2\cos\theta}{\sin\theta} d\theta = 2 \cot\theta \, d\theta$$

Integrating, we get

$$\log r = \log \sin^2\theta + \text{constant}$$

$$\Rightarrow r = A \sin^2\theta , \ \psi = \text{constant}$$

The equation ψ = const. shows that the streamlines lie in planes which pass through the axis of symmetry $\theta = 0$.

5. Irrotational and Rotational Motion, Vortex Lines

5.1. Vorticity. If $\bar{q}$ = (u, v, w) be the velocity vector of a fluid particle, then the vector $\bar{\xi}$ defined by

$$\bar{\xi} = \text{curl } \bar{q} = \nabla \times \bar{q}$$

is called the **vortex vector** or **vorticity** and it's components are (ξ_1, ξ_2, ξ_3), given by

$$\xi_1 = \frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}, \xi_2 = \frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}, \ \xi_3 = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}$$

5.2. Vortex Motion (or Rotational Motion). The fluid motion is said to be rotational if

$$\bar{\xi} = \text{curl } \bar{q} \neq \bar{0}$$

5.3. Irrotational Motion. If $\bar{\xi} = \text{curl } \bar{q} = \bar{0}$, then the fluid motion is said to be irrotational or of potential kind and then $\bar{q} = -\nabla\phi$.

5.4. Vortexline. It is a curve in the fluid such that the tangent at any point on the curve has the direction of the vorticity vector $\bar{\xi}$.

The differential equations of vortexlines are given by $\bar{\xi} \times d\bar{r} = \bar{0}$

$$\text{i.e. } \frac{dx}{\xi_1} = \frac{dy}{\xi_2} = \frac{dz}{\xi_3}$$

where $\bar{\xi}. = (\xi_1, \xi_2, \xi_3)$.

5.5. Vortex Tube. It is the locus of vortex line drawn at each point of a closed curve i.e. vortex tube is the surface formed by drawing vortex lines through each point of a closed curve in the fluid.

A vortex tube with small cross-section is called a **vortex filament**.

5.6. Flow. Let A and B be two points in the fluid.

Then $\int_B^A \bar{q} \cdot d\bar{r}$ is called the flow along any path from A to B

If motion is irrotational then $\bar{q} = -\nabla\phi$ and

$$\text{flow} = -\int_A^B \nabla\phi \cdot d\bar{r} = -\int_A^B d\phi = \phi(A) - \phi(B)$$

5.7. Circulation It is the flow round a closed curve. If C be the closed curve in a moving fluid, then circulation Γ about C is given by

$$\Gamma = \oint_C \bar{q} \cdot d\bar{r} = \int_S \hat{n} \cdot \text{curl}\,\bar{q}\, dS = \int_S \hat{n} \cdot \bar{\xi}\, dS.$$

If the motion is irrotational, then $\bar{q} = -\nabla\phi$ and thus,

$$\Gamma = -\oint_C \nabla\varphi \cdot d\bar{r} = -\oint_C d\varphi = \varphi(A) - \varphi(A) = 0,$$

where A is any point on the curve C. This shows that for an irrotational motion, circulation is zero.

5.8. Theorem :-The necessary and sufficient condition such that the vortex lines are at right angles to the stream lines, is

$$(u, v, w) = \mu\left(\frac{\partial\phi}{\partial x}, \frac{\partial\phi}{\partial y}, \frac{\partial\phi}{\partial z}\right)$$

i.e. $\bar{q} = \mu\nabla\phi$, where μ and ϕ are functions of x, y, z and t.

Proof. Necessary condition:- We know that the differential equation

$\bar{q}\cdot d\bar{r} = 0$ is integrable if $\left(\frac{\partial Q}{\partial z}-\frac{\partial R}{\partial y}\right)+\ldots+=0$	$\because$ pdx + Qdy + Rdz = 0 is integrable if

$$\bar{q}\cdot \text{curl}\,\bar{q} = 0 \qquad \text{(exactness condition)}$$

i.e. $\quad \bar{q}\cdot\bar{\xi} = 0, \quad \bar{\xi} = \text{curl }\bar{q}$

This shows that the streamlines are at right angles to the vortex lines. Thus the streamlines and vortex lines are at right angles to each other if the differential equation $\bar{q}\cdot d\bar{r} = 0$ is integrable.

The exactness condition $\bar{q}\cdot\text{curl}\bar{q} = 0$ implies that $\bar{q} = -\nabla\phi$.

But $\text{curl}\,\bar{q} = \text{curl}(-\nabla\phi) = \bar{0}$. Thus the vortexlines do not exist. The equations $\bar{q}\cdot d\bar{r} = 0$ are therefore not exact.

So, there exists an integrating factor μ(function of x, y, z, t) such that

$$\mu^{-1}\bar{q}\cdot d\bar{r} = 0 \text{ is integrable.}$$

If this differential equation is integrable, then we can write

$\mu^{-1}\bar{q}\cdot d\bar{r} = d\phi$, where ϕ is a scalar function of x, y, z, t

$\Rightarrow \mu^{-1}\bar{q}\cdot d\bar{r} = \nabla\phi\cdot d\bar{r} \qquad | \because d\phi = \nabla\phi\cdot d\bar{r}$

$\Rightarrow \bar{q} = \mu\,\nabla\phi.$

Sufficient condition :- Let us take $\bar{q} = \mu\,\nabla\phi \quad \Rightarrow \nabla\phi = \mu^{-1}\bar{q}$

Then, curl $\bar{q}$ = curl $(\mu\nabla\phi)$

$\Rightarrow \bar{\xi} = \nabla\times(\mu\nabla\phi) = \mu(\nabla\times\nabla\phi) + \nabla\mu\times\nabla\phi = \nabla\mu\times\nabla\phi$

Therefore,

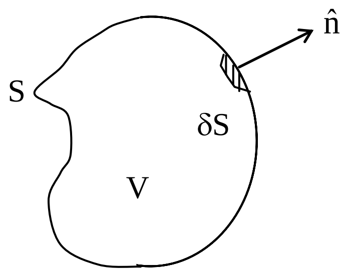
$$\bar{q}\cdot\bar{\xi} = (\nabla\mu\times\nabla\phi)\cdot\bar{q} = \nabla\mu\cdot(\nabla\phi\times\bar{q})$$

$$= \nabla\mu\cdot(\mu^{-1}\bar{q}\times\bar{q}) = 0$$

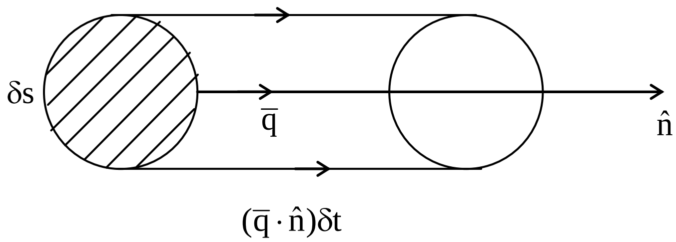
This shows that the directions of streamlines and vortexlines are at right angles to each other.

6. Equation of Continuity

6.1. Equation of Continuity by Euler's Method (Equation of conservation of Mass): Equation of continuity is obtained by using the fact that the mass contained inside a given volume of fluid remains constant throughout the motion. Consider a region of fluid in which there is no inlets (sources) or outlets (sinks) through which the fluid can enter or leave the region. Let S be the surface enclosing volume V of this region and let $\hat{n}$ denotes the unit vector normal to an element δS of S drawn outwards.
Let $\overline{q}$ be the fluid velocity and ρ be the fluid density.



First, we consider the mass of fluid which leaves V by flowing across an element δS of S in time δt. This quantity is exactly that which is contained in a small cylinder of cross-section δS of length $(\overline{q}\cdot\hat{n})\,\delta t$.
Thus, mass of the fluid is = density $\times$ Volume = $\rho\ (\overline{q}\cdot\hat{n})\,St.\ \delta S$



Hence the rate at which fluid leaves V by flowing across the element $S\delta$ is

$$\rho\,(\overline{q}\cdot\hat{n})\,\delta S.$$

Summing over all such elements δS, we obtain the rate of flow of fluid coming out of V across the entire surface S. Hence, the rate at which mass flows out of the region V is

$$\int_S \rho(\overline{q}\cdot\hat{n})dS = \int_S (\rho\overline{q})\cdot\hat{n}dS$$

By Gauss divergence the
$$\int_S \overline{F}\cdot\hat{n}\,dS = \int_V \nabla\cdot\overline{F}dV.$$

$$= \int_V \operatorname{div}(\rho\overline{q})dV \qquad (1)$$

Now, the mass M of the fluid possessed by the volume V of the fluid is

$M = \int_V \rho\, dV$, where $\rho = \rho(x, y, z, t)$ with (x, y, z) the Cartesian co-ordinates of a general point of V, a fixed region of space. Since the space co-ordinates are independent of time t, therefore the rate of increase of mass within V is

$$\frac{dM}{dt} = \frac{d}{dt}\left(\int_V \rho dV\right) = \int_V \frac{\partial\rho}{\partial t} dV \qquad | \text{ V does not change w.r.t. time} \quad (2)$$

But the considered region is free from source or sink i.e. the mass is neither created nor destroyed, therefore the total rate of change of mass is zero and thus from (1) & (2), we get

$$\int_V \frac{\partial\rho}{\partial t} dV. + \int_V \operatorname{div}(\rho\overline{q})dV = 0$$

$$\Rightarrow \qquad \int_V \left[\frac{\partial\rho}{\partial t} + \operatorname{div}(\rho\overline{q})\right] dV = 0$$

Since V is arbitrary, we conclude that at any point of the fluid which is neither a source nor a sink,

$$\frac{\partial\rho}{\partial t} + .\operatorname{div}(\rho\overline{q}) = 0$$

i.e. $$\frac{\partial\rho}{\partial t} + .\nabla.(\rho\overline{q}) = 0 \qquad (3)$$

Equation (3) is known as equation of continuity.

Corollary (1). We know that

$$\operatorname{div}(\rho\overline{q}) = \rho \operatorname{div}\overline{q} + .\overline{q}.(\operatorname{grad}\rho)$$

Therefore, (3) takes the form

$$\frac{\partial\rho}{\partial t} + \rho(\nabla\cdot\overline{q}) + (\overline{q}\cdot\nabla)\rho = 0 \qquad (4)$$

Corollary (2). We know that the differential operator $\frac{D}{Dt}$ is given by

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + (\overline{q}\cdot\nabla)$$

Therefore, from (4), we obtain the equation of continuity as $\frac{D\rho}{Dt} + .\rho(\nabla\cdot\overline{q}) = 0$

i.e. $$\frac{D\rho}{Dt} + \rho \operatorname{div} \bar{q} = 0 \qquad (5)$$

Corollary (3). Equation (5) can be written as

$$\frac{1}{\rho}\frac{D\rho}{Dt} + \operatorname{div} \bar{q} = 0$$

$\Rightarrow$ $$\frac{D}{Dt}(\log \rho) + \operatorname{div} \bar{q} = 0 \qquad (6)$$

Corollary (4). When the motion of fluid is steady, then $\frac{\partial \rho}{\partial t} = 0$ and thus the equation of continuity (3) becomes

$\operatorname{div}(\rho\bar{q}) = 0$ |Here ρ is not a function of time i.e. ρ = ρ(x, y, z) (7)

Corollary (5). When the fluid is incompressible, then ρ = constant and thus $\frac{D\rho}{Dt} = 0$.

The equation of continuity becomes

$$\operatorname{div} \bar{q} = 0 \qquad (8)$$

which is same for homogeneous and incompressible fluid.

Corollary (6). If in addition to homogeneity and incompressibility, the flow is of potential kind such that $\bar{q} = -\nabla\phi$, then the equation of continuity becomes single word

$$\operatorname{div}(-\nabla\phi.) = 0 \quad \Rightarrow \nabla.(\nabla\phi.) = 0 \quad \Rightarrow \nabla^2\phi = 0 \qquad (9)$$

which is known as the Laplace equation.

6.2. Equation of continuity in Cartesian co-ordinates :- Let (x, y, z) be the rectangular Cartesian co-ordinates.

Let $$\bar{q} = u\,\hat{i} + v\,\hat{j} + w\,\hat{k} \qquad (1)$$

and $$\nabla = \frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k} \qquad (2)$$

Then, the equation of continuity $\frac{\partial \rho}{\partial t} + \operatorname{div}(\rho\bar{q}) = 0$ can be written as

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x}(\rho u) + \frac{\partial}{\partial y}(\rho v) + \frac{\partial}{\partial z}(\rho w) = 0 \qquad (3)$$

i.e. $$\frac{\partial \rho}{\partial t} + u\frac{\partial \rho}{\partial x} + v\frac{\partial \rho}{\partial y} + w\frac{\partial \rho}{\partial z} + \rho\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right) = 0 \qquad (4)$$

which is the required equation of continuity in Cartesian co-ordinates.

Corollary (1). If the fluid motion is steady, then $\frac{\partial \rho}{\partial t} = 0$ and the equation (3) becomes

$$\frac{\partial}{\partial x}(\rho u) + \frac{\partial}{\partial y}(\rho v) + \frac{\partial}{\partial z}(\rho w) = 0 \qquad (5)$$

Corollary (2). If the fluid is incompressible, then ρ = constant and the equation of continuity is $\nabla \cdot \bar{q} = 0$

i.e. $$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0 \qquad (6)$$

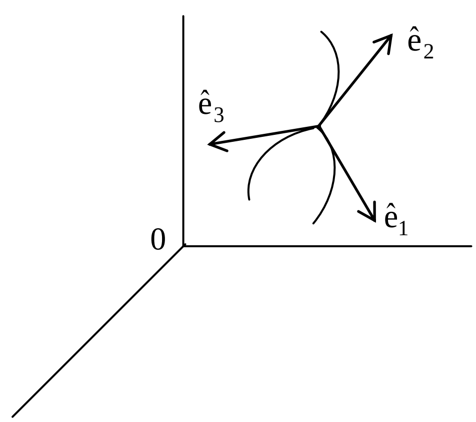
Corollary (3). If the fluid is incompressible and of potential kind, then equation of continuity is

$$\nabla^2 \phi = 0$$

i.e. $$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} + \frac{\partial^2 \phi}{\partial z^2} = 0, \text{ where } \bar{q} = -\nabla \phi .$$

6.3. Equation of continuity in orthogonal curvilinear co-ordinates: Let (u_1, u_2, u_3) be the orthogonal curvilinear co-ordinates and $\hat{e}_1, \hat{e}_2, \hat{e}_3$ be the unit vectors tangent to the co-ordinate curves.

Let $\bar{q} = q_1 \hat{e}_1 + q_2 \hat{e}_2 + q_3 \hat{e}_3$ (1)



The general equation of continuity is

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \bar{q}) = 0 \qquad (2)$$

We know from vector calculus that for any vector point function $\bar{f}$ = (f_1, f_2, f_3),

$$\nabla \cdot \bar{f} = \frac{1}{h_1 h_2 h_3}\left[\frac{\partial}{\partial u_1}(h_2 h_3 f_1) + \frac{\partial}{\partial u_2}(h_3 h_1 f_2) + \frac{\partial}{\partial u_3}(h_1 h_2 f_3)\right] \qquad (3)$$

where h_1, h_2, h_3 are scalars.

Using (3), the equation of continuity (2) becomes

$$\frac{\partial\rho}{\partial t}+\frac{1}{h_1h_2h_3}\left[\frac{\partial}{\partial u_1}(h_2h_3\rho q_1)+\frac{\partial}{\partial u_2}(h_3h_1\rho q_2)+\frac{\partial}{\partial u_3}(h_1h_2\rho q_3)\right] \quad (4)$$

Corollary (1). When motion of fluid is steady, then equation (4) becomes

$$\frac{\partial}{\partial u_1}(h_2h_3\rho q_1)+\frac{\partial}{\partial u_2}(h_3h_1\rho q_2)+\frac{\partial}{\partial u_3}(h_1h_2\rho q_3)=0 \qquad (5)$$

Corollary (2). When the fluid is incompressible, the equation of continuity is (ρ = const)

$$\frac{\partial}{\partial u_1}(h_2h_3q_1)+\frac{\partial}{\partial u_2}(h_3h_1q_2)+\frac{\partial}{\partial u_3}(h_1h_2q_3)=0 \qquad (6)$$

Corollary (3). When fluid is incompressible and irrotational then ρ = const

$\bar{q}=-\nabla\phi=-\left(\frac{1}{h_1}\frac{\partial}{\partial u_1},\frac{1}{h_2}\frac{\partial}{\partial u_2},\frac{1}{h_3}\frac{\partial}{\partial u_3}\right)\phi$ and the equation of continuity

becomes

$$\frac{\partial}{\partial u_1}\left(\frac{h_2h_3}{h_1}\frac{\partial\phi}{\partial u_1}\right)+\frac{\partial}{\partial u_2}\left(\frac{h_1h_3}{h_2}\frac{\partial\phi}{\partial u_2}\right)+\frac{\partial}{\partial u_3}\left(\frac{h_1h_2}{h_3}\frac{\partial\phi}{\partial u_3}\right)=0 \qquad (7)$$

Now, we shall write equation (4) in cylindrical & spherical polar co-ordinates.

6.4. Equation of continuity in cylindrical co-ordinates (r, θ, z) . Here,

$$u_1 \equiv r,\ u_2 \equiv \theta,\ u_3 \equiv z \ \text{ and } h_1 = 1,\ h_2 = r,\ h_3 = 1$$

The equation of continuity becomes

$$\frac{\partial\rho}{\partial t}+\frac{1}{r}\left[\frac{\partial}{\partial r}(r\rho q_1)+\frac{\partial}{\partial\theta}(\rho q_2)+\frac{\partial}{\partial z}(r\rho q_3)\right]=0$$

i.e.

$$\frac{\partial\rho}{\partial t}+\frac{1}{r}\frac{\partial}{\partial r}(r\rho q_1)+\frac{1}{r}\frac{\partial}{\partial\theta}(\rho q_2)+\frac{\partial}{\partial z}(\rho q_3)=0 \qquad (8)$$

Corollary (1). When the fluid motion is steady, then equation (8) becomes

$$\frac{\partial}{\partial r}(r\rho q_1)+\frac{\partial}{\partial\theta}(\rho q_2)+r\frac{\partial}{\partial z}(\rho q_3)=0 \qquad (9)$$

Corollary (2). For incompressible fluid, equation of continuity is

$$\frac{\partial}{\partial r}(rq_1)+\frac{\partial}{\partial\theta}(q_2)+r\frac{\partial q_3}{\partial z}=0 \qquad (10)$$

Corollary (3). When the fluid is incompressible and is of potential kind, then equation (8) takes the form

$$\frac{\partial}{\partial r}\left(r\frac{\partial \phi}{\partial r}\right)+\frac{\partial}{\partial \theta}\left(\frac{1}{r}\frac{\partial \phi}{\partial \theta}\right)+\frac{\partial}{\partial z}\left(r\frac{\partial \phi}{\partial z}\right)=0 \qquad (11)$$

where $\bar{q}=-\nabla\phi$; ∇ is expressed in cylindrical co-ordinates.

6.5. Equation of continuity in spherical co-ordinates (r, θ, ψ). Here,

$(u_1, u_2, u_3) \equiv (r, \theta, \psi)$ and $h = 1, h_2 = r, h_3 = r\sin\theta$

The equation of continuity becomes

$$\frac{\partial \rho}{\partial t}+\frac{1}{r^2\sin\theta}\left[\frac{\partial}{\partial r}(r^2\sin\theta\rho q_1)+\frac{\partial}{\partial \theta}(r\sin\theta\rho q_2)+\frac{\partial}{\partial \psi}(r\rho q_3)\right]=0$$

$$\Rightarrow \frac{\partial \rho}{\partial t}+\frac{1}{r^2\sin\theta}\left[\sin\theta\frac{\partial}{\partial r}(r^2\rho q_1)+r\frac{\partial}{\partial \theta}(\sin\theta\rho q_2)+r\frac{\partial}{\partial \psi}(\rho q_3)\right]=0 \quad (12)$$

Corollary (1). For steady case, equation (12) becomes

$$\sin\theta\frac{\partial}{\partial r}(r^2\rho q_1)+r\frac{\partial}{\partial \theta}(\sin\theta.\rho q_2)+r\frac{\partial}{\partial \psi}(\rho q_3)=0 \qquad (13)$$

Corollary (2). For incompressible fluid, we have

$$\sin\theta\frac{\partial}{\partial r}(r^2 q_1)+r\frac{\partial}{\partial \theta}(\sin\theta.q_2)+r\frac{\partial q_3}{\partial \psi}=0 \qquad (14)$$

Corollary (3). When fluid is incompressible and of potential kind, then equation of continuity is

$$\frac{\partial}{\partial r}\left(r^2\sin\theta\frac{\partial \phi}{\partial r}\right)+\frac{\partial}{\partial \theta}\left(\sin\theta\frac{\partial \phi}{\partial \theta}\right)+\frac{\partial}{\partial \psi}\left(\frac{1}{\sin\theta}.\frac{\partial \phi}{\partial \psi}\right)=0 \qquad (15)$$

where $\bar{q}=-\nabla\phi$; ∇ is expressed in spherical co-ordinates.

6.6. Symmetrical forms of motion and equation of continuity for them. We have the following three types of symmetry which are special cases of cylindrical and spherical polar co-ordinates.

(i) Cylindrical Symmetry :- In this type of symmetry, with suitable choice of cylindrical polar co-ordinates (r, θ, z), every physical quantity is independent of both θ and z so that

$$\frac{\partial}{\partial \theta}=\frac{\partial}{\partial z}=0 \text{ and } \bar{q}=\bar{q}(r,t)$$

For this case, the equation of continuity in cylindrical co-ordinates, reduces to

$$\frac{\partial \rho}{\partial t}+\frac{1}{r}\frac{\partial}{\partial r}(\rho q_1 r)=0 \qquad (1)$$

If the flow is steady, then equation (1) becomes

$$\frac{\partial}{\partial r}(\rho q_1 r) = 0 \qquad \Rightarrow \rho q_1 r = \text{constant} = F(t), \text{ (say)}.$$

Further, if the fluid is incompressible then q_1 r = constant = G(t), (say).

(ii) Spherical Symmetry :- In this case, the motion of fluid is symmetrical about the centre and thus with the choice of spherical polar co-ordinates (r, θ, ψ), every physical quantity is independent of both θ & ψ. so that

$$\frac{\partial}{\partial \theta} = \frac{\partial}{\partial \psi} = 0 \text{ and } \bar{q} = \bar{q}\,(r, t)$$

The equation of continuity, for such symmetry, reduces to

$$\frac{\partial \rho}{\partial t} + \frac{1}{r^2}.\frac{\partial}{\partial r}(\rho q_1 r^2) = 0 \qquad (2)$$

For steady motion, it becomes

$$\frac{\partial}{\partial r}(\rho q_1 r^2) = 0 \qquad \Rightarrow \rho q_1\, r^2 = \text{const} = F(t), \text{ (say)}$$

and for incompressible fluid, it has the form $q_1\, r^2$ = constant = G(t), (say).

(iii) Axial Symmetry :- (a) In cylindrical co-ordinates (r, θ, z), axial symmetry means that every physical quantity is independent of θ i.e. $\frac{\partial}{\partial \theta} = 0$ and thus the equation of continuity becomes

$$\frac{\partial \rho}{\partial t} + \frac{1}{r}\left[\frac{\partial}{\partial r}(\rho q_1 r) + \frac{\partial}{\partial z}(\rho q_3 r)\right] = 0$$

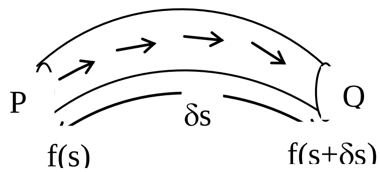
(b) In spherical co-ordinates (r, θ, ψ), axial symmetry means that every physical quantity is independent of ψ i.e. $\frac{\partial}{\partial \psi} = 0$ and the equations of continuity, for this case, reduces to

$$\frac{\partial \rho}{\partial t} + \frac{1}{r^2}\frac{\partial}{\partial r}(\rho q_1 r^2) + \frac{1}{r\sin\theta}\frac{\partial}{\partial \theta}(\rho q_2 \sin\theta) = 0.$$

6.7. Example. If σ(s) is the cross-sectional area of a stream filament, prove that the equation of continuity is

$\frac{\partial}{\partial t}(\rho\sigma) + \frac{\partial}{\partial s}(\rho\sigma q) = 0,$ where δs is an element of arc of the filament and q is the fluid speed.

Solution. Let P and Q be the points on the end sections of the stream filament.



The rate of flow of fluid out of volume of filament is

$$(\rho q\sigma)_Q - (\rho q\sigma)_P = \frac{\partial}{\partial s}(\rho q\sigma)_P\,\delta s$$

where we have retained the terms upto first order only, since δs is infinitesimally small

Now, the fluid speed is along the normal to the cross-section. At time t, the mass within the segment of filament is $\rho\sigma\delta s$ and its rate of increase is

$$\frac{\partial}{\partial t}(\rho\sigma\delta s) = \frac{\partial}{\partial t}(\rho\sigma)\delta s \qquad (2)$$

Using law of conservation of mass, we have from (1) & (2)

$$\frac{\partial}{\partial t}(\rho\sigma)\delta s + \frac{\partial}{\partial s}(\rho q\sigma)\delta s = 0 \qquad | \text{ Total rate} = 0$$

i.e. $$\frac{\partial}{\partial t}(\rho\sigma) + \frac{\partial}{\partial s}(\rho\sigma q) = 0 \qquad (3)$$

which is the required equation at any point P of the filament.

6.8. Deduction :- For steady incompressible flow, $\frac{\partial}{\partial t}(\rho\sigma) = 0$ and equation (3) reduces to

$$\frac{\partial}{\partial s}(\rho\sigma q) = 0 \quad \Rightarrow \frac{\partial}{\partial s}(\sigma q) = 0 \quad \Rightarrow \sigma\, q = \text{constant}$$

which shows that for steady incompressible flow product of velocity and cross-section of stream filament is constant. This result means that the volume of fluid a crossing every section per unit time is constant

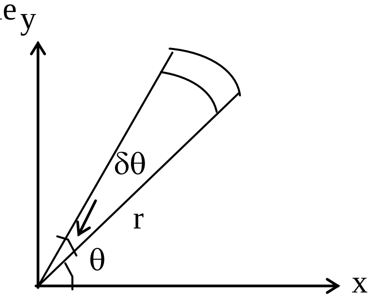
$$\left(\sigma q = c \Rightarrow \sigma\frac{\text{distance}}{t} = c \Rightarrow \frac{\text{volume}}{t} = c\right)$$

6.9. Example. A mass of a fluid moves in such a way that each particle describes a circle in one plane about a fixed axis, show that the equation of continuity is

$$\frac{\partial\rho}{\partial t}+\frac{\partial}{\partial\theta}(\rho\omega)=0,$$

where ω is the angular velocity of a particle whose azimuthal angle is θ at time t.

Solution. Here, the motion is in a plane i.e. we have a two dimensional case and the particle describe a circle



Therefore, z = constant, r = constant

$$\Rightarrow \qquad \frac{\partial}{\partial z}=0, \qquad \frac{\partial}{\partial r}=0 \qquad (1)$$

i.e. there is only rotation.
We know that the equation of continuity in cylindrical co-ordinates (r, θ, z) is

$$\frac{\partial\rho}{\partial t}+\frac{1}{r}\frac{\partial}{\partial r}(r\rho q_1)+\frac{1}{r}\frac{\partial}{\partial\theta}(\rho q_2)+\frac{\partial}{\partial z}(\rho q_3)=0 \qquad (2)$$

Using (1), we get

$$\frac{\partial\rho}{\partial t}+\frac{1}{r}\frac{\partial}{\partial\theta}(\rho q_2)=0$$

$$\Rightarrow \qquad \frac{\partial\rho}{\partial t}+\frac{1}{r}\frac{\partial}{\partial\theta}(\rho r\omega)=0, \text{ where } q=q_2=r\omega.$$

$$\Rightarrow \qquad \frac{\partial\rho}{\partial t}+\frac{\partial}{\partial\theta}(\rho\omega)=0$$

Hence the result

6.10. Example. A mass of fluid is in motion so that the lines of motion lie on the surface of co-axial cylinders, show that the equation of continuity is

$$\frac{\partial\rho}{\partial t}+\frac{1}{r}\frac{\partial}{\partial\theta}(\rho v_\theta)+\frac{\partial}{\partial z}(\rho v_z)=0$$

where v_θ, v_z are the velocities perpendicular and parallel to z.

Solution. We know that the equation of continuity in cylindrical co-ordinates (r, θ, z) is given by

$$\frac{\partial\rho}{\partial t}+\frac{1}{r}\frac{\partial}{\partial r}(\rho r v_r)+\frac{1}{r}\frac{\partial}{\partial\theta}(\rho v_\theta)+\frac{\partial}{\partial z}(\rho v_z)=0, \text{ where } \bar{q}=(v_r, v_\theta, v_z)$$

Since the lines of motion (path lines) lie on the surface of cylinder, therefore the component of velocity in the direction of dr is zero i.e. $v_r = 0$
Thus, the equation of continuity in the present case reduces to

$$\frac{\partial\rho}{\partial t}+\frac{1}{r}\frac{\partial}{\partial v}(\rho v_\theta)+\frac{\partial}{\partial z}(\rho v_z)=0$$

Hence the result

6.11. Example. The particles of a fluid move symmetrically in space with regard to a fixed centre, prove that the equation of continuity is

$$\frac{\partial\rho}{\partial t}+u\frac{\partial\rho}{\partial r}+\frac{\rho}{r^2}\cdot\frac{\partial}{\partial r}(r^2 u)=0.$$

where u is the velocity at a distance r

Solution. First, derive the equation of continuity in spherical co-ordinates. Now, the present case is the case of spherical symmetry, since the motion is symmetrical w.r.t. a fixed centre.
Therefore, the equation of continuity is

$$\frac{\partial\rho}{\partial t}+\frac{1}{r^2}\cdot\frac{\partial}{\partial r}(\rho q_1 r^2)=0 \qquad \Big| \because \frac{\partial}{\partial\theta}=\frac{\partial}{\partial\psi}=0$$

$$\Rightarrow \qquad \frac{\partial\rho}{\partial t}+\frac{1}{r^2}\cdot\frac{\partial}{\partial r}(\rho q_1 r^2)=0, \text{ where } q_1 \equiv u$$

$$\Rightarrow \qquad \frac{\partial\rho}{\partial t}+\frac{1}{r^2}\cdot\frac{\partial\rho}{\partial r}ur^2+\frac{1}{r^2}.\rho.\frac{\partial}{\partial r}(ur^2)=0$$

$$\Rightarrow \qquad \frac{\partial\rho}{\partial t}+u.\frac{\partial\rho}{\partial r}+\frac{\rho}{r^2}\frac{\partial}{\partial r}(r^2 u)=0$$

Hence the result

6.12. Example. If the lines of motion are curves on the surfaces of cones having their vertices at the origin and the axis of z for common axis, prove that the equation of continuity is

$$\frac{\partial\partial}{\partial t}+\frac{\partial}{\partial r}(\rho q_r)+\frac{2\rho}{r}q_r+\frac{\operatorname{cosec}\theta}{r}\frac{\partial}{\partial\psi}(\rho q_\psi)=0$$

Solution. First derive the equation of continuity in spherical co-ordinates (r, θ, ψ) as

$$\frac{\partial\rho}{\partial t}+\frac{1}{r^2\sin\theta}\left[\sin\theta\frac{\partial}{\partial r}(\rho q_1 r^2)+r\frac{\partial}{\partial\theta}(\rho q_2\sin\theta)+r\frac{\partial}{\partial\psi}(\rho q_3)\right]=0$$

In the present case, it is given that lines of motion lie on the surfaces of cones, therefore velocity perpendicular to the surface is zero i.e. $q_2 = 0$
Therefore, the equation of continuity becomes.

$$\frac{\partial\rho}{\partial t}+\frac{1}{r^2}\frac{\partial}{\partial r}(\rho q_r r^2)+\frac{1}{r\sin\theta}.\frac{\partial}{\partial\psi}(\rho q_\psi)=0 \text{ where } (q_1, q_2, q_3) \equiv$$

(q_r, q_θ, q_ψ)

$$\Rightarrow \qquad \frac{\partial\rho}{\partial r}+\frac{1}{r^2}\left[r^2\frac{\partial}{\partial r}(\rho q_r)+\rho q_r(2r)\right]+\frac{1}{r\sin\theta}\frac{\partial}{\partial\psi}(\rho q_\psi)=0$$

$$\Rightarrow \qquad \frac{\partial\rho}{\partial t}+\frac{\partial}{\partial r}(\rho q_r)\frac{2\rho}{r}q_r+\frac{\operatorname{cosec}\theta}{r}\frac{\partial}{\partial\psi}(\rho q_\psi)=0$$

Hence the result

6.13. Example. Show that polar form of equation of continuity for a two dimensional incompressible fluid is

$$\frac{\partial}{\partial r}(ru)+\frac{\partial v}{\partial\theta}=0$$

If u = $\frac{-\mu\cos\theta}{r^2}$, then find v and the magnitude of the velocity $\bar{q}$, where $\bar{q}$ = (u, v)

Solution. First derive the equation of continuity in polar co-ordinates (r, θ) in two dimensions as

$$\frac{\partial\rho}{\partial t}+\frac{1}{r}\frac{\partial}{\partial r}(\rho r q_1)+\frac{1}{r}\frac{\partial}{\partial\theta}(\rho q_2)=0 \qquad\qquad | z=0$$

In the present case ρ = constant
Therefore, the equation of continuity reduces to

$$\frac{\rho}{r}\frac{\partial}{\partial r}(ru)+\frac{\rho}{r}\frac{\partial}{\partial\theta}(v)=0, \text{where } \bar{q} = (q_1, q_2, q_3) \equiv (u, v, w)$$

i.e. $$\frac{\partial}{\partial r}(ru)+\frac{\partial v}{\partial\theta}=0$$

Hence the result.

Now u = $\frac{-\mu\cos\theta}{r^2}$ $\Rightarrow \frac{\partial}{\partial r}\left(\frac{-\mu\cos\theta}{r^2}r\right)+\frac{\partial v}{\partial\theta}=0$

$$\Rightarrow \frac{\mu\cos\theta}{r^2}+\frac{\partial v}{\partial\theta}=0 \quad \Rightarrow \frac{\partial v}{\partial\theta}=\frac{-\mu\cos\theta}{r^2}$$

Integrating w.r.t θ, we get

$$v = \frac{-\mu \sin\theta}{r^2}$$

and thus $|\bar{q}| = q = \sqrt{u^2 + v^2} = \frac{\mu}{r^2}$

6.14. Equation of Continuity by Lagrange's Method. Let initially a fluid element be at (a, b, c) at time $t = t_0$ when its volume is dV_0 and density is ρ_0. After time t, let the same fluid element be at (x, y, z) when its volume is dV and density is ρ. Since mass of the fluid element remains invariant during its motion, we have

$$\rho_0\, dV_0 = \rho dV \text{ i.e. } \rho_0\, da\, db\, dc = \rho\, dx\, dy\, dz$$

or
$$\rho_0\, da\, db\, dc = \rho \frac{\partial(x, y, z)}{\partial(a, b, c)} da\, db\, dc$$

or
$$\rho J = \rho_0 \qquad (1)$$

where $J = \dfrac{\partial(x, y, z)}{\partial(a, b, c)}$

which is the required equation of continuity.

6.15. Remark. By simple property of Jacobians, we get

$$\frac{dJ}{dt} = J \nabla \cdot \bar{q}$$

Thus (1) gives $\dfrac{d}{dt}(\rho J) = 0 \;\Rightarrow \dfrac{d\rho}{dt} J + \rho \dfrac{dJ}{dt} = 0$

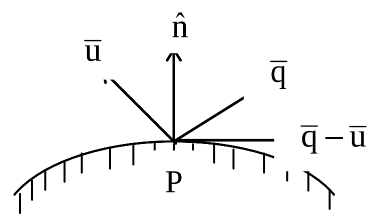
$$\Rightarrow \frac{d\rho}{dt} J + \rho J\, \nabla \cdot \bar{q} = 0 \qquad \Rightarrow \frac{d\rho}{dt} + \rho \nabla \cdot \bar{q} = 0 \qquad \text{or} \frac{D\rho}{Dt} + \rho \nabla \cdot \bar{q} = 0$$

which is the Euler's equation of continuity.

7. Boundary Surfaces

Physical conditions that should be satisfied on given boundaries of the fluid in motion, are called boundary conditions. The simplest boundary condition occurs where an ideal and incompressible fluid is in contact with rigid impermeable boundary, e.g., wall of a container or the surface of a body which is moving through the fluid.

Let P be any point on the boundary surface where the velocity of fluid is $\bar{q}$ and velocity of the boundary surface is $\bar{u}$.



The velocity at the point of contact of the boundary surface and the liquid must be tangential to the surface otherwise the fluid will break its contact with the boundary surface. Thus, if $\hat{n}$ be the unit normal to the surface at the point of contact, then

$$(\bar{q}-\bar{u})\cdot\hat{n}=0 \;\Rightarrow \bar{q}\cdot\hat{n}=\bar{u}\cdot\hat{n} \qquad (1)$$

In particular, if the boundary surface is at rest, then $\bar{u}=\bar{0}$ and the condition becomes

$$\bar{q}\cdot\hat{n}=0 \qquad (2)$$

Another type of boundary condition arrives at a free surface where liquid borders a vacuum eg. the interface between liquid and air is usually regarded as free surface. For this free surface, pressure p satisfies

$$P = \Pi \qquad (3)$$

where Π denotes the pressure outside the fluid i.e. the atmospheric pressure. Equation (3) is a dynamic boundary condition.

Third type of boundary condition occurs at the boundary between two immissible ideal fluids in which the velocities are $\bar{q}_1$ & $\bar{q}_2$ and pressures are p_1 & p_2 respectively.

Now, we find the condition that a given surface satisfies to be a boundary

surface.

7.1. Article. To obtain the differential equation satisfied by boundary surface of a fluid in motion

or

To find the condition that the surface.

$$F(\bar{r},t)=F(x,y,z,t)=0$$

may represent a boundary surface :-

If $\bar{q}$ be the velocity of fluid and $\bar{u}$ be the velocity of the boundary surface at a point P of contact, then

$$(\bar{q}-\bar{u})\cdot\hat{n}=0 \;\Rightarrow \bar{q}\cdot\hat{n}=\bar{u}\cdot\hat{n} \qquad (1)$$

where $\bar{q}-\bar{u}$ is the relative velocity and $\hat{n}$ is a unit vector normal to the surface at P.
The equation of the given surface is

$$F(\bar{r},t)=F(x,y,z,t)=0 \qquad (2)$$

We know that a unit vector normal to the surface (2) is given by

$$\hat{n}=\frac{\nabla F}{|\nabla F|}$$

Thus, from (1), we get $\qquad \bar{q}\cdot\nabla F=\bar{u}\cdot\nabla F \qquad (3)$

since the boundary surface is itself in motion, therefore at time (t + δt), it's equation is given by

$$F(\bar{r}+\delta\bar{r},t+\delta t)=0. \qquad (4)$$

From (2) & (4), we have

$$F(\bar{r}+\delta\bar{r},t+\delta t)-F(\bar{r},t)=0$$

i.e. $F(\bar{r}+\delta\bar{r},t+\delta t)-F(\bar{r},t+\delta t)+F(\bar{r},t+\delta t)-F(\bar{r},t)=0$

By Taylor's series, we can have

$$(\delta\bar{r}\cdot\nabla)F(\bar{r},t+\delta t)+\delta t\frac{\partial}{\partial t}\{F(\bar{r},t)\}=0$$

$$\because F(x+\delta x,y+\delta y+z+\delta z)=F(x,y,z)+\delta x\frac{\partial F}{\partial x}+\delta y\frac{\partial F}{\partial y}+\delta z\frac{\partial F}{\partial z}+\ldots$$

$$= F(x,y,z)+\delta\bar{r}\cdot\nabla F$$

$$\Rightarrow \qquad \left(\frac{\delta\bar{r}}{\delta t}\cdot\nabla\right)F(\bar{r},t+\delta t)+\frac{\partial F}{\partial t}=0$$

Taking limit as $\delta t\to 0$, we get

$$\left(\frac{d\bar{r}}{dt}.\nabla\right)F+\frac{\partial F}{\partial t}=0$$

$$\Rightarrow \qquad \frac{\partial F}{\partial t}+(\bar{q}.\nabla)F=0 \qquad \text{i.e.}\frac{DF}{Dt}=0 \qquad (5)$$

which is the required condition for any surface F to be a boundary surface

Corollary (1) If $\bar{q}$ = (u, v, w), then the condition (5) becomes

$$\frac{\partial F}{\partial t}+u\frac{\partial F}{\partial x}+v\frac{\partial F}{\partial y}+w\frac{\partial F}{\partial z}=0$$

In case, the surface is rigid and does not move with time, then $\frac{\partial F}{\partial t} = 0$ and the boundary condition is

$$u\frac{\partial F}{\partial x} + v\frac{\partial F}{\partial y} + w\frac{\partial F}{\partial z} = 0 \qquad \text{i.e.} (\bar{q} \cdot \nabla)F = 0$$

Corollary (2) The boundary condition

$$\frac{\partial F}{\partial t} + u\frac{\partial F}{\partial x} + v\frac{\partial F}{\partial y} + w\frac{\partial F}{\partial z} = 0$$

is a linear equation and its solution gives

$$\frac{dt}{1} = \frac{dx}{u} = \frac{dy}{v} = \frac{dz}{w} \qquad \left| \frac{D}{Dt} \equiv \frac{d}{dt} \text{in Lagrangian view} \right.$$

$$\Rightarrow \qquad \frac{dx}{dt} = u, \frac{dy}{dt} = v, \frac{dz}{dt} = w$$

which are the equations of path lines.
Hence once a particle is in contact with the surface, it never leaves the surface.

Corollary (3) From equation (5), we have

$$\bar{q} \cdot \nabla F = \frac{-\partial F}{\partial t}$$

$$\Rightarrow \qquad \bar{q} \cdot \frac{\nabla F}{|\nabla F|} = \frac{-\partial F/\partial t}{|\nabla F|}$$

$$\Rightarrow \qquad \bar{q} \cdot \hat{n} = \frac{-\partial F/\partial t}{|\nabla F|}$$

which gives the normal velocity.

Also from (1), we get

$$\bar{u} \cdot \hat{n} = \frac{-\partial F/\partial t}{|\nabla F|} \qquad \left| \because \ \bar{q} \cdot \hat{n} = \bar{u} \cdot \hat{n}. \right.$$

which gives the normal velocity of the boundary surface.

7.2. Example. Show that the ellipsoid

$$\frac{x^2}{a^2k^2t^{2n}} + kt^n\left[\left(\frac{y}{b}\right)^2 + \left(\frac{z}{c}\right)^2\right] = 1$$

is a possible form of the boundary surface of a liquid.

Solution. The surface F(x, y, z, t) = 0 can be a possible boundary surface, if it satisfies the boundary condition.

$$\frac{DF}{Dt} = \frac{\partial F}{\partial t} + u\frac{\partial F}{\partial x} + v\frac{\partial F}{\partial y} + w\frac{\partial F}{\partial z} = 0 \qquad (1)$$

where u, v, w satisfy the equation of continuity

$$\nabla \cdot \bar{q} = 0 \text{ i.e.} \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0 \qquad (2)$$

Here, $F(x, y, z, t) \equiv \dfrac{x^2}{a^2k^2t^{2n}} + k\,t^n\left[\left(\dfrac{y}{b}\right)^2 + \left(\dfrac{z}{c}\right)^2\right] - 1 = 0$

Therefore, $\dfrac{\partial F}{\partial t} = -\dfrac{x^2.2n}{a^2k^2t^{2n+1}} + nkt^{n-1}\left[\left(\dfrac{y}{b}\right)^2 + \left(\dfrac{z}{c}\right)^2\right]$

$$\frac{\partial F}{\partial x} = \frac{2x}{a^2k^2t^{2n}}, \frac{\partial F}{\partial y} = \frac{2kt^n y}{b^2}, \frac{\partial F}{\partial z} = \frac{2kt^n z}{c^2}.$$

Thus, from (1), we get

$$\frac{-x^2}{a^2k^2}\frac{2n}{t^{2n+1}} + nk\,t^{n-1}\left[\left(\frac{y}{b}\right)^2 + \left(\frac{z}{c}\right)^2\right]$$

$$+ \frac{2.xu.}{a^2k^2t^{2n}.} + \frac{2kt^n yv}{b^2} + \frac{2kt^n.zw}{c^2} = 0$$

or $\left(u - \dfrac{nx}{t}\right)\dfrac{2x}{a^2k^2t^{2n}} + \left(v + \dfrac{ny}{2t}\right)\dfrac{2kyt^n}{b^2} + \left(w + \dfrac{nz}{2t}\right)\dfrac{2kzt^n}{c^2} = 0$

which will hold. if we take

$$u - \frac{-nx}{t} = 0, \quad v + \frac{ny}{2t} = 0, \quad w + \frac{nz}{2t} = 0$$

i.e. $u = \dfrac{nx}{t}, \quad v = -\dfrac{ny}{2t}, \quad w = -\dfrac{nz}{2t}$ (3)

It will be a justifiable step if equation (2) is satisfied.

i.e. $\dfrac{n}{t} + \dfrac{-n}{2t} + \dfrac{-n}{2t} = 0$.

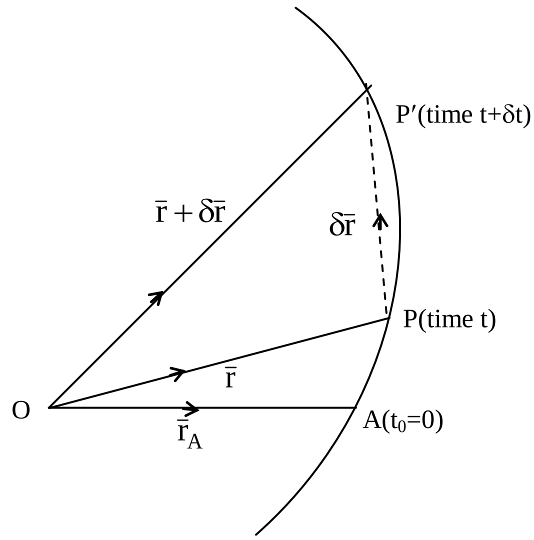
which is true.

Hence the given ellipsoid is a possible form of boundary surface of a liquid.

8. Acceleration at a Point of a Fluid

Suppose that a fluid particle is moving along a curve C, initially it being at point A(t_0 = 0) with position vector $\bar{r}_A$. Let P and P′ be its positions at time t and t + δt with position vectors $\bar{r}$ and $\bar{r}+\delta\bar{r}$ respectively.

Therefore, $\delta\bar{r} = \overline{PP'}$



The points A, P, P′ are geometrical points of region occupied by fluid and they coincide with the locations of the same fluid particle at times t_0, t, t + δt respectively. Let $\bar{f}$ be the acceleration of the particle at time t when it coincides with P. By definition

$$\bar{f} = \underset{\delta t \to 0}{\mathrm{Lt}} \frac{(\text{Change in particle velocity in time } \delta t)}{\delta t} \qquad (1)$$

But the particle vel. at time t is $\bar{q}(\bar{r}, t)$ and at time t+δt it is $\bar{q}(\bar{r}+\delta\bar{r}, t+\delta t)$.

Thus (1) becomes

$$\bar{f} = \underset{\delta t \to 0}{\mathrm{Lt}} \frac{[\bar{q}(\bar{r}+\delta\bar{r}, t+\delta t) - \bar{q}(\bar{r}, t)]}{\delta t} \qquad (2)$$

Now,

$$\frac{\bar{q}(\bar{r}+\delta\bar{r}, t+\delta t) - \bar{q}(\bar{r}, t)}{\delta t} = \frac{\bar{q}(\bar{r}+\delta\bar{r}, t+\delta t) - \bar{q}(\bar{r}, t+\delta t)}{\delta t} + \frac{\bar{q}(\bar{r}, t+\delta t) - \bar{q}(\bar{r}, t+\delta t)}{\delta t} \qquad (3)$$

Since $\bar{r}$ is independent of time t, therefore

$$\underset{\delta t \to 0}{\mathrm{Lt}} \frac{\bar{q}(\bar{r}, t+\delta t) - \bar{q}(\bar{r}, t)}{\delta t} = \frac{\partial \bar{q}}{\partial t} \qquad (4)$$

Using Taylor's expansion, we get

$$\bar{q}(\bar{r}+\delta\bar{r}, t+\delta t) - \bar{q}(\bar{r}, t+\delta t) = (\delta\bar{r}\cdot\nabla)\bar{q}(\bar{r}, t+\delta t) + \in \quad (5)$$

where $|\in| = 0[(\delta\bar{r})^2]$

[$\because$ $F(x+\delta x, y+\delta y, z+\delta z) - F(x,y,z) = \left(\delta x\frac{\partial}{\partial x} + \delta y\frac{\partial}{\partial y} + \delta z\frac{\partial}{\partial z}\right)(F(x,y,z)$

$$+\frac{1}{\underline{|2}}\left(\delta x\frac{\partial}{\partial x} + \delta y\frac{\partial}{\partial y} + \delta z\frac{\partial}{\partial z}\right)^2 . F(x,y,z) + \ldots\ldots$$

and

$$\delta x\frac{\partial}{\partial x} + \delta y\frac{\partial}{\partial y} + \delta z\frac{\partial}{\partial z} = (\delta\bar{r}\cdot\nabla), \text{ where}$$

$$\delta\bar{r} = \delta x\,\hat{i} + \delta y\,\hat{j} + \delta z\,\hat{k}, \nabla \equiv \frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}\,]$$

But $\delta\bar{r}$ is merely the displacement of the fluid particle in time δt, therefore,

$$\delta\bar{r} = \bar{q}(\bar{r}, t)\delta t \quad (6)$$

Thus, from (5), we obtain

$$\underset{\delta t\to 0}{\mathrm{Lt}}\ \frac{\bar{q}(\bar{r}+\delta\bar{r}, t+\delta t) - \bar{q}(\bar{r}, t+\delta t)}{\delta t} = (\bar{q}\cdot\nabla)\bar{q} \quad (7)$$

where R. H. S. of (4) & (7) are evaluated at $P(\bar{r}, t)$. Hence, from (2), the acceleration of fluid at P in vector form is given by

$$\bar{f} = \frac{\partial\bar{q}}{\partial t} + (\bar{q}\cdot\nabla)\bar{q} \quad (8)$$

8.1. Remark. We have obtained the acceleration i.e. rate of change of velocity $\bar{q}$. The same procedure can be applied to find the rate of change of any physical property associated with the fluid, such as density. Thus, if F = $F(\bar{r}, t)$ is any scalar or vector quantity associated with the fluid, it's rate of change at time t is given by

$$\frac{DF}{Dt} = \frac{\partial F}{\partial t} + (\bar{q}\cdot\nabla)F$$

The operator $\frac{D}{Dt} \equiv \frac{\partial}{\partial t} + (\bar{q} \cdot \nabla)$ is Lagrangian and operators on R.H.S. are Eulerian since $\bar{r}$ is independent of t. $\frac{D}{Dt}$ is also called material derivative.

In particular, if F = ρ, the density of the fluid, then

$$\frac{D\rho}{Dt} = \frac{\partial \rho}{\partial t} + (\bar{q} \cdot \nabla)\rho$$

which is the general equation of motion for unsteady flow.

8.2. Components of Acceleration in Cartesian co-ordinates. Let u,v,w be the Cartesian components of $\bar{q}$ and f_1, f_2, f_3 that of $\bar{f}$ i.e. $\bar{q}$ = (u,v,w), $\bar{f}$ = (f_1, f_2, f_3).
Then from equation.

$$\bar{f} = \frac{\partial \bar{q}}{\partial t} + (\bar{q} \cdot \nabla)\bar{q}, \qquad (1)$$

we get

$$f_1 = \frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z}$$

$$f_2 = \frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z}$$

$$f_3 = \frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z}$$

which are the required Cartesian components of $\bar{f}$.

In tensor form with co-ordinates x_i and velocity components q_i (i = 1, 2, 3), the above set of equations can be written as

$$f_i = \frac{\partial q_i}{\partial t} + q_j q_{i,j}, \quad \text{where}\, q_{i,j} = \frac{\partial q_i}{\partial x_j}$$

8.3. Components of Acceleration Curvilinear co-ordinates. Before obtaining the acceleration components in curvilinear co-ordinates; we obtain a more suitable form of equation (1). as

$$\bar{f} = \frac{\partial \bar{q}}{\partial t} + \nabla\left(\frac{1}{2}\bar{q}^{\,2}\right) - \bar{q} \times (\nabla \times \bar{q})$$

$$= \frac{\partial \bar{q}}{\partial t} + \nabla\left(\frac{1}{2}\bar{q}^{\,2}\right) + \bar{\xi}\times\bar{q} \quad , \quad \text{where } \bar{\xi} = \text{curl } \bar{q} = \nabla\times\bar{q} \ .$$

We have

$$(\bar{q}\cdot\nabla)\bar{q} = (\bar{q}\cdot\hat{i})\frac{\partial \bar{q}}{\partial x} + (\bar{q}.\hat{j})\frac{\partial \bar{q}}{\partial y} + (\bar{q}\cdot\hat{k})\frac{\partial \bar{q}}{\partial z} \qquad (2)$$

For any three vectors $\bar{A}, \bar{B}, \bar{C},$ we have

$$\bar{A}\times(\bar{B}\times\bar{C}) = (\bar{A}\cdot\bar{C})\bar{B} - (\bar{A}\cdot\bar{B})\bar{C}$$

i.e.

$$(\bar{A}\cdot\bar{B})\bar{C} = (\bar{A}\cdot\bar{C})\bar{B} - \bar{A}\times(\bar{B}\times\bar{C})$$

In particular, taking $\bar{A} = \bar{q},\ \bar{B} = \hat{i},\ \bar{C} = \frac{\partial \bar{q}}{\partial x}$, we get

$$(\bar{q}\cdot\hat{i})\frac{\partial \bar{q}}{\partial x} = \left(\bar{q}\cdot\frac{\partial \bar{q}}{\partial x}\right)\hat{i} - \bar{q}\times\left(\hat{i}\times\frac{\partial \bar{q}}{\partial x}\right)$$

$$= \hat{i}\frac{\partial}{\partial x}\left(\frac{1}{2}\bar{q}^2\right) - \bar{q}\times\left(\hat{i}\times\frac{\partial \bar{q}}{\partial x}\right) \qquad (3)$$

Similarly,

$$(\bar{q}\cdot\bar{j})\frac{\partial \bar{q}}{\partial y} = \hat{j}\frac{\partial}{\partial y}\left(\frac{1}{2}\bar{q}^2\right) - \bar{q}\times\left(\hat{j}\times\frac{\partial \bar{q}}{\partial y}\right) \qquad (4)$$

$$(\bar{q}\cdot\hat{k})\frac{\partial \bar{q}}{\partial z} = \hat{k}\frac{\partial}{\partial z}\left(\frac{1}{2}\bar{q}^2\right) - \bar{q}\times\left(\hat{k}\times\frac{\partial \bar{q}}{\partial z}\right) \qquad (5)$$

Adding (3), (4) and (5), we get

$$(\bar{q}\cdot\nabla)\bar{q} = \nabla\left(\frac{1}{2}\bar{q}^2\right) - \bar{q}\times\Sigma\left(\hat{j}\times\frac{\partial \bar{q}}{\partial x}\right)$$

$$= \nabla\left(\frac{1}{2}\bar{q}^2\right) - \bar{q}\times(\nabla\times\bar{q})$$

Thus, from (1), we obtain

$$\bar{f} = \frac{d\bar{q}}{dt} = \frac{\partial \bar{q}}{\partial t} + \nabla\left(\frac{1}{2}\bar{q}^2\right) - \bar{q}\times(\nabla\times\bar{q})$$

$$= \frac{\partial \bar{q}}{\partial t} + \nabla\left(\frac{1}{2}\bar{q}^2\right) + \bar{\xi}\times\bar{q} \qquad (6)$$

Now, let (u_1, u_2, u_3) denote the orthogonal curvilinear co-ordinates.

Also let $\bar{q} = (q_1, q_2, q_3, \bar{f} = (f_1, f_2, f_3)$ $\bar{\xi} = (\xi_1, \xi_2, \xi_3)$, where the terms have their usual meaning. We know that the expression for the operator ∇ in curvilinear co-ordinates is

$$\nabla \equiv \left(\frac{1}{h_1}\frac{\partial}{\partial u_1}, \frac{1}{h_2}\frac{\partial}{\partial u_2}, \frac{1}{h_3}\frac{\partial}{\partial u_3}\right),$$

where h_1, h_2, h_3 are scalar factors.

The components of $\bar{\xi} = \text{curl}\ \bar{q}$ in the curvilinear system are given by

$$\left.\begin{aligned} \xi_1 &= \frac{1}{h_2 h_3}\left[\frac{\partial}{\partial u_2}(h_3 q_3) - \frac{\partial}{\partial u_3}(h_2 q_2)\right] \\ \xi_2 &= \frac{1}{h_3 h_1}\left[\frac{\partial}{\partial u_3}(h_1 q_1) - \frac{\partial}{\partial u_1}(h_3 q_3)\right] \\ \xi_3 &= \frac{1}{h_1 h_2}\left[\frac{\partial}{\partial u_1}(h_2 q_2) - \frac{\partial}{\partial u_2}(h_1 q_1)\right] \end{aligned}\right\} \tag{7}$$

Using these results in (6), we find that

$$\left.\begin{aligned} f_1 &= \frac{\partial q_1}{\partial t} + \frac{1}{2h_1}\frac{\partial}{\partial u_1}\left(q_1^2 + q_2^2 + q_3^2\right) + (\xi_2 q_3 - \xi_3 q_2) \\ f_2 &= \frac{\partial q_2}{\partial t} + \frac{1}{2h_2}\frac{\partial}{\partial u_2}\left(q_1^2 + q_2^2 + q_3^2\right) + (\xi_3 q_1 - \xi_1 q_3) \\ f_3 &= \frac{\partial q_3}{\partial t} + \frac{1}{2h_3}\frac{\partial}{\partial u_3}\left(q_1^2 + q_2^2 + q_3^2\right) + (\xi_1 q_2 - \xi_2 q_1) \end{aligned}\right\} \tag{8}$$

which are the components of acceleration in curvilinear co-ordinates.

Now, we write the components of acceleration in cylindrical (r, θ, z) and spherical (r, θ, ψ) co-ordinates.

8.4. Components of Acceleration in Cylindrical Co-ordinates (r, θ, z).

Here,

$$u_1 \equiv r,\ u_2 \equiv \theta\ u_3 \equiv z. \quad \text{and} \quad h_1 = 1,\ h_2 = r,\ h_3 = 1$$

Therefore, $$\nabla \equiv \left(\frac{\partial}{\partial r}, \frac{1}{r}\frac{\partial}{\partial \theta}, \frac{\partial}{\partial z} \right)$$

and

$$\xi_1 = \frac{1}{r}\left[\frac{\partial q_3}{\partial \theta} - \frac{\partial}{\partial z}(r q_2) \right] = \frac{1}{r}\frac{\partial q_3}{\partial \theta} . - \frac{\partial q_2}{\partial z}$$

$$\xi_2 = \frac{\partial}{\partial z}(q_1) - \frac{\partial}{\partial r}(q_3) = \frac{\partial q_1}{\partial z} - \frac{\partial q_3}{\partial r}$$

$$\xi_3 = \frac{1}{r}\left[\frac{\partial}{\partial r}(r q_2) - \frac{\partial q_1}{\partial \theta} \right] = \frac{\partial q_2}{\partial r} + \frac{q_2}{r} - \frac{1}{r}\frac{\partial q_1}{\partial \theta}$$

Thus,

$$f_1 = \frac{\partial q_1}{\partial t} + \frac{1}{2}\frac{\partial}{\partial r}\left(q_1{}^2 + q_2{}^2 + q_3{}^2 \right) + \left(q_3 \frac{\partial q_1}{\partial z} - q_1 \frac{\partial q_3}{\partial r} \right)$$

$$- \left(q_2 \frac{\partial q_2}{\partial r} + \frac{q_2{}^2}{r} - \frac{q_2}{r}\frac{\partial q_1}{\partial \theta} \right)$$

$$= \frac{\partial q_1}{\partial t} + q_1 \frac{\partial q_1}{\partial r} + q_2 \frac{\partial q_2}{\partial r} + q_3 \frac{\partial q_3}{\partial r} + q_3 \frac{\partial q_1}{\partial z} - q_1 \frac{\partial q_3}{\partial r}$$

$$- q_2 \frac{\partial q_2}{\partial r} - \frac{q_2^2}{r} + \frac{q_2}{r}\frac{\partial q_1}{\partial \theta}$$

$$= \frac{\partial q_1}{\partial t} + q_1 \frac{\partial q_1}{\partial r} + \frac{q_2}{r}\frac{\partial q_1}{\partial \theta} + q_3 \frac{\partial q_1}{\partial z} - \frac{q_2^2}{r}$$

If we define the differential operator

$$\frac{D}{Dt} \equiv \frac{d}{dt} = \frac{\partial}{\partial t} + q_1 \frac{\partial}{\partial r} + \frac{q_2}{r}\frac{\partial}{\partial \theta} + q_3 \frac{\partial}{\partial z}, \text{ then}$$

$$\left.\begin{aligned} f_1 &= \frac{Dq_1}{Dt} - \frac{q_2^2}{r} \equiv \frac{Du}{Dt} - \frac{v^2}{r} \\ \text{Similarly,} \quad f_2 &= \frac{Dq_2}{Dt} + \frac{q_1 q_2}{r} \equiv \frac{Dv}{Dt} - \frac{uv}{r} \end{aligned}\right\} \quad (9)$$

$$f_3 = \frac{Dq_3}{Dt} \equiv \frac{Dw}{Dt}$$

where $(q_1, q_2, q_3) \equiv (u, v, w)$

Equation (9) gives the required components of acceleration in cylindrical co-ordinates.

8.5. Components of Acceleration in Spherical Co-ordinates (r, θ, ψ). Here,

$$u_1 \equiv r,\ u_2 \equiv \theta,\ u_3 \equiv \psi \quad \text{and } h_1 = 1,\ h_2 = r,\ h_3 = r\sin\theta$$

Therefore, $$\nabla \equiv \left(\frac{\partial}{\partial r}, \frac{1}{r}\frac{\partial}{\partial \theta}, \frac{1}{r\sin\theta}\frac{\partial}{\partial \psi}\right)$$

and

$$\xi_1 = \frac{1}{r^2\sin\theta}\left[\frac{\partial}{\partial\theta}(r\sin\theta q_3) - \frac{\partial}{\partial\psi}(rq_2)\right]$$

$$= \frac{1}{r^2\sin\theta}\left[r\left(\cos\theta q_3 + \sin\theta\frac{\partial q_3}{\partial\theta}\right) - r\frac{\partial q_2}{\partial\psi}\right]$$

$$= \frac{1}{r\sin\theta}\left[q_3\cos\theta + \sin\theta.\frac{\partial q_3}{\partial\theta} - \frac{\partial q_2}{\partial\psi}\right]$$

$$\xi_2 = \frac{1}{r\sin\theta}\left[\frac{\partial}{\partial\psi}(q_1) - \frac{\partial}{\partial r}(r\sin\theta q_3)\right]$$

$$= \frac{1}{r\sin\theta}\left[\frac{\partial q_1}{\partial\psi} - \sin\theta q_3 - r\sin\theta\frac{\partial q_3}{\partial r}\right]$$

$$\xi_3 = \frac{1}{r}\left[\frac{\partial}{\partial r}(rq_2) - \frac{\partial}{\partial\theta}(q_1)\right] = \frac{1}{r}\left[q_2 + r\frac{\partial q_2}{\partial r} - \frac{\partial q_1}{\partial\theta}\right]$$

Thus,

$f_1 =$

$$\frac{\partial q_1}{\partial t} + \frac{1}{2}\frac{\partial}{\partial r}\left(q_1^2 + q_2^2 + q_3^2\right) + \frac{q_3}{r\sin\theta}\left[\frac{\partial q_1}{\partial\psi} - q_3\sin\theta - r\sin\theta\frac{\partial q_3}{\partial r}\right]$$

$$-\frac{q_2}{r}\left[q_2 + r\frac{\partial q_2}{\partial r} - \frac{\partial q_1}{\partial \theta}\right]$$

$$= \frac{\partial q_1}{\partial t} + q_1\frac{\partial q_1}{\partial r} + q_2\frac{\partial q_2}{\partial r} + q_3\frac{\partial q_3}{\partial r} + \frac{q_3}{r\sin\theta}\frac{\partial q_1}{\partial \psi} - \frac{q_3^2}{r}$$

$$-q_3\frac{\partial q_3}{\partial r} - \frac{q_2^2}{r} - q_2\frac{\partial q_3}{\partial r} + \frac{q_2}{r}\frac{\partial q_1}{\partial \theta}$$

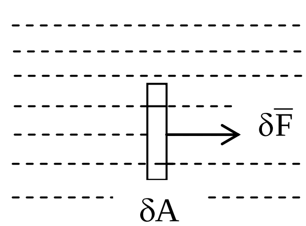
$$= \frac{Dq_1}{Dt} - \frac{q_2^2}{r} - \frac{q_3^2}{r}, \text{ where } \frac{D}{Dt} = \frac{\partial}{\partial t} + q_1\frac{\partial}{\partial r} + \frac{q_2}{r}\frac{\partial}{\partial \theta} + \frac{q_3}{r\sin\theta}\frac{\partial}{\partial \psi}$$

$$\left.\begin{aligned}
&\text{i.e.} && f_1 = \frac{Dq_1}{Dt} - \frac{q_2^2 + q_3^2}{r} \equiv \frac{Du}{Dt} - \frac{v^2 + w^2}{r}\\
&\text{Similarly,} && f_2 = \frac{Dq_2}{Dt} + \frac{q_1 q_2 - q_3^2\cot\theta}{r} \equiv \frac{Dv}{Dt} + \frac{uv - w^2\cot\theta}{r}\\
& && f_3 = \frac{Dq_3}{Dt} + \frac{q_1 q_3 + q_2 q_3\cot\theta}{r} \equiv \frac{Dw}{Dt} + \frac{w(u + v\cot\theta)}{r}
\end{aligned}\right\} \quad (10)$$

Equation (10) gives the required comps of acceleration in spherical co-ordinates.

8.6. Pressure at a point of a Moving Fluid. Let P be a point in a ideal (inviscid) fluid moving with velocity $\bar{q}$. We insert an elementary rigid plane area δA into this fluid at point P. This plane area also moves with the velocity $\bar{q}$ of the local fluid at P.

If $\delta\bar{F}$ denotes the force exerted on one side of δA by the fluid particles on the other side,

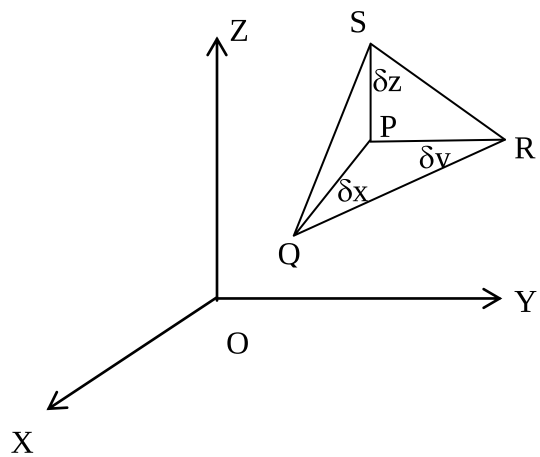


then this force will act normal to δA.

Further, if we assume that $\underset{\delta A \to 0}{\mathrm{Lt}} \dfrac{\delta \overline{F}}{\delta A}$ exists uniquely, then this limit is called the (hydrodynamic) fluid pressure at point P and is denoted by p.

8.7.Theorem :- Prove that the pressure p at a point P in a moving inviscid fluid is same in all direction.

Proof :- Let $\overline{q}$ be the velocity of the fluid. We consider am elementary tetrahedron PQRS of the fluid at a point P of the moving fluid, Let the edges of the tetrahedron be PQ = δx, PR = δy, PS = δz at time t, where δx, δy, δz are taken along the co-ordinate axes OX, OY, OZ respectively. This tetrahedron is also moving with the velocity $\overline{q}$ of the local fluid at P.



Let p be the pressure on the face QRS where area is δs. Suppose that < *l*, m, n> are the d.c.'s of the normal to δs drawn outwards from the tetrahedron. Then,

$$l\delta s = \text{projection of the area } \delta s \text{ on yz-plane.}$$

$$= \text{area of face PRS (triangle)}$$

$$= \frac{1}{2}\delta y.\delta z = \frac{\delta y \delta z}{2}$$

Similarly,

$$m\delta s = \text{area of face PQS} \;=\; \frac{1}{2}\delta z.\delta x = \frac{\delta z \delta x}{2}$$

and

$$n\delta s = \text{area of face PQR} \;=\; \frac{1}{2}\delta x.\delta y = \frac{\delta x \delta y}{2}$$

The total force exerted by the fluid, outside the tetrahedron, on the face QRS is

$$= -p\delta s\ (l\hat{i} + m\hat{j} + n\hat{k})$$

$$= -p\ (l\delta s\,\hat{i} + m\delta s\,\hat{j} + n\delta s\,\hat{k})$$

$$= -\frac{p}{2}(\delta y\delta z\,\hat{i} + \delta z\delta x\,\hat{j} + \delta x\delta y\,\hat{k})$$

Let p_x, p_y, p_z be the pressures on the faces PRS, PQS, PRQ. The forces exerted on these faces by the exterior fluid are

$$\frac{1}{2}p_x\delta y\delta z\ \hat{i},\ \frac{1}{2}p_y\delta z\delta x\hat{j},\ \frac{1}{2}p_z\delta x\delta y\hat{k}\ \text{ respectively.}$$

Thus, the total surface force on the tetrahedron is

$$-\frac{p}{2}(\delta y\delta z\,\hat{i} + \delta z\delta x\,\hat{j} + \delta x\delta y\,\hat{k}) + \frac{1}{2}p_x\delta y\delta z\,\hat{i}$$

$$+\ \frac{1}{2}p_y\ \delta z\,\delta x\,\hat{j} + \frac{1}{2}p_z\delta x\delta y\,\hat{k}$$

$$= \frac{1}{2}\left[(p_x - p)\delta y\delta z\,\hat{i} + (p_y - p)\delta z\delta x\,\hat{j} + (p_z - p)\delta x\delta y\,\hat{k}\right] \qquad (1)$$

In addition to surface force (fluid forces), the fluid may be subjected to body forces which are due to external causes such as gravity. Let $\overline{F}$ be the mean body force per unit mass within the tetrahedron.

Volume of the tetrahedron PQRS is $\frac{1}{3}h\ \delta s$ i.e. $\frac{1}{6}\delta x\ \delta y\ \delta z$, where h is the perpendicular from P on the face QRS.

Thus, the total force acting on the tetrahedron PQRS is = $\frac{1}{6}e\,\overline{F}\,\delta x\ \delta y\ \delta z$ (2)

Where ρ is the mean density of the fluid.

From (1) and (2), the net force acting on the tetrahedron is

$$\frac{1}{2}\left[p_x - p)\delta y\delta z\,\hat{i} + (p_y - p)\delta z\delta x\,\hat{j} + (p_z - p)\delta x\delta y\,\hat{k}\right] + \frac{1}{6}\rho\,\overline{F}\delta x\delta y\delta z$$

Now, the acceleration of the tetrahedron is $\frac{D\overline{q}}{Dt}$ and the mass $\frac{1}{6}\rho\ \delta x\ \delta y\ \delta z$ of fluid inside it is constant.

Thus, the equation of motion of the fluid contained in the tetrahedron is

$$\frac{1}{2}\left[(p_x - p)\delta y\delta z\,\hat{i} + (p_y - p)\delta z\delta x\,\hat{j} + (P_z - p)\delta x\delta y\,\hat{k}\right] + \frac{1}{6}\rho\,\overline{F}\delta x\delta y\delta z$$

$$= \frac{1}{6}\rho\delta x\delta y\delta z\left(\frac{D\overline{q}}{Dt}\right). \qquad (\ \overline{f} = m\ \ \overline{a})$$

i.e.

$$(p_x - p)\ l$$

$$\delta s\,\hat{i} + (p_y - p)m\,\delta s\,\hat{j} + (p_z - p)n\delta s\,\hat{k} + \frac{1}{3}\rho\,\overline{F}h\delta s = \frac{1}{3}\rho h\delta s\frac{D\overline{q}}{Dt}$$

On dividing by Ss and letting the tetrahedron shrink to zero about P, in which case $h \to 0$, it follows that

$$p_x - p = 0,\ p_y - p = 0,\ p_z - p = 0$$

i.e.

$$p_x = p_y = p_z = p. \qquad (3)$$

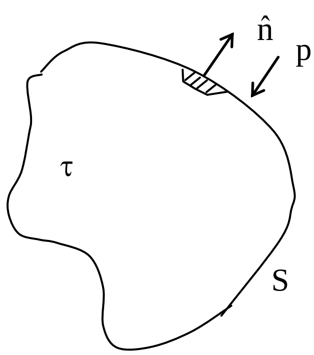
Since the choice of axes is arbitrary, the relation (3) establishes that at any point P of a moving ideal fluid, the pressure p is same in all directions.

9. Equations of Motion

9.1. Euler's Equation of Motion of an Ideal Fluid (Equation of Conservation of Momentum). To obtain Euler's dynamical equation, we shall make use of Newton's second law of motion.

Consider a region τ of fluid bounded by a closed surface S which consists of the same fluid particles at all times. Let $\overline{q}$ be the velocity and ρ be the density of the fluid.

Then $\rho\ d\tau$ is an element of mass within S and it remains constant.



The linear momentum of volume τ is

$$\overline{M} = \int_\tau \overline{q}\,\rho d\tau \qquad | \text{ mass} \times \text{velocity} = \text{momentum.}$$

Rate of change of momentum is

$$\frac{d\overline{M}}{dt} = \frac{d}{dt}\int_\tau \overline{q}\,\rho d\tau = \int_\tau \frac{d\overline{q}}{dt}\rho d\tau \qquad (1)$$

The fluid within τ is acted upon by two types of forces

The first type of forces are the surface forces which are due to the fluid exterior to τ.
Since the fluid is ideal, the surface force is simply the pressure p directed along the inward normal at all point of S.
The total surface force on S is

$$\int_S p(-\hat{n})dS = -\int_S p\,\hat{n}dS = \int_\tau \nabla p d\tau \qquad \text{(By Gauss div. Theorem)} \qquad (2)$$

The second type of forces are the body forces which are due to some external agent. Let $\overline{F}$ be the body force per unit mass acting on the fluid. Then $\overline{F}\ \rho d\tau$ is the body force on the element of mass e$d\tau$ and the total body force on the mass within τ is

$$\int_\tau \overline{F}\rho d\tau \qquad (3)$$

By Newton's second law of motion, we have

Rate of change of momentum = total force

$$\Rightarrow \qquad \int_\tau \frac{d\overline{q}}{dt}\rho d\tau = \int_\tau \overline{F}\rho d\tau - \int_\tau \nabla p d\tau$$

$$\Rightarrow \qquad \int_\tau \left(\frac{d\overline{q}}{dt}\rho - \overline{F}\rho + \nabla p\right)d\tau = 0$$

Since $d\tau$ is arbitrary, we get

$$\frac{d\overline{q}}{dt}\rho - \overline{F}\rho + \nabla p = 0$$

i.e.

$$\frac{d\overline{q}}{dt} = \overline{F} - \frac{1}{\rho}\nabla p \qquad (4)$$

which holds at every point of the fluid and is known as Euler's dynamical equation for an ideal fluid.

9.2. Remark. The above method for obtaining the Euler's equation of motion, is also known as **flux method**.

9.3. Other Forms of Euler's Equation of Motion. (i) We know that

$$\frac{d}{dt} \equiv \frac{D}{Dt} = \frac{\partial}{\partial t} + \bar{q} \cdot \nabla ,$$

therefore equation (4) becomes.

$$\frac{\partial \bar{q}}{\partial t} + (\bar{q} \cdot \nabla)\bar{q} = \bar{F} - \frac{1}{\rho}\nabla p \tag{5}$$

But $(\bar{q} \cdot \nabla)\bar{q} = \nabla\left(\frac{1}{2}\bar{q}^2\right) + \bar{\xi} \times \bar{q}, \ \bar{\xi} = \text{curl } \bar{q}$

Therefore, Euler's equation becomes

$$\frac{\partial \bar{q}}{\partial t} + \nabla\left(\frac{1}{2}\bar{q}^2\right) + \bar{\xi} \times \bar{q} = \bar{F} - \frac{1}{\rho}\nabla p. \tag{6}$$

Equation (6) is called Lamb's hydrodynamical equation

(ii) Cartesian Form. Let $\bar{q} = (u, v, w), \bar{F} = (X, Y, Z)$ and $\nabla p = \left(\frac{\partial p}{\partial x}, \frac{\partial p}{\partial y}, \frac{\partial p}{\partial z}\right)$,

then equation (5) gives

$$\left.\begin{aligned}
\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} &= X - \frac{1}{\rho}\frac{\partial p}{\partial x} \\
\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z} &= Y - \frac{1}{\rho}\frac{\partial p}{\partial x} \\
\frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z} &= Z - \frac{1}{\rho}\frac{\partial p}{\partial z}
\end{aligned}\right\} \tag{7}$$

Equation (7) are the required equations in Cartesian form.

(iii) Equations of Motion in Cylindrical Co-ordinates. (r, θ, z). Here,

$$\bar{q} = (u, v, w),\ d\bar{r} = (dr, rd\theta, dz)$$

$$\nabla p = \left(\frac{\partial p}{\partial r}, \frac{1}{r}\frac{\partial p}{\partial \theta}, \frac{\partial p}{\partial z}\right)$$

Let $\overline{F}$ = (Fr, Fθ, Fz).

Also, the acceleration components in cylindrical co-ordinates are

$$\frac{d\overline{q}}{dt} = \left(\frac{du}{dt} - \frac{v^2}{r}, \frac{dv}{dt} + \frac{uv}{r}\frac{dw}{dt}\right)$$

Thus, the equation of motion

$$\frac{d\overline{q}}{dt} = \overline{F} - \frac{1}{\rho}\nabla p. \text{ becomes}$$

$$\left.\begin{aligned} \frac{du}{dt} - \frac{v^2}{r} &= F_r - \frac{1}{\rho}\frac{\partial p}{\partial r} \\ \frac{dv}{dt} + \frac{vu}{r} &= F_\theta - \frac{1}{r\rho}\frac{\partial p}{\partial \theta} \\ \frac{dw}{dt} &= F_z - \frac{1}{\rho}\frac{\partial p}{\partial z} \end{aligned}\right\} \qquad (8)$$

(iv) Equations of Motion in Spherical co-ordinates (r, θ, ψ). Here,

$$\overline{q} = (u, v, w),\ d\overline{r} = (dr, rd\theta,\ r\sin\theta d\psi)$$

$$\nabla p = \left(\frac{\partial p}{\partial r}, \frac{1}{r}\frac{\partial p}{\partial \theta}, \frac{1}{r\sin\theta}\frac{\partial p}{\partial \psi}\right)$$

Let $\overline{F}$ = (Fr, Fθ, Fψ). The components of acceleration in spherical co-ordinates are

$$\frac{d\overline{q}}{dt} = \left(\frac{du}{dt} - \frac{v^2 + w^2}{r}, \frac{dv}{dt} - \frac{w^2\cot\theta}{r} + \frac{uv}{r}, \frac{dw}{dt} + \frac{vw\cot\theta}{r}\right)$$

Thus, the equation of motion take the form

$$\frac{du}{dt}-\frac{v^2+w^2}{r}=Fr-\frac{1}{\rho}\frac{\partial p}{\partial r}$$
$$\frac{dv}{dt}-\frac{w^2\cot\theta}{r}+\frac{uv}{r}=F\theta-\frac{1}{\rho r}\frac{\partial p}{\partial\theta} \qquad (9)$$
$$\frac{dw}{dt}+\frac{vw\cot\theta}{r}=F_\psi-\frac{1}{\rho r\sin\theta}\frac{\partial p}{\partial\psi}$$

9.4. Remark :- The two equations, the equation of continuity and the Euler's equation of motion, comprise the equations of motion of an ideal fluid. Thus the equations

$$\frac{\partial\rho}{\partial t}+\operatorname{div}(\rho\bar{q})=0$$

and $\dfrac{\partial\bar{q}}{\partial t}+(\bar{q}\cdot\nabla)\bar{q}=\bar{F}-\dfrac{1}{\rho}\nabla p$

are fundamental to any theoretical study of ideal fluid flow. These equations are solved subject to the appropriate boundary and initial conditions dictated by the physical characteristics of the flow.

9.5. Lagrange's Equation of Motion. Let initially a fluid element be at (a, b, c) at time $t = t_0$ when its volume is dV_0 and density is ρ_0. After time t, let the same fluid element be at (x, y, z) when its volume is dV and density is ρ . The equation of continuity is

$$\rho J = \rho_0 \qquad (1)$$

where $J = \dfrac{\partial(x,y,z)}{\partial(a,b,c)}$

The components of acceleration are

$$\ddot{x}=\frac{\partial^2 x}{\partial t^2},\ \ddot{y}=\frac{\partial^2 y}{\partial t^2},\ \ddot{z}=\frac{\partial^2 z}{\partial t^2}$$

Let the body force $\bar{F}$ be conservative so that we can express it in terms of a body force potential function Ω as

$$\bar{F} = -\nabla\Omega \qquad (2)$$

By Euler's equation of motion,

$$\frac{d\bar{q}}{dt} = \bar{F} - \frac{1}{\rho}\nabla p = -\nabla\Omega - \frac{1}{\rho}\nabla p \tag{3}$$

Its Cartesian equivalent is

$$\left.\begin{aligned} \frac{\partial^2 x}{\partial t^2} &= -\frac{\partial \Omega}{\partial x} - \frac{1}{\rho}\frac{\partial p}{\partial x} \\ \frac{\partial^2 y}{\partial t^2} &= -\frac{\partial \Omega}{\partial y} - \frac{1}{\rho}\frac{\partial p}{\partial y} \\ \frac{\partial^2 z}{\partial t^2} &= -\frac{\partial \Omega}{\partial z} - \frac{1}{\rho}\frac{\partial p}{\partial z} \end{aligned}\right\} \tag{4}$$

We note that a, b, c, t are the independent variables and our object is to determine x, y, z in terms of a, b, c, t and so investigate completely the motion.

To deduce equations containing only differentiations w.r.t. the independent variables a, b, c, t we multiply the equations in (4) by $\partial x/\partial a$, $\partial y/\partial a$, $\partial z/\partial a$ and add to get

$$\frac{\partial^2 x}{\partial t^2}\frac{\partial x}{\partial a} + \frac{\partial^2 y}{\partial t^2}\frac{\partial y}{\partial a} + \frac{\partial^2 z}{\partial t^2}\frac{\partial z}{\partial a} = -\frac{\partial \Omega}{\partial a} - \frac{1}{\rho}\frac{\partial p}{\partial a} \tag{5}$$

Similarly, we get

$$\frac{\partial^2 x}{\partial t^2}\frac{\partial x}{\partial b} + \frac{\partial^2 y}{\partial t^2}\frac{\partial y}{\partial b} + \frac{\partial^2 z}{\partial t^2}\frac{\partial z}{\partial b} = -\frac{\partial \Omega}{\partial b} - \frac{1}{\rho}\frac{\partial p}{\partial b} \tag{6}$$

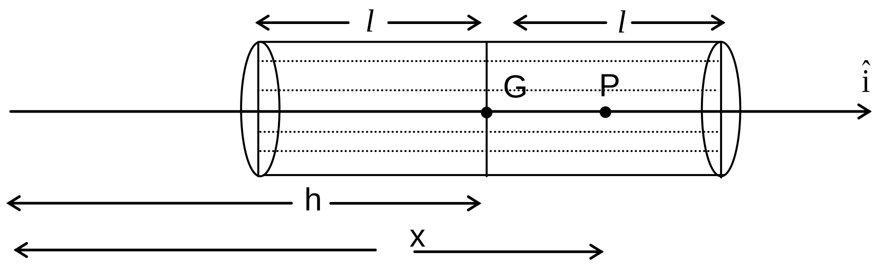
$$\frac{\partial^2 x}{\partial t^2}\frac{\partial x}{\partial c} + \frac{\partial^2 y}{\partial t^2}\frac{\partial y}{\partial c} + \frac{\partial^2 z}{\partial t^2}\frac{\partial z}{\partial c} = -\frac{\partial \Omega}{\partial c} - \frac{1}{\rho}\frac{\partial p}{\partial c} \tag{7}$$

These equations (5), (6), (7) together with equation (1) constitute Lagarange's Hydrodynamical Equations.

9.6. Example. A homogeneous incompressible liquid occupies a length $2l$ of a straight tube of uniform small bore and is acted upon by a body force which is such that the fluid is attracted to a fixed point of the tube, with a force varying as the distance from the point. Discuss the motion and determine the velocity and pressure within the liquid.

Solution. We note that the small bore of the tube permits us to ignore any variation of velocity across any cross-section of the tube and to suppose that the flow is unidirectional.

We u be the velocity along the tube and p be the pressure at a general point P at distance x from the centre of force O. Also, let h be the distance of the centre of mass G of the fluid, as shown in the figure.



Equations of motion of the fluid are :

(i) Equation of Continuity : Here, $\bar{q} = (u,0,0)$
Therefore, equation of continuity becomes

$$\frac{\partial u}{\partial x} = 0 \Rightarrow u = u(t) \qquad (1)$$

(ii) Euler's Equation : In this case, it becomes

$$\frac{\partial u}{\partial t} + \frac{u\partial u}{\partial x} + X - \frac{1}{\rho}\frac{\partial p}{\partial x} = -\mu x - \frac{1}{\rho}\frac{\partial p}{\partial x}$$

$$\Rightarrow \quad \frac{\partial u}{\partial t} = -\mu x - \frac{1}{\rho}\frac{\partial p}{\partial x} \qquad (2) \qquad | \text{ using (1)}$$

where $-\mu x\,\hat{i}$ is the body force per unit mass, μ being a positive constant.

We observe that equation (2) can be written as

$$\frac{du}{dt} = -\mu x - \frac{1}{\rho}\frac{dp}{dx} \qquad (3)$$

Integrating w.r.t. x, we get

$$x\frac{du}{dt} = -\mu\frac{x^2}{2} - \frac{p}{\rho} + C \qquad (4)$$

where C is a constant and at most can be a function of t only. w.r.t. (x, y, z)

Let Π be the pressure at the free surfaces $x = h-l$ and $x = h + l$ of the liquid. Then using these boundary conditions, equation (4) becomes

$$(h-l)\frac{du}{dt} = -\frac{1}{2}\mu(h-l)^2 - \frac{\Pi}{\rho} + C$$

$$(h+l)\ \frac{du}{dt} = -\frac{1}{2}\mu(h-l)^2 - \frac{\Pi}{\rho} + C$$

which on subtraction give

$$\frac{du}{dt} = -\mu h \qquad (5)$$

But in the fluid motion all fluid particles move with the same velocity u and u $= \frac{dh}{dt}$

$\therefore$ Equation (5) becomes

$$\frac{d^2h}{dt^2} = -\mu h \qquad (6)$$

Now, we solve the different equation (6), which can be written as

$$(D^2 + \mu)\ h = 0$$

Here auxiliary equation is

$$D^2 + \mu = 0 \ \Rightarrow D = \pm\sqrt{\mu}\ i$$

Therefore, the solution of (6) is

$$h = A \cos\left(\sqrt{\mu}\ t + \in\right)$$

where A and $\in$ are constants which can be determined from initial conditions.

To Calculate Pressure :– We have from (3) & (5)

$$-\mu x - \frac{1}{\rho}\frac{dp}{dx} = -\mu h$$

$$\Rightarrow \frac{1}{\rho}\frac{dp}{dx} = \mu(h-x)$$

Integrating w.r.t. x , we get

$$\frac{p}{\rho} = \frac{\mu(h-x)^2}{2(-1)} + D \qquad (7)$$

The boundary condition x = h – l, p = Π gives

$$\frac{\Pi}{\rho} = \mu.\frac{l^2}{-2} + D$$

i.e. $$D = \Pi/\rho + \frac{\mu l^2}{2}$$

Therefore, equation (7) becomes

$$\frac{p}{\rho} = \frac{\mu(h-x)^2}{-2} + \Pi/\rho + \frac{\mu l^2}{2}$$

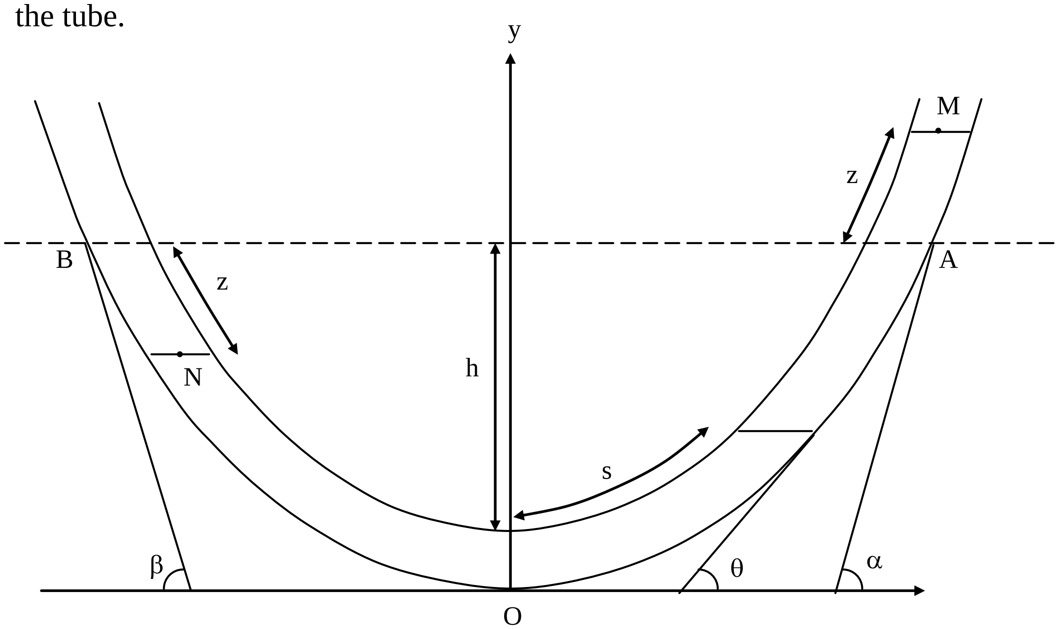
$$= \frac{\Pi}{\rho} - \frac{\mu}{2}\left[(h-x)^2 - l^2\right]$$

$$= \frac{\Pi}{\rho} - \frac{\mu}{2}\left[(h-x+l)(h-x-l)\right]$$

9.7. Example. Homogeneous liquid is in motion in a vertical plane, within a curved tube of uniform small bore, under the action of gravity. Calculate the period of oscillation.

Solution. Let O be the lowest point of the tube, AB the equilibrium level of the liquid and h the height of AB above O. Let α and β be respectively the inclinations of the tube to the horizontal at A and B and θ be the inclination of the tube at a distance s along the tube from O. Let a and b denote the arc lengths of OA and OB respectively and suppose that at time t, the liquid is displaced through a small distance z along the tube from its equilibrium position.

Due to the assumption of uniform small bore the flow is unidirectional along the tube.



Let the velocity be u(s, t).

The equation of continuity gives $\frac{\partial u}{\partial s} = 0$ (1)

$\Rightarrow$ u is independent of s
Euler's equation of motion becomes

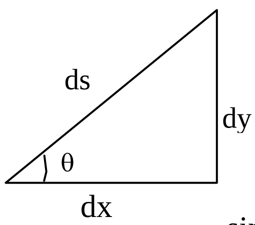
$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial s} = -g\sin\theta - \frac{1}{\rho}\frac{\partial p}{\partial s}$$

Using equation (1), this gives

$$\frac{du}{dt} \equiv \frac{\partial u}{\partial t} = -g\sin\theta - \frac{1}{\rho}\frac{dp}{ds}$$

i.e. $$\frac{du}{dt} = -g\frac{dy}{ds} - \frac{1}{\rho}\frac{dp}{ds} \qquad (2)$$

Integrating it w.r.t. s, we find



$\sin\theta = \dfrac{dy}{ds}$

$$s\frac{du}{dt} = -gy - \frac{p}{\rho} + C \qquad (3)$$

where C may be a function of time t at the most.
The boundary conditions at free surface are

(i) p = Π for y = h + z sin α, s = OM = a + z at M

(ii) p = Π for y = h – z sin β, s = ON = –(b–z) at N .

Using these boundary conditions in (3), we get

$$(a + z)\ \frac{du}{dt} = -g(h + z\sin\alpha) - \frac{\Pi}{\rho} + C$$

$$-(b-z)\ \frac{du}{dt} = -g(h - z\sin\beta) - \frac{\Pi}{\rho} + C$$

Subtracting these we get

$$(a+b)\frac{du}{dt} = -gz(\sin\alpha + \sin\beta) \qquad (4)$$

Since

$$u = \frac{dz}{dt} \Rightarrow \frac{du}{dt} = \frac{d^2z}{dt^2},$$

equation (4) becomes

$$(a+b)\frac{d^2z}{dt^2} = -gz(\sin\alpha + \sin\beta)$$

$$\Rightarrow \qquad \frac{d^2z}{dt^2} = -\mu z, \qquad (5)$$

where $$\mu = \frac{g(\sin\alpha + \sin\beta)}{a+b}$$

We observe that equation (5) represents the simple harmonic motion. It's period T is given by

$$T = \frac{2\pi}{\sqrt{\mu}} = 2\pi\left[\frac{a+b}{g(\sin\alpha + \sin\beta)}\right]^{\frac{1}{2}}.$$

10. Bernoulli's Equation (Theorem)

10.1. For Steady Flow. We shall obtain a special form of Euler's dynamical equation in terms of pressure. The Euler's dynamical equation is

$$\frac{d\overline{q}}{dt} = \overline{F} - \frac{1}{\rho}\nabla p \qquad (1)$$

where $\overline{q}$ is velocity, $\overline{F}$ is the body force, p and ρ are pressure and density respectively.

$\overline{F}$ be conservative so that it can be expressed in terms of a body force potential function Ω as

$$\overline{F} = -\nabla\,\Omega \qquad (2)$$

When the flow is steady, then $\frac{\partial\overline{q}}{\partial t} = 0$ (3)

Therefore, in case of steady motion with a conservative body force equation (1), on using (2) and (3), gives

$$\nabla\left(\frac{1}{2}\bar{q}^2\right) - \bar{q}\times\bar{\xi} = -\nabla\Omega - \frac{1}{\rho}\nabla p$$

$$\left|\begin{array}{l} \because \dfrac{d\bar{q}}{dt} = \dfrac{\partial\bar{q}}{\partial t} + (\bar{q}.\nabla)\bar{q} \\ \text{or}\,\dfrac{d\bar{q}}{dt} = \dfrac{\partial\bar{q}}{\partial t} + \nabla\left(\dfrac{1}{2}\bar{q}^2\right) - \bar{q}\times\bar{\xi} \ \text{ and } \dfrac{\partial q}{\partial t} = 0 \end{array}\right.$$

$$\Rightarrow \qquad \nabla\left(\frac{1}{2}\bar{q}^2 + \Omega\right) + \frac{1}{\rho}\nabla p = \bar{q}\times\bar{\xi} \tag{4}$$

Further, if we suppose that the liquid is barotropic i.e. density is a function of pressure p only, then we can write

$$\frac{1}{\rho}\nabla p = \nabla\int\frac{dp}{\rho}$$

Using this in (4), we get

$$\nabla\left[\frac{1}{2}\bar{q}^2 + \Omega + \int\frac{dp}{\rho}\right] = \bar{q}\times\bar{\xi}. \tag{5}$$

Multiplying (5) scalarly by $\bar{q}$ and noting that

$\bar{q}\cdot(\bar{q}\times\bar{\xi}) = (\bar{q}\times\bar{q})\cdot\bar{\xi} = 0$, we get

$$\bar{q}\cdot\nabla\left[\frac{1}{2}\bar{q}^2 + \Omega + \int\frac{dp}{\rho}\right] = 0 \tag{6}$$

If $\hat{s}$ is a unit vector along the streamline through general point of the fluid and s measures distance along this stream line, then since $\hat{s}$ is parallel to $\bar{q}$, therefore equation (6) gives

$$\frac{\partial}{\partial s}\left[\frac{1}{2}\bar{q}^2 + \Omega + \int\frac{dp}{\rho}\right] = 0 \qquad \left|\begin{array}{l} \because \hat{s} \text{ is parallel to } \bar{q} \\ \bar{q} = k\hat{s} \\ \hat{s}\nabla \equiv \dfrac{\partial}{\partial s} \end{array}\right.$$

Hence along any particular streamline, we have

$$\frac{1}{2}\bar{q}^2 + \Omega + \int\frac{dp}{\rho} = C \tag{7}$$

where C is constant which takes different values for different streamlines. Equation (7) is known as Bernaull's equation. This result applies to steady flow of ideal. barotropic fluids in which the body forces are conservative.
Now, if $\hat{s}$ is a unit vector taken along a vortexline, then, similarly, we get

$\frac{1}{2}\bar{q}^2 + \Omega + \int \frac{dp}{\rho} = C$ along any particular vortexline. (Here, we multiply scalarly by $\bar{\xi}$)

10.2. Remark. (i) If $\bar{q} \times \bar{\xi} = \bar{0}$ i.e. if $\bar{q}$ & $\bar{\xi}$ are parallel, then streamlines and vortex lines coincide and $\bar{q}$ is said to be **Beltrami vector.**

If $\bar{\xi} = \bar{0}$, the flow is irrotational.
For both of these flow patterns,

$$\frac{1}{2}\bar{q}^2 + \Omega + \int \frac{dp}{\rho} = C$$

where C is same at all points of the fluid.
(ii) For homogeneous incompressible fluids, ρ is constant and

$$\int \frac{dp}{\rho} = \frac{p}{\rho}.$$

The Bernoulli's equation becomes

$$\frac{p}{\rho} + \frac{1}{2}\bar{q}^2 + \Omega = C$$

so that if $\bar{q}$ is known, the pressure can be calculated.

10.3. For Unsteady Irrotational Flow. Here also, we suppose that the body forces are conservative i.e. $\bar{F} = -\nabla\Omega$

For irrotational flow, $\bar{q} = -\nabla\phi \Rightarrow \nabla \times \bar{q} = 0$
The equation of motion

$$\frac{\partial \bar{q}}{\partial t} + \nabla\left(\frac{1}{2}\bar{q}^2\right) - \bar{q} \times (\nabla \times \bar{q}) = \bar{F} - \frac{1}{\rho}\nabla p \qquad (1)$$

in the present case becomes.

$$-\nabla\left(\frac{\partial \phi}{\partial t}\right) + \nabla\left(\frac{1}{2}\bar{q}^2\right) = -\nabla\Omega - \frac{1}{\rho}\nabla p$$

$$\Rightarrow \quad \nabla\left(\frac{1}{2}\bar{q}^2+\Omega+\int\frac{dp}{\rho}-\frac{\partial\phi}{\partial t}\right)=0 \qquad \text{| Barotropic fluid.}$$

Integrating, we get

$$\frac{1}{2}\bar{q}^2+\Omega+\int\frac{dp}{\rho}-\frac{\partial\phi}{\partial t}=f(t) \tag{2}$$

which is the required equation.

If the liquid is homogeneous, then $\int\frac{dp}{\rho}=\frac{p}{\rho}$ and the equation (2) become

$$\frac{1}{2}\bar{q}^2+\Omega+\frac{p}{\rho}-\frac{\partial\phi}{\partial t}=f(t).$$

Further, for study case,

$$\frac{\partial\phi}{\partial t}=0,\ f(t)=\text{const}$$

$$\frac{1}{2}\bar{q}^2+\Omega+\frac{p}{\rho}=\text{const}$$

10.4. Example. A long straight pipe of length L has a slowly tapering circular cross section. It is inclined so that its axis makes and angle α to the horizontal with its smaller cross-section downwards. The radius of the pipe at its upper end is twice that of at its lower end and water is pumped at a steady rate through the pipe to emerge at atmospheric pressure. It the pumping pressure is twice the atmospheric pressure, show that the fluid leaves the pipe with a speed U give by

$$U^2=\frac{32}{15}\left[gL\sin\alpha+\frac{\Pi}{\rho}\right],$$

where Π is atmospheric pressure

Solution. The assumption that the pipe is slowly tapering means that any variation in the velocity over any cross-section can be ignored. Let the velocity at the wider and of the pipe be V and the emerging velocity be U (velocity at the lower end). The only body force is that of gravity, so $\bar{F}=-g\,\hat{j}$ and consequently $\Omega = gy$

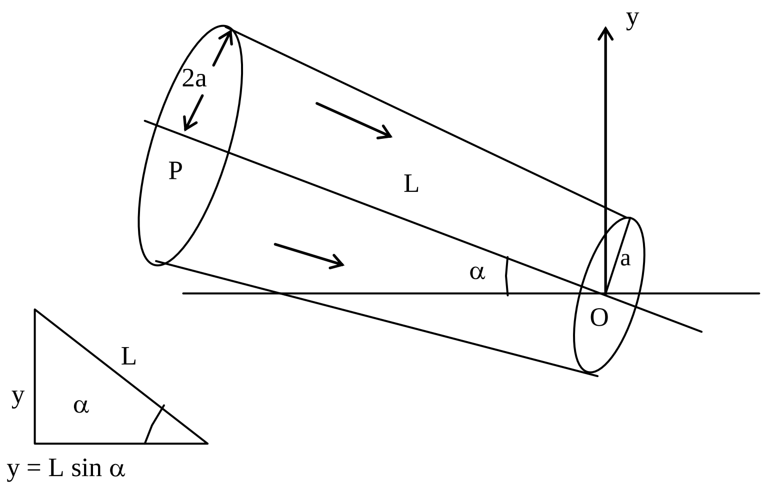
$$\because \bar{F} = -\nabla\Omega \;\Rightarrow -q\,\hat{j} = -\nabla\Omega = -\frac{\partial\Omega}{\partial x}\hat{i} - \frac{\partial\Omega}{\partial y}\hat{j} - \frac{\partial\Omega}{\partial z}\hat{k}$$

$$\Rightarrow -g = -\frac{\partial\Omega}{\partial y} \;\Rightarrow \Omega = gy$$

Bernoulli's equation, $\frac{p}{\rho} + \frac{1}{2}q^2 + \Omega = C$ | $\because$ For water ρ is const.

becomes $\frac{p}{\rho} + \frac{1}{2}q^2 + gy = C$ (1)

Applying this equation of the two ends of the pipe, we get



$$\frac{2\Pi}{\rho} + \frac{1}{2}.V^2 + gL\sin\alpha = \frac{\Pi}{\rho} + \frac{1}{2}U^2 \qquad (2) \qquad |\text{for}$$

lower end y = 0

Let a and 2a be the radii of the lower and upper ends respectively, then by the principle of conservation of mass

$$\pi(2a)^2V = \pi\, a^2\, U$$

$$\Rightarrow V = \frac{U}{4} \qquad (3)$$

From (2) and (3), we obtain

$$\Pi + \frac{1}{2}\rho\left(\frac{U^2}{16}\right) + g\rho L\sin\alpha = \frac{1}{2}\rho U^2$$

$$\Rightarrow \qquad \frac{1}{2}\rho\left(U^2 - \frac{U^2}{16}\right) = \Pi + g\rho L \sin\alpha.$$

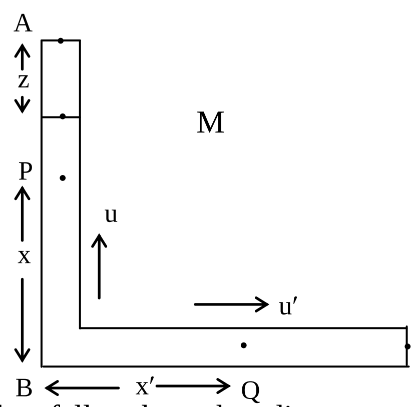
$$\Rightarrow \frac{15}{32}\rho U^2 = \Pi + g\rho L \sin\alpha.$$

$$\Rightarrow \qquad U^2 = \frac{32}{15}\left[gL\sin\alpha + \frac{\Pi}{\rho}\right]$$

Hence the result.

10.5. Example. A straight tube ABC, of small bore, is bent so as to make the angle ABC a right angle and AB equal to BC. The end C is closed and the tube is placed with end A upwards and AB vertical, and is filled with liquid. If the end C be opened, prove that the pressure at any point of the vertical tube is instantaneously diminished one-half. Also find the instantaneous change of pressure at any point of the horizontal tube, the pressure of the atmospheric being neglected.

Solution. Let AB = BC = a



When the liquid in AB has fallen through a distance z at time t, then let P be any point in the vertical column such that

$$AM = z, \; BP = x, \; BM = a-z$$

If u and p be the velocity and pressure at P, then equation of motion is

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = -g - \frac{1}{\rho}\frac{\partial p}{\partial x} \qquad (1) \qquad |u \equiv u(x, t)$$

and equation of continuity is

$$\frac{\partial u}{\partial x} = 0 \quad \text{i.e. } u = u(t)$$

Therefore, equation (1) becomes

$$\frac{\partial u}{\partial t} = -g - \frac{1}{\rho}\frac{\partial p}{\partial x}$$

Integrating. w.r.t. x , we get

$$x\frac{\partial u}{\partial t} = -gx. - \frac{1}{\rho}p + C \qquad (2)$$

Using the boundary condition p = 0 at x = a–z, we get

$$C = (a-z)\frac{\partial u}{\partial t} + g(a-z)$$

Therefore, equation (2) becomes

$$x\frac{\partial u}{\partial t} = -gx - \frac{p}{\rho} + (a-z)\frac{\partial u}{\partial t} + g(a-z)$$

i.e. $$\frac{p}{\rho} = -(x-a+z)\left(\frac{\partial u}{\partial t} + g\right) \qquad (3)$$

Now, we take a point Q in BC, where BQ = x′ and let u′, p′ be the velocity and pressure at Q, then

$$\frac{p'}{\rho} = -(x'-a)\frac{\partial u'}{\partial t} \quad | \text{ z = 0 and g is not effecting} \qquad (4)$$

Equating the pressure at B, when x = 0, x′ = 0, we get

$$(a-z)\left(\frac{\partial u}{\partial t} + g\right) = a\frac{\partial u'}{\partial t} \qquad | \text{ From (3) \& (4)}$$

$$= -a\frac{\partial u}{\partial t} \qquad | \because u' = -u$$

Initially, when C is just opened, then z = 0, t = 0 and we have

$$a\left[\left(\frac{\partial u}{\partial t}\right)_{t=0} + g\right] = -a\left(\frac{\partial u}{\partial t}\right)_{t=0}$$

$$\Rightarrow \qquad \left(\frac{\partial u}{\partial t}\right)_{t=0} = \frac{-g}{2} \quad \text{i.e.} \left(\frac{\partial u}{\partial t}\right)_{0} = -g/2 \qquad (5)$$

Therefore, from equation (3), initially, the pressure at P is given by

$$\frac{p_0}{\rho} = -(x-a)\left[\left(\frac{\partial u}{\partial t}\right)_0 + g\right] \qquad |p_0 \equiv (p)_{t=0}$$

$$= \frac{-g}{2}(x-a)$$

$$\Rightarrow \qquad p_0 = \frac{1}{2}\rho g(a-x) \qquad (6)$$

But when the end C is closed, the liquid is at rest and the hydrostatic pressure at P is

$$p_1 = \rho gh = \rho g\,(a-x) \qquad |h = AP = a-x \qquad (7)$$

From (6) and (7), we get

$$p_0 = \frac{1}{2}p_1$$

Thus, the pressure is diminished to one-half.
Now, from (4), initial pressure at Q is given by

$$\frac{p'_0}{\rho} = -(x'-a)\left(\frac{\partial u'}{\partial t}\right)_{t=0} = (x'-a)\left(\frac{\partial u}{\partial t}\right)_{t=0} = (a-x')\frac{g}{2}$$

$$\Rightarrow \quad p_0' = \frac{1}{2}\rho g(a-x')$$

When the end C is closed, the initial pressure (hydrostatic) p_2 at Q (or B or C) is $\rho g\,a$.
Therefore, instantaneous change in pressure

$$= p_2 - p_0' = \rho g a - \frac{1}{2}\rho g(a-x') = \frac{1}{2}\rho g(a+x')$$

10.6. Example. A sphere is at rest in an infinite mass of homogeneous liquid of density ρ, the pressure at infinity being Π. Show that, if the radius R of the sphere varies in any manner, the pressure at the surface of the sphere at any time is

$$\Pi + \frac{\rho}{2}\left[\frac{d^2(R^2)}{dt^2} + \left(\frac{dR}{dt}\right)^2\right]$$

Solution. In the incompressible liquid, outside the sphere, the fluid velocity $\bar{q}$ will be radial and thus will be a function of r, the radial distance from the centre of the sphere (the origin), and time t only.

The equation of continuity in spherical polar co-ordinates becomes

$$\frac{1}{r^2}\frac{d}{dr}(r^2u) = 0 \qquad (1)$$

$\because \bar{q} = (u,0,0),\ u = u(r,t),\ \nabla \equiv \left(\frac{\partial}{\partial r},0,0\right)$

$\nabla \cdot \bar{q} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2u).$

i.e. spherical symmetry

$$\Rightarrow r^2u = \text{constant} = f(t)$$

On the surface of the sphere,

$$r = R,\ u = \dot{R}$$

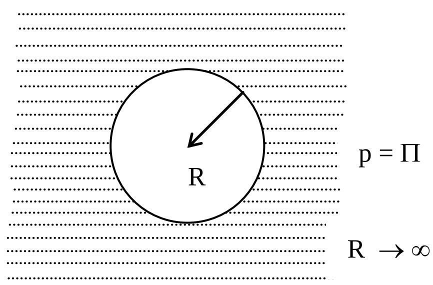
Therefore,

$$f(t) = R^2\ \dot{R}$$

and thus

$$r^2u = R^2\dot{R} \qquad (2)$$

$\therefore$



We observe that $u \to 0$ as $n \to \infty$, as required.

From (1), it is clear that curl $\bar{q} = \bar{0}$

$\Rightarrow$ the motion is irrotational and $\bar{q} = -\nabla\phi$

$$\Rightarrow\ u = -\frac{\partial\phi}{\partial r} \quad \Rightarrow\ -\frac{\partial\phi}{\partial r} = \frac{f}{r^2} \qquad |\text{ From (2)}$$

$$\Rightarrow\ \phi = f/r \qquad (3)$$

The pressure equation for irrotational non-steady fluid motion in the absence of body forces is

$$\frac{p}{\rho} + \frac{1}{2}\bar{q}^2 - \frac{\partial\phi}{\partial t} = C(t)$$

i.e. $$\frac{p}{\rho}+\frac{1}{2}u^2-\frac{\partial\phi}{\partial t}=C(t). \qquad (4)$$

where C(t) is a function of time t.

As $r\to\infty$, $p\to\Pi$, $u = f/r^2 \to 0$, $\phi\to 0$

so that $C(t) = \Pi/\rho$ for all t (5)

Therefore, from (2), (3), (4) and (5), we get

$$\frac{p}{\rho}=\frac{\Pi}{\rho}+\frac{\partial}{\partial t}(f/r)-\frac{1}{2}\left(\frac{R^2\dot{R}}{r^2}\right)^2 \qquad (6)$$

But $$\frac{\partial f}{\partial t}=\frac{d}{dt}(R^2\dot{R})=\ddot{R}R^2+2R\dot{R}^2$$

At the surface of the sphere, we have r = R and equation (6) gives

$$\frac{p}{\rho}=\frac{\Pi}{\rho}+\frac{1}{R}\left(2R\dot{R}^2+\ddot{R}R^2\right)-\frac{1}{2}\dot{R}^2$$

$$\Rightarrow \qquad \frac{p}{\rho}=\frac{\Pi}{\rho}+2\dot{R}^2+R\ddot{R}-\frac{1}{2}\dot{R}^2$$

$$=\frac{\Pi}{\rho}+\frac{1}{2}(3\dot{R}^2+2R\ddot{R}) \qquad (7)$$

Now,

$$\frac{d^2(R^2)}{dt^2}+\left(\dot{R}\right)^2=\frac{d}{dt}(2R\dot{R})+(\dot{R})^2$$

$$=(2R\ddot{R}+2\dot{R}^2)+\dot{R}^2$$

$$=2R\ddot{R}+3\dot{R}^2$$

Therefore, from (7), we obtain

$$p=\Pi+\frac{1}{2}\rho\left[\frac{d^2(R^2)}{dt^2}+\left(\frac{dR}{dt}\right)^2\right]$$

Hence the result.

10.7. Example. An infinite mass of ideal incompressible fluid is subjected to a force $\mu r^{-7/3}$ per unit mass directed towards the origin. If initially the fluid is at rest and there is a cavity in the form of the sphere r = a in it, show that the cavity will be completely filled after an interval of time $\pi a^{5/3}(10\mu)^{-1/2}$.

Solution. The motion is entirely radial and consequently irrotational and the present case is the case of spherical symmetry. The equation of continuity is

$$\frac{1}{r^2}\frac{d}{dr}(r^2 u) = 0 \Rightarrow r^2 u = \text{constant} = f(t) \qquad (1)$$

On the surface of the sphere, r = R, $\dot{R}$ = v (say)
Therefore,

$$r^2 \dot{r} = f(t) = R^2 \dot{R}$$

$$\Rightarrow \dot{f}(t) = R^2 \ddot{R} + \dot{R}2R\dot{R} = R^2 \frac{dv}{dt} + 2Rv^2$$

$$\Rightarrow \frac{\dot{f}(t)}{R} = 2v^2 + R\frac{dv}{dt} = 2v^2 + R\frac{dv}{dR}\frac{dR}{dt}$$

$$= 2v^2 + Rv\frac{dv}{dR} \qquad (2)$$

The Euler's equation of motion, in radial direction, using $\dot{r} = u$, is

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial r} = F_r - \frac{1}{\rho}\frac{\partial p}{\partial r}$$

But $$\frac{\partial u}{\partial t} = \frac{\partial}{\partial t}\left(\frac{f(t)}{r^2}\right) = \frac{\dot{f}(t)}{r^2},\ F_r = -\mu r^{-7/3}$$

So, we need to integrate the Euler's equation

$$\frac{\dot{f}(t)}{r^2} + \frac{\partial}{\partial r}\left(\frac{1}{2}u^2\right) = \frac{-\mu}{r^{7/3}} - \frac{\partial}{\partial r}\left(\frac{p}{\rho}\right) \qquad (3)$$

Let us assume that the cavity has radius R at time t and its velocity then is $\dot{R} = v$. Integrating (3) over the whole liquid (r = R to r = ∞) at time t, we obtain

$$\left[\frac{-\dot{f}(t)}{r}\right]_R^\infty + \left[\frac{1}{2}u^2\right]_v^0 = \frac{3\mu}{4}\left[\frac{1}{r^{4/3}}\right]_R^\infty - \left[\frac{p}{\rho}\right]_R^\infty$$

Since the fluid is at rest at infinity, $u_\infty = 0$. Also $p_\infty = 0$, $p_R = 0$ (cavity), thus we get

$$\frac{\dot{f}(r)}{R} - \frac{1}{2}v^2 = -\frac{3\mu}{4}\frac{1}{R^{4/3}}$$

$$\Rightarrow 2Rv\,\frac{dv}{dR} + 3v^2 = -\frac{3\mu}{2}\frac{1}{R^{4/3}} \qquad | \text{ using (2)}$$

To make it exact, we multiply by R^2 so that

$$2R^3v\,\frac{dv}{dR} + 3R^2v^2 = -\frac{3\mu}{2}R^{2/3}$$

$$\Rightarrow \frac{d(R^3v^2)}{dR} = -\frac{3m}{2}R^{2/3}$$

Integrating, we get

$$R^3\,v^2 = A - \frac{9m}{10}R^{5/3} \qquad (4)$$

When $R = a$, $\dot{R} \equiv v = 0$, which gives $A = \frac{9\mu}{10}a^{5/3}$.

Now, we take $v = \dot{R} < 0$ because as the cavity fills, R decreases with time. Thus (4) gives

$$\frac{dR}{dt} = -\left(\frac{9\mu}{10}\right)^{1/2}\left(\frac{a^{5/3} - R^{5/3}}{R^3}\right)^{1/2}$$

Therefore,

$$\left(\frac{9\mu}{10}\right)^{1/2} t = -\int_a^0 \frac{R^{3/2}dR}{\left(a^{5/3} - R^{5/3}\right)^{1/2}}$$

$$= \frac{6a^{5/3}}{5}\int_0^{\pi/2} \sin^2\theta\, d\theta \qquad | \; R^{5/3} = a^{5/3}\sin^2\theta \text{ i.e. } R = a\,(\sin\theta)^{6/5}$$

$$= \frac{3\pi a^{5/3}}{10}$$

Thus,

$$t = \pi\, a^{5/3}\,(10\mu)^{-1/2}\,.$$

Hence the result.

11. Impulsive Motion

Impulsive motion occurs in a fluid when there is rapid but finite charge in the fluid velocity $\bar{q}$ over a short interval δt of time t, or a high pressure on a boundary acting over time δt, or the rapid variation in the velocity of a rigid body immersed in the fluid. Such type of actions are termed as **impulsive actions.**

The situation of impulsive action is effectively modeled mathematically by letting the body force or pressure approach to infinity while $\delta t \to 0$ in such a way that the integral of body force or pressure over the time interval δt remains finite in this limit.

It the flow is incompressible, infinitely rapid propagation of the effect of the impulsive action takes place, so that an impulsive pressure is produced instantaneously throughout the fluid. Here, we consider only the incompressible fluid with constant density ρ. The impulsive body force $\bar{I}$ and impulsive pressure P are defined as

$$\begin{aligned} \bar{I} &= \underset{\delta t \to 0}{\mathrm{Lt}} \int_t^{t+\delta t} \bar{F}\,dt \\ P &= \underset{\delta t \to 0}{\mathrm{Lt}} \int_t^{t+\delta t} p\,dt \end{aligned} \qquad (1)$$

We note that finite body forces such as gravity do not contribute to the impulsive body force $\bar{I}$.

To determine the equation of impulsive motion, we consider the Euler's

equation

$$\frac{D\bar{q}}{Dt} \equiv \frac{d\bar{q}}{dt} \equiv \frac{\partial \bar{q}}{\partial t} + (\bar{q}\cdot\nabla)\bar{q} = \bar{F} - \frac{1}{\rho}\nabla p$$

Integrating w.r.t. time t from t to t + δt and taking limit as $\delta t \to 0$, we get

$$\begin{aligned} \underset{\delta t \to 0}{\mathrm{Lt}} \int_t^{t+\delta t} \frac{D\bar{q}}{Dt}\,dt &\equiv \underset{\delta t \to 0}{\mathrm{Lt}} \int_t^{t+\delta t} \frac{\partial \bar{q}}{\partial t}\,dt + \underset{\delta t \to 0}{\mathrm{Lt}} \int_t^{t+\delta t} (\bar{q}\cdot\nabla)dt \\ &= \underset{\delta t \to 0}{\mathrm{Lt}} \int_t^{t+\delta t} \bar{F}\,dt - \frac{1}{\rho} \underset{\delta t \to 0}{\mathrm{Lt}} \int_t^{t+\delta t} \nabla p\,dt \qquad (2) \end{aligned}$$

Assuming that fluid is accelerated impulsively at t = 0 and since we expect a finite change in $\bar{q}$ as a result of the impulse, we get from (1) and (2)

$$\bar{q}' - \bar{q} = \bar{I} - \frac{1}{\rho}\nabla P \qquad (3)$$

where $\bar{q}$ and $\bar{q}'$ denote respectively the fluid velocity before and after the impulsive action.
Thus, the equation of impulsive motion is

$$\rho(\bar{q}' - \bar{q}) = \rho\bar{I} - \nabla P \qquad (4)$$

which holds at each point of the fluid.
In cartesian co-ordinates, (4) can be expressed as

$$\rho(u' - u) = \rho X' - \frac{\partial P}{\partial x}$$

$$\rho(v' - v) = \rho Y' - \frac{\partial P}{\partial y}$$

$$\rho(w' - w) = \rho Z' - \frac{\partial P}{\partial z}$$

where

$$\bar{q} = (u, v, w), \bar{q}' = (u', v', w'),\ \bar{I} = (X', Y', Z')$$

When there is no externally applied impulse, then $\bar{I} = \bar{0}$ and equation (4) becomes

$$-\nabla P = \rho(\bar{q}' - \bar{q}) \qquad (5)$$

Further, if the motion is irrotational, then $\bar{q} = -\nabla\phi,\ \bar{q}' = -\nabla\phi'$, where ϕ and ϕ' denote the velocity potential just before and just after the impulsive action, then (5) becomes

$$P = \rho(\phi' - \phi) \qquad (6)$$

Where we have ignored the constant of integration since an extra pressure, constant throughout the liquid, would not effect the impulsive motion.

11.1. Corollary. If the fluid is at rest prior to the impulsive action, then the velocity $\bar{q}$ generated in the fluid by the impulse is given by

$$\bar{q} = \bar{I} - \frac{1}{\rho}\nabla P \qquad (7)$$

| In (3), put $\bar{q} = \bar{0}$ and $\bar{q}' \equiv \bar{q}$

For this case, equation (5) can be put as

$$-\nabla P = \rho\,\overline{q} \tag{8}$$

and equation (6) becomes

$$P = \rho\phi \tag{9}$$

Equations (6) and (9) give the relation between impulsive pressure P and the velocity potential ϕ.

11.2. Remark. From the above discussion, we observe that, likewise, an irrotational motion can be brought to rest by applying an impulsive pressure $-\rho\phi$ throughout the fluid.

11.3. Example. A sphere of radius a is surrounded by an infinite liquid of density ρ, the pressure at infinity being Π. The sphere is suddenly annihilated. Show that the pressure at distance r from the centre immediately falls to $\pi\left(1-\frac{a}{r}\right)$. Show further that if the liquid is brought to rest by impinging on a concentric sphere of radius $\frac{a}{2}$, the impulsive pressure sustained by the surface of the sphere is $\sqrt{7\Pi\rho a^2/6}$.

Solution. Let v′ be the velocity at a distance r′ from the centre of the sphere at any time t and p be the pressure. The equation of continuity (case of spherical symmetry) is

$$\frac{1}{r'^2}\frac{d}{dr'}(r'^2\, v') = 0 \qquad \Rightarrow r'^2\, v^2 = f(t) \tag{1}$$

Equation of motion is

$$\frac{\partial v'}{\partial t} + v'\frac{\partial v'}{\partial r'} = \frac{-1}{\rho}\frac{\partial p}{\partial r'} \qquad |\text{ No body force}$$

or
$$\frac{\dot{f}(t)}{r'^2} + v'\frac{\partial v'}{\partial r'} = -\frac{1}{\rho}\frac{\partial p}{\partial r'}$$

Integrating w.r.t. r′, we get

$$-\frac{\dot{f}(t)}{r'} + \frac{1}{2}v'^2 = -\frac{p}{\rho} + C$$

Since $r' \to \infty \Rightarrow p = \Pi$, $v' = 0$ so that $C = \Pi/\rho$.

Thus $$\frac{-\dot{f}(t)}{r'} + \frac{1}{2}v'^2 = \frac{\Pi - p}{\rho} \qquad (2)$$

When, sphere is suddenly annihilated i.e. $r' = a$, $v' = 0$, $p = 0$, then

$$-\frac{\dot{f}(t)}{a} = \Pi/\rho \qquad \text{i.e.} \quad \dot{f}(t) = -\frac{\Pi a}{\rho} \qquad (3)$$

The velocity v' vanishes just after annihilation, so from (2) and (3), we get

$$\frac{\Pi a}{\rho r'} = \frac{\Pi - p}{\rho} \qquad \Rightarrow \frac{a\Pi}{r'} = \Pi - p$$

Thus, the pressure at the time of annihilation ($r' = r$) is

$$\frac{a\Pi}{r} = \Pi - p \qquad \Rightarrow p = \Pi\left(1 - \frac{a}{r}\right)$$

which proves the first result.
Now, let P be the impulsive pressure at a distance r', then from the relation $-\nabla P = \rho \bar{q}$, we get

$$-\frac{dP}{dr'} = \rho v' \quad \Rightarrow dP = -\rho v'\, dr'$$

From the equation of continuity, we have

$$r^2 v = r'^2 v' = f(t) \qquad (4)$$

So $$dP = -\rho v\,(r^2/r'^2)\, dr' \qquad (5)$$

where r is the radius of the inner surface and v is the velocity there.
Integrating (5), we get

$$P = \rho v\,(r^2/r') + C_1$$

When $r' \to \infty$, $P = 0$ so that $C_1 = 0$

Thus $$P = \rho v(r^2/r') \qquad (6)$$

Equation (6) determines the impulsive pressure P at a distance r'. The velocity v at the inner surface of the sphere ($p = 0$) is obtained from (2) as

$$-\frac{\dot{f}(t)}{r}+\frac{1}{2}v^2=\frac{\Pi}{\rho} \qquad (7)$$

From (4), $\dot{f}(t)=\frac{d}{dt}(r^2v)=r^2\frac{dv}{dt}+v.2r\frac{dr}{dt}=r^2\frac{dv}{dr}\frac{dr}{dt}+2rv^2$

$$\Rightarrow \qquad \frac{\dot{f}(t)}{r}=rv\frac{dv}{dr}+2v^2$$

Thus (7) becomes

$$rv\,\frac{dv}{dr}+2v^2-\frac{1}{2}v^2=-\frac{\Pi}{\rho}$$

or
$$rv\frac{dv}{dr}+\frac{3}{2}v^2=\frac{-\Pi}{\rho}$$

$$\Rightarrow \qquad 2r^3v\,\frac{dv}{dr}+3v^2r^2=\frac{-2\Pi}{\rho}r^2 \qquad |\text{Multiplying by } r^2$$

$$\Rightarrow \qquad \frac{d(r^3v^2)}{dr}=-\frac{2\Pi}{\rho}r^2$$

Integrating, we get

$$r^3v^2=-\frac{2\Pi}{3\rho}r^3+C_2$$

Since r = a, v = 0 so we find $C_2=\frac{2\Pi a^3}{3\rho}$

Therefore, $r^3\,v^2=\frac{2\Pi}{3\rho}(a^3-r^3)$

The velocity v at the surface of the sphere r = a/2, on which the liquid strikes, is

$$v^2=\frac{2\Pi}{3\rho}\frac{a^3-(a/2)^3}{(a/2)^3}=\frac{14}{3}\frac{\Pi}{\rho}$$

From relation (6), using r = a/2, we get

$$P=\frac{\rho}{4}\sqrt{\frac{14}{3}\frac{\Pi}{\rho}}.\frac{a^2}{r'} \qquad (8)$$

which determines the impulsive pressure at a distance r’ from the centre of the sphere.

Thus, the impulsive pressure at the surface of the sphere of radius a/2 is given by

$$P = \frac{\rho}{4}\sqrt{\frac{14}{3}\frac{\Pi}{\rho}}\,\frac{a^2}{a/2} = \sqrt{7\Pi\rho a^2/6}$$

Hence the result

12. Stream Function

When motion is the same in all planes parallel to xy plane (say) and there is no velocity parallel to the z-axis, i.e. when u, v are functions of x, y, t only and w = 0, we may regard the motion as **two-dimensional** and consider only the cases confined to the xy plane. When we speak of the flow across a curve in this plane, we shall mean the flow across unit length of a cylinder whose trace on the xy plane is the curve in question, the generators of the cylinder being parallel to the z-axis.

For a two-dimensional motion in xy-plane, $\bar{q}$ is a function of x, y, t only and the differential equation of the streamlines (lines of flow) are

$$\frac{dx}{u} = \frac{dy}{v} \text{ i.e. } vdx - udy = 0 \tag{1}$$

and the corresponding equation of continuity is

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \tag{2}$$

We note that equation (2) is the condition of exactness of (1), it follows that (1) must be an exact differential, $d\psi$(say). Thus

$$vdx - udy = d\psi = \frac{\partial\psi}{\partial x}dx + \frac{\partial\psi}{\partial y}dy$$

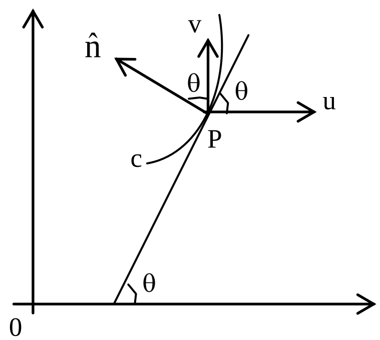
so that $$u = -\frac{\partial\psi}{\partial y},\ v = \frac{\partial\psi}{\partial x}$$

This function ψ is called the **stream function** or the **current function** or **Lagrange's stream function.**

Obviously, the streamlines are given by the solution of (1) i.e. $d\psi = 0$ i.e. ψ = constant. (For unsteady flow, streamlines are given by $\psi = f(t)$)
Thus, the stream function is constant along a streamline.

From the above discussion, it is clear that the existence of stream function is merely a consequence of the continuity and incompressibility of the fluid. The stream function always exists in all types of **two dimensional** motion whether rotational or irrotational. However, it should be noted again that velocity potential exists only for **irrotational motion** whether two dimensional or three dimensional.

12.1. Physical Interpretation of Stream Function :-



Let P be a point on a curve C in xy-plane. Let an element ds of the curve makes an angle θ with x-axis. The direction cosines of the normal at P are

$$(\cos(90+\theta), \cos\theta, 0)$$

i.e. $$(-\sin\theta, \cos\theta, 0).$$

The flow across the curve C from right to left is

$$= \int_C \bar{q}\cdot\hat{n}\,ds, \text{ where } \hat{n} = -\sin\theta\,\hat{i} + \cos\theta\,\hat{j},$$

$$\bar{q} = u\,\hat{i} + v\,\hat{j}$$

$$= \int_C (-u\sin\theta + v\cos\theta)\,ds$$

$$= \int_C \left(\frac{\partial\psi}{\partial y}\sin\theta + \frac{\partial\psi}{\partial x}\cos\theta\right)ds \qquad \left|\; u = -\frac{\partial\psi}{\partial y},\; v = \frac{\partial\psi}{\partial x}\right.$$

$$= \int_C \left(\frac{\partial\psi}{\partial x}\frac{dx}{ds} + \frac{\partial\psi}{\partial y}\frac{dy}{ds}\right)ds \qquad \left|\; \cos\theta = \frac{dx}{ds},\; \sin\theta = \frac{dy}{ds}\right.$$

$$= \int_C \left(\frac{\partial\psi}{\partial x}dx + \frac{\partial\psi}{\partial y}dy\right)$$

$$= \int_C d\psi = (\psi_B - \psi_A)$$

where ψ_A and ψ_B are the values of ψ at the initial and final points of the curve. Thus, the difference of the values of a stream function at any two points represents the flow across that curve, joining the two points.

12.2. Corollary. If we suppose that the curve C be the streamline, then no fluid crosses its boundary, then

$$(\psi_B - \psi_A) = 0 \quad \Rightarrow \psi_B = \psi_A$$

i.e. ψ is constant along c.

12.3. Relation Between ϕ and ψ (i.e. C−R equations) :-

We know that the velocity potential ϕ is given by

$$\bar{q} = -\nabla\phi = -\left(\frac{\partial\phi}{\partial x}, \frac{\partial\phi}{\partial y}\right)$$

i.e. $$u = -\frac{\partial\phi}{\partial x}, v = -\frac{\partial\phi}{\partial y} \qquad (1)$$

Also, the stream function ψ is given by

$$u = -\frac{\partial\psi}{\partial y}, v = \frac{\partial\psi}{\partial x} \qquad (2)$$

From (1) and (2), we get

$$\frac{\partial\phi}{\partial x} = \frac{\partial\psi}{\partial y} \text{ and } \frac{\partial\phi}{\partial y} = -\frac{\partial\psi}{\partial x} \qquad (3)$$

Equations in (3) imply

$$\nabla^2\phi = 0 \text{ and } \nabla^2\psi = 0$$

i.e. ϕ and ψ are harmonic functions.

Again, from (3), we get

$$\nabla\phi = \text{grad } \phi = -\bar{q} = -(u\,\hat{i} + v\,\hat{j})$$

$$= -\left(-\frac{\partial\psi}{\partial y}\hat{i} + \frac{\partial\psi}{\partial x}\hat{j}\right)$$

$$= \frac{\partial\psi}{\partial y}\hat{i} - \frac{\partial\psi}{\partial x}\hat{j}$$

$$= \frac{\partial\psi}{\partial y}(\hat{j}\times\hat{k}) + \frac{\partial\psi}{\partial x}(\hat{i}\times\hat{k})$$

$$= \left(\frac{\partial\psi}{\partial x}\hat{i} + \frac{\partial\psi}{\partial y}\hat{j}\right)\times\hat{k}$$

$$= \nabla\psi\times\hat{k} = \text{grad } \psi\times\hat{k}$$

i.e. grad ϕ = (grad ψ)$\times\hat{k}$ $= -\hat{k}\times$ grad ψ

i.e. $$\nabla\phi = \nabla\psi \times \hat{k} \tag{4}$$

Again, from (3), we note that

$$\frac{\partial\phi}{\partial x}\frac{\partial\psi}{\partial x} = \frac{\partial\psi}{\partial y}\left(-\frac{\partial\phi}{\partial y}\right)$$

$$\Rightarrow \qquad \frac{\partial\phi}{\partial x}\frac{\partial\psi}{\partial x} + \frac{\partial\phi}{\partial y}\frac{\partial\phi}{\partial y} = 0$$

i.e. $$\nabla\phi \cdot \nabla\psi = 0 \tag{5}$$

Thus, for irrotational incompressible two-dimensional flow (steady or unsteady), $\phi(x, y)$, $\psi(x, y)$ are harmonic functions and the family of curves ϕ = constant (equipotentials) and ψ = constant (streamlines) intersect orthogonally.

12.4. Exercise. Show that $u = 2c\, xy$, $v = c(a^2 + x^2 - y^2)$ are the velocity components of a possible fluid motion. Determine the stream function and the streamlines.

12.5. Remark. We shall consider the study of two dimensional motion later on. At present we continue discussing three dimensional irrotational flow of incompressible fluids.

UNIT – II

1. Three Dimensional Irrotational Flow

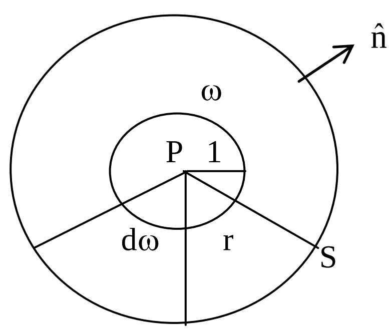
1.1. Acyclic and Cyclic Irrotational Motion. An irrotational motion is called acyclic if the velocity potential ϕ is a single valued function i.e. when at every field point, a unique velocity potential exists, otherwise the irrotational motion is said to be cyclic. Clearly, only acyclic irrotational motion is possible in a simply connected region.

For a possible fluid motion, even if ϕ is multivalued at a particular point, the velocity at that point must be single-valued. Hence if we obtain two different values of ϕ, these values can only differ by a constant.

At present, we restrict ourself to acyclic irrotational motion for which we prove a number of results related to ϕ.

1.2.Mean Value of Velocity Potential Over Spherical Surfaces. Theorem : The mean value of a ϕ over any spherical surface S drawn in the fluid throughout whose interior $\nabla^2\phi = 0$, is equal to the value of ϕ at the centre of the sphere.

Proof. Let $\phi(P)$ be the value of ϕ at the centre P of a spherical surface S of radius r, wholly lying in the liquid and let $\bar{\phi}$ denotes the mean value of ϕ over S. Let us draw another concentric sphere ω of unit radius. Then a cone with vertex P which intercepts area dS from the sphere S, intercepts an area $d\omega$ from the sphere ω and we have



$$\frac{dS}{d\omega} = \frac{r^2}{1^2} \Rightarrow dS = r^2 d\omega \qquad (1)$$

Now, by definition

$$\bar{\phi} = \frac{\int_S \phi dS}{\int_S dS} = \frac{1}{4\pi r^2}\int_S \phi dS$$

$$= \frac{1}{4\pi r^2}\int_S \phi r^2 d\omega = \frac{1}{4\pi}\int_S \phi d\omega$$

$$\Rightarrow \qquad \frac{\partial \bar{\phi}}{\partial r} = \frac{1}{4\pi}\int_S \frac{\partial \phi}{\partial r} d\omega = \frac{1}{4\pi}\int_S \frac{\partial \phi}{\partial r}\frac{dS}{r^2}$$

$$= \frac{1}{4\pi r^2}\int_S \frac{\partial \phi}{\partial r} dS \qquad (2)$$

| $\because$ r^2 is constant on S

Since the normal $\hat{n}$ to the surface is along the radius r, therefore on S, we have

$$\frac{\partial \phi}{\partial r} = \frac{\partial \phi}{\partial n} = \nabla\phi.\hat{n} \qquad (3)$$

From (2) & (3), we find

$$\frac{\partial \bar{\phi}}{\partial r} = \frac{1}{4\pi r^2}\int_S \nabla\phi.\hat{n} dS$$

$$= \frac{1}{4\pi r^2}\int_\tau \text{div}(\nabla\phi) d\tau \qquad \text{| Gauss theorem}$$

$$= \frac{1}{4\pi r^2}\int_\tau \nabla^2\phi d\tau = 0, \qquad |\ \nabla^2\phi = 0$$

where τ is the volume enclosed by the surface S.

Thus $\frac{\partial \bar{\phi}}{\partial r} = 0 \Rightarrow \bar{\phi}$ = constant.

This shows that $\bar{\phi}$ is independent of choice of r and hence mean value of ϕ is same over all spherical surfaces having the same centre P. When S shrinks to point P, then $\bar{\phi} = \phi(P)$

1.3. Corollary. The velocity potential ϕ can not have a maximum or minimum value in the interior of any region throughout which $\nabla^2\phi = 0$.

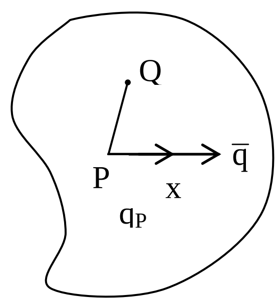
Proof. If possible, suppose that ϕ has a maximum value $\phi(P)$ at a point P. We draw a sphere with centre P and radius $\in$, where $\in$ is small. Then the mean value $\bar{\phi}$ of ϕ must be less than $\phi(P)$ i.e. $\bar{\phi} < \phi(P)$ as $\phi(P)$ is maximum. This is a contradiction to the mean potential theorem in which $\bar{\phi} = \phi(P)$. Thus ϕ cannot have a maximum value. Similarly ϕ cannot have a minimum value.

1.4. Theorem. In an irrotational motion the maximum value of the fluid velocity occurs at the boundary.

Proof. Let P be any interior point of the fluid and Q be a neighbouring point also lying in the fluid. Let us take the direction of x-axis along the direction of $\bar{q}$ at P. Let q_P and q_Q denote the speed of particles at P & Q respectively.

Then $$q_P^2 = \left(\frac{\partial\phi}{\partial x}\right)_P^2$$

and $$q_Q^2 = \left(\frac{\partial\phi}{\partial x}\right)_Q^2 + \left(\frac{\partial\phi}{\partial y}\right)_Q^2 + \left(\frac{\partial\phi}{\partial z}\right)_Q^2$$



Since $\nabla^2\phi = 0 \Rightarrow \frac{\partial}{\partial x}(\nabla^2\phi) = 0 \Rightarrow \nabla^2\left(\frac{\partial\phi}{\partial x}\right) = 0$

$\Rightarrow \frac{\partial\phi}{\partial x}$ satisfies Laplace equation. Therefore, by mean value theorem (corollary), $\frac{\partial\phi}{\partial x}$ cannot be maximum or minimum at P. Thus, there are points such as Q in the neighbourhood of P such that

$$\left(\frac{\partial\phi}{\partial x}\right)_Q^2 > \left(\frac{\partial\phi}{\partial x}\right)_P^2 \Rightarrow q_Q^2 > q_P^2$$

$\Rightarrow q_P$ cannot be maximum in the interior of fluid and its maximum value $|\bar{q}|$, if any, must therefore occur on the boundary.

1.5. Note. q = | $\bar{q}$ | may be minimum in the interior of the fluid as $\bar{q} = \bar{0}$ at the stagnation point. i.e. q is minimum at stagnation points.

1.6. Corollary. In steady irrotational flow, the pressure has its minimum value on the boundary.

Proof. From Bernoulli's equation, we have

$$\frac{p}{\rho} + \frac{1}{2}q^2 = \text{constant} \tag{1}$$

Equation (1) shows that p is least when q^2 is greatest and by above theorem, q^2 is greatest at the boundary. Thus, the minimum value of p must occur only on the boundary.

1.7. Note. The maximum value of p occurs at the stagnation points, where $\bar{q} = \bar{0}$.

1.8. Theorem. If liquid of infinite extent is in irrotational motion and is bounded internally by one or more closed surfaces S, the mean value of ϕ over a large sphere Σ, of radius R, which encloses S, is of the form

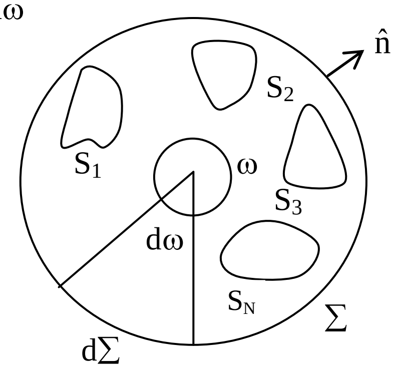
$$\bar{\phi} = \frac{M}{R} + C$$

where M and C are constants, provided that the liquid is at rest at infinity.

Proof. Suppose that the volume of fluid acrossing each of internal surfaces contained within Σ, per unit time, is a finite quantity say $-4\pi M$ (i.e. $-4\pi M$ represents the flux of fluid across Σ or S). Since the fluid velocity at any point of Σ is $\frac{\partial \phi}{\partial R}$ radially outwards, the equation of continuity gives

$$\int_{\Sigma} \frac{\partial \phi}{\partial R} d\Sigma = -4\pi M \tag{1}$$

But $\qquad d\Sigma = R^2 d\omega$



Therefore,

$$\frac{1}{4\pi}\int_{\Sigma}\frac{\partial\phi}{\partial R}R^2 d\omega = -M$$

$$\Rightarrow \qquad \frac{1}{4\pi}\int_{\Sigma}\frac{\partial\phi}{\partial R}d\omega = \frac{-M}{R^2}$$

$$\Rightarrow \qquad \frac{1}{4\pi}\frac{\partial}{\partial R}\int_{\Sigma}\phi d\omega = \frac{-M}{R^2}$$

Integrating w.r.t. R, we get

$$\frac{1}{4\pi}\int_{\Sigma}\phi d\omega = \frac{M}{R} + C$$

where C is independent of R.

$$\Rightarrow \qquad \frac{1}{4\pi}\int_{\Sigma}\phi\left(\frac{d\Sigma}{R^2}\right) = \frac{M}{R} + C$$

$$\Rightarrow \qquad \frac{\int_{\Sigma}\phi d\Sigma}{4\pi R^2} = \frac{M}{R} + C$$

$$\Rightarrow \qquad \bar{\phi} = \frac{M}{R} + C \qquad (2)$$

To show that C is an absolute constant, we have to prove that it is independent of co-ordinates of centre of sphere Σ. Let the centre of the sphere Σ be displaced by distance δx in an arbitrary direction while keeping R constant, then from (2),

$$\frac{\partial\bar{\phi}}{\partial x} = \frac{\partial C}{\partial x} \qquad (3)$$

$| \because$ R is constant

Also, $$\frac{\partial\bar{\phi}}{\partial x} = \frac{\partial}{\partial x}\left[\frac{1}{4\pi}\int_{\Sigma}\phi dw\right] = \frac{1}{4\pi}\int_{\Sigma}\frac{\partial\phi}{\partial x}dw$$

$= 0$, since $\frac{\partial\phi}{\partial x} = 0$ on Σ when $R\to\infty$ as the liquid is at rest at

infinity.

$\therefore$ From (3), we get

$$\frac{\partial C}{\partial x} = 0 \Rightarrow C \text{ is an absolute constant.}$$

Hence

$$\bar{\phi} = \frac{M}{R} + C\text{, where M and C are constants.}$$

1.9. Corollary. When closed surfaces within Σ are rigid then no flow can take place across them, therefore, in that case M = 0 and $\bar{\phi} = C$.

This shows that mean value of ϕ over any sphere enclosing solid rigid boundaries is constant.

2. Kinetic Energy of Irrotational Flow

We shall prove that K.E. is given by

$$T = \frac{\rho}{2}\int_S \phi \frac{\partial \phi}{\partial n} dS,$$

where ϕ is the velocity potential.

We know that if τ be the finite region occupied by the fluid, then the K.E. is given by

$$T = \frac{1}{2}\int_\tau \rho \bar{q}^{\,2} d\tau = \frac{1}{2}\int_\tau \rho(\bar{q}.\bar{q})d\tau$$

$$= \frac{1}{2}\int_\tau \rho(\nabla\phi.\nabla\phi)d\tau \qquad | \ \bar{q} = -\nabla\phi$$

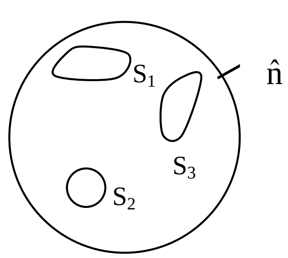
If fluid density is constant, then

$$T = \frac{\rho}{2}\int_\tau (\nabla\phi.\nabla\phi)d\tau \qquad (1)$$

Now, div $(\phi\ \nabla\phi) = \nabla.(\phi\nabla\phi) = \nabla\phi.\ \nabla\phi. + \phi\ (\nabla.\ \nabla\phi)$

$= \nabla\phi.\ \nabla\phi + \phi\ \nabla^2\phi$

$= \nabla\phi.\ \nabla\phi. \quad | \because \nabla^2\phi = 0$



Therefore, from (1) & (2), we get

$$T = \frac{\rho}{2}\int_{\tau} \operatorname{div}(\phi\nabla\phi)\, d\tau = \frac{\rho}{2}\int_{S} \phi\nabla\phi.\ \hat{n}\, dS \qquad | \text{ By Gauss theorem}$$

$$= \frac{\rho}{2}\int_{S} \phi(\nabla\phi.\hat{n})dS = \frac{\rho}{2}\int_{S} \phi\frac{\partial\phi}{\partial n}dS, \text{ where } S = S_0 + S_1 + S_2 + \ldots + S_n$$

denotes the sum of the outer boundary surface S_0 and the inverse boundaries $S_1, S_2, \ldots, S_n$ and $\hat{n}$ is unit normal to S drawn out of the fluid on each boundary.

Also $T = -\frac{\rho}{2}\int_{S} \phi\frac{\partial\phi}{\partial n}dS$, where $\hat{n}$ is unit normal to S drawn inside the fluid on each boundary.

2.1. Kelvin's Minimum Energy Theorem. The kinetic energy of irrotational motion of a liquid occupying a finite simply connected region is less than that of any other motion of the liquid which is consistent with the same normal velocity of the boundary.

Proof. Let T be the K.E., $\bar{q}$ be the fluid velocity and ϕ be the velocity potential of the given irrotational motion. Let τ be the region occupied by the fluid and S be the surface of this region, then

$$T = \frac{\rho}{2}\int_{\tau} \bar{q}^2 d\tau = \frac{\rho}{2}\int_{\tau} (-\nabla\phi)^2\, d\tau$$

$$= \frac{\rho}{2}\int_{S} \phi\frac{\partial\phi}{\partial n}dS \qquad (1)$$

Let T_1 be the K.E. and $\bar{q}_1$ be the velocity of any other motion of the fluid consistent with the same normal velocity of the boundary S (or consistent with the same kinetic boundary condition)
For both the motions, the continuity equation is satisfied i.e.

$$\nabla\cdot\bar{q} = 0 = \nabla\cdot\bar{q}_1 \qquad (2)$$

The boundaries have the same normal velocity

i.e. $\bar{q}\cdot\hat{n} = \bar{q}_1\cdot\hat{n}$

i.e. $$(\bar{q}_1 - \bar{q}) \cdot \hat{n} = 0 \qquad (3)$$

Now, let us consider

$$T_1 - T = \frac{\rho}{2}\int_\tau (q_1^2 - q^2)\, d\tau$$

$$= \frac{\rho}{2}\int_\tau \left[2\bar{q}\cdot(\bar{q}_1 - \bar{q}) + (\bar{q}_1 - \bar{q})^2\right] d\tau$$

$$= \frac{\rho}{2}\int_\tau 2\bar{q}\cdot(\bar{q}_1 - \bar{q})\,d\tau + \frac{\rho}{2}\int_\tau (\bar{q}_1 - \bar{q})^2\, d\tau$$

$$= -\rho\int_\tau \nabla\phi\,.\,(\bar{q}_1 - \bar{q})\, d\tau + \frac{\rho}{2}\int_\tau (\bar{q}_1 - \bar{q})^2\, d\tau \qquad (4)$$

From vector calculus, we have

$$\nabla\cdot[\phi(\bar{q}_1 - \bar{q})] = \nabla\phi\cdot(\bar{q}_1 - \bar{q}) + \phi\,\nabla\cdot(\bar{q}_1 - \bar{q})$$

i.e. $$\nabla\phi\cdot(\bar{q}_1 - \bar{q}) = \nabla\cdot[\phi(\bar{q}_1 - \bar{q})] - \phi\nabla\cdot(\bar{q}_1 - \bar{q})$$

Therefore, from (4), we find

$$T_1 - T = -\rho\int_\tau \nabla\cdot[\phi(\bar{q}_1 - \bar{q})]\,d\tau + \rho\int_\tau \phi\nabla\cdot\phi(\bar{q}_1 - \bar{q})\,d\tau$$

$$+ \frac{\rho}{2}\int_\tau (\bar{q}_1 - \bar{q})^2\, d\tau$$

$$= -\rho\int_S \phi(\bar{q}_1 - \bar{q})\cdot\hat{n}\,dS + \rho\int_\tau \phi\nabla\cdot(\bar{q}_1 - \bar{q})\,d\tau$$

$$+ \frac{\rho}{2}\int_\tau (\bar{q}_1 - \bar{q})^2 d\tau \qquad \text{|By Gauss theorem}$$

$$= \frac{\rho}{2}\int_\tau (\bar{q}_1 - \bar{q})^2 d\tau \qquad \text{| using (2) \& (3)}$$

$$> 0$$

$$\Rightarrow \quad T_1 > T\,.$$

Hence the theorem.

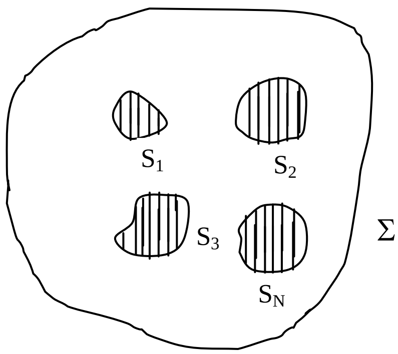
2.2. Kinetic Energy of Infinite Liquid. Theorem : An infinite liquid is in irrotational motion which is at rest at infinity and is bounded internally by solid surface (s)S. Show that the K.E. of the moving fluid is

$$T = \frac{1}{2} e \int_S \phi \frac{\partial \phi}{\partial n} dS$$

where $S = S_1 + S_2 + \ldots S_N$ denotes the sum of the inner boundaries $S_1, S_2, \ldots, S_N$ and $\hat{n}$ is normal to S drawn out of the fluid on each boundary.

Proof. Let Σ be a large surface enclosing the surface (s) S and τ be the region bounded by S internally and by Σ externally.

Using the result of K.E. for finite liquids, we find that the K.E. T* for finite region τ is given by



$$T^* = \frac{\rho}{2} \int_S \phi \frac{\partial \phi}{\partial n} dS + \frac{\rho}{2} \int_\Sigma \phi \frac{\partial \phi}{\partial n} dS \qquad (1)$$

Now, div $\bar{q} = \nabla^2 \phi = 0$ throughout τ and the divergence theorem accordingly gives

$$\int_\tau \text{div}\, \bar{q}\, d\tau = 0 \Rightarrow \int_{S \cup \Sigma} \hat{n} \cdot \bar{q}\, dS = 0$$

$$\Rightarrow \int_{S \cup \Sigma} \hat{n} \cdot \nabla \phi dS = 0 \Rightarrow \int_{S \cup \Sigma} \frac{\partial \phi}{\partial n} dS = 0$$

$$\Rightarrow \int_S \frac{\partial \phi}{\partial n} dS + \int_\Sigma \frac{\partial \phi}{\partial n} dS = 0 \qquad (2)$$

Since the surface S is solid, there is no flow across it, hence

$$\int_S \frac{\partial \phi}{\partial n} dS = 0 \qquad (3)$$

Therefore, from (2), we get

$$\int_{\Sigma} \frac{\partial \phi}{\partial n} dS = 0 \qquad (4)$$

For the surface Σ, as Σ goes to infinity, the liquid is at rest

$$\Rightarrow \quad \bar{q} = 0 \Rightarrow \nabla\phi = 0 \Rightarrow \phi = \text{constant} = C \text{ (say)} \qquad (5)$$

Hence, as Σ goes to ∞, the K.E. of the liquid is

$$T^* \to T = \frac{\rho}{2}\int_S \phi \frac{\partial \phi}{\partial n} dS + \frac{\rho}{2} c \int_{\Sigma} \frac{\partial \phi}{\partial n} dS \qquad | \text{ Using (5)}$$

$$= \frac{\rho}{2}\int_S \phi \frac{\partial \phi}{\partial n} dS \qquad | \text{ Using (4)}$$

Hence the result

2.3. Remark. We note that the K.E. for finite and infinite liquid has the same expression.

2.4. Theorem. Show that acyclic irrotational motion is impossible in a finite volume of fluid bounded by rigid surfaces at rest or in infinite fluid at rest at infinity and bounded internally by rigid bodies at rest.

Proof. If possible suppose that acyclic irrotational motion is possible and let ϕ be the velocity potential. Then, K.E. of the fluid is

$$T = \frac{\rho}{2}\int_{\tau} (\nabla\phi)^2 d\tau = \frac{\rho}{2}\int_S \phi \frac{\partial \phi}{\partial n} dS \qquad (1)$$

Where S is the sum of all the rigid boundaries when τ is finite or the sum of internal rigid boundaries when τ is infinite.

Now, since the boundaries are rigid, then at every point of S, the normal velocity is zero

i.e. $\quad \dfrac{\partial \phi}{\partial n} = 0$ at each point of S $\qquad (2)$

From (1) & (2), we get

$$\int_{\tau} q^2 \, d\tau = 0 \Rightarrow q^2 = 0 \Rightarrow \bar{q} = \bar{0} \text{ at each point of } \tau.$$

$\Rightarrow$ liquid is at rest.

Hence there is no motion of fluid.

$\Rightarrow$ acyclic irrotational motion is impossible.

2.5. Corollary. If the solid boundaries in motion are instantaneously brought to rest, show that the motion of the fluid will instantaneously cease to be irrotational.

Proof. If possible, assume that the motion remains irrotational, then the K.E. is given by

$$T = \frac{1}{2}\rho\int_{\tau} \bar{q}^2 d\tau = \frac{1}{2}\rho\int_{S} \phi\frac{\partial\phi}{\partial n} dS \qquad (1)$$

When the surface S (solid boundary) is brought to rest instantaneously, then

$\bar{q} = \bar{0}$ at each point of S.

$\Rightarrow$ ϕ = constant at each point of S.

$\Rightarrow$ $\frac{\partial\phi}{\partial n} = 0$ constant at each point of S.

$\Rightarrow$ $\bar{q} = \bar{0}$ in τ

$\Rightarrow$ there is no motion.

Thus the motion is no longer irrotational.

2.6. Uniqueness Theorems. Theorem 1: If the region occupied by the fluid is finite, then only one irrotational motion of the fluid exists when the boundaries have prescribed velocities. OR Show that there cannot be two different forms of acyclic irrotational motion of a given liquid whose boundaries have prescribed velocities.

Proof. If possible, let ϕ_1 and ϕ_2 be two different velocity potentials representing two motions, then

$$\nabla^2\phi_1 = 0 = \nabla^2\phi_2 \qquad (1)$$

Since the kinetic conditions at the boundaries are satisfied by both flows, therefore at each point of S,

$$\frac{\partial\phi_1}{\partial n} = \frac{\partial\phi_2}{\partial n} \qquad (2)$$

Let $\phi = \phi_1 - \phi_2$

$\Rightarrow \nabla^2\phi = \nabla^2\phi_1 - \nabla^2\phi_2 = 0$ at each point of fluid. and $\frac{\partial\phi}{\partial n} = \frac{\partial\phi_1}{\partial n} - \frac{\partial\phi_2}{\partial n} = 0$ at each point of S.
$\Rightarrow \phi$ represents a possible irrotational motion.

Also, the K.E. is given by

$$\frac{\rho}{2}\int_\tau (\bar{q})^2 d\tau = \frac{\rho}{2}\int_S \phi\frac{\partial\phi}{\partial n} dS = 0 \qquad \Big| \frac{\partial\phi}{\partial n} = 0$$

$\Rightarrow \bar{q} = \bar{0}$ at each point of fluid.

$\Rightarrow \nabla\phi = 0$ at each point of fluid.

$\Rightarrow \nabla\phi_1 - \nabla\phi_2 = 0 \Rightarrow \nabla\phi_1 = \nabla\phi_2$

which shows that the motions are the same. Moreover ϕ is unique apart from an additive constant which gives rise to no velocity and thus can be taken as zero (without loss of generality)

Theorem II. If the region occupied by the fluid is infinite and fluid is at rest at infinity, prove that only one irrotational motion is possible when internal boundaries have prescribed velocities.

Proof. If possible, let there be two irrotational motions given by two different velocity potentials ϕ_1 & ϕ_2. The conditions on boundaries are

$$\frac{\partial\phi_1}{\partial n} = \frac{\partial\phi_2}{\partial n} \qquad (1)$$

and $\bar{q}_1 = \bar{q}_2 = \bar{0}$ at infinity (2)

Let us write $\phi = \phi_1 - \phi_2$ (3)

$\Rightarrow \nabla^2\phi = \nabla^2\phi_1 - \nabla^2\phi_2 = 0 - 0 = 0$

$\Rightarrow$ motion given by ϕ is also irrotational.
Further from (3), we get

$$\frac{\partial\phi}{\partial n} = \frac{\partial\phi_1}{\partial n} - \frac{\partial\phi_2}{\partial n} = 0 \qquad | \text{ using (1)}$$

$\Rightarrow \bar{q}\cdot\hat{n} = 0 \qquad \Rightarrow \bar{q} = \bar{0}$ on the surface

Also,

$$\overline{q} = -\nabla\phi = -\nabla\phi_1 + \nabla\phi_2$$
$$= \overline{q}_1 - \overline{q}_2 = \overline{0} \text{ at } \infty \qquad | \text{ using (2)}$$

Therefore $\overline{q} = \overline{0}$ everywhere on the surface and also at infinity.

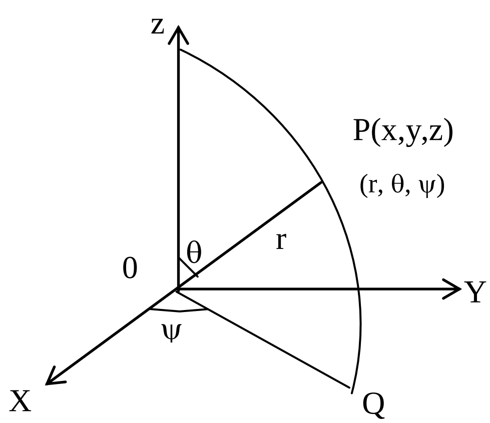
Hence we get ϕ = constant $\Rightarrow \phi_1 - \phi_2$ = constant (4)

Without loss of generality, we can take the constant on R.H.S. of (4) to be zero (it gives no motion) and thus we get $\phi_1 = \phi_2$

2.7. Remark. The above two uniqueness theorems are useful in finding solutions of $\nabla^2\phi = 0$ subject to prescribed boundary conditions.

3. Axially Symmetric Flows

A potential flow which is axially symmetric about the axis $\theta = 0, \pi$ (i.e. z-axis is taken as the axis of symmetry) has the property that at any point P, all the scalar and vector quantities associated with the flow are independent of azinuthal angle ψ such that $\frac{\partial}{\partial\psi} \equiv 0$, where (r, θ, ψ) are spherical polar co-ordinates.



The equation of continuity div $\overline{q} = 0$ for steady flow of an incompressible fluid becomes.

$$\frac{1}{r^2}\frac{\partial}{\partial r}(r^2 q_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta\, q_\theta) = 0 \qquad (1)$$

For irrotational motion $\overline{q} = -\nabla\phi$, where ϕ is velocity potential and thus

$$q_r = -\frac{\partial\phi}{\partial r},\ q_\theta = -\frac{1}{r}\frac{\partial\phi}{\partial\theta}$$

From equation (1), we have

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\phi}{\partial r}\right)+\frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\phi}{\partial\theta}\right)=0 \qquad (2)$$

Let a solution of (2) in separable variables r, θ has the form

$$\phi = -R(r)\,\Theta(\theta) \qquad (3)$$

Using (3) in (2), we get

$$\frac{\partial}{\partial r}\left[r^2\frac{\partial}{\partial r}(R\Theta)\right]+\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left[\sin\theta\frac{\partial}{\partial\theta}(R\Theta)\right]=0$$

$$\Rightarrow \qquad \Theta\frac{\partial}{\partial r}\left(r^2\frac{\partial R}{\partial r}\right)+\frac{R}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\Theta}{\partial\theta}\right)=0$$

$$\Rightarrow \qquad \frac{1}{R}\frac{d}{dr}\left(r^2\frac{dR}{dr}\right)=-\frac{1}{\Theta\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{d\Theta}{d\theta}\right) \qquad (4)$$

The L.H.S. of (4) is a function of r only while the R.H.S. is a function of θ only. The equation can therefore be satisfied if and only if either side is a constant, say n(n+1) and thus we get

$$\frac{1}{R}\frac{d}{dr}\left(r^2\frac{dR}{dr}\right)=n(n+1) \qquad (5)$$

and

$$\frac{d}{d\theta}\left(\sin\theta\frac{d\Theta}{d\theta}\right)+n(n+1)\Theta\sin\theta=0 \qquad (6)$$

To solve (5), we put

$$R = r^m \Rightarrow \frac{dR}{dr}=m\,r^{m-1}$$

Thus (5) $\Rightarrow \dfrac{1}{r^m}\dfrac{d}{dr}\left(r^2 m\,r^{m-1}\right)=n(n+1)$

$$\Rightarrow \; m\,\frac{d}{dr}\left(r^{m+1}\right)=r^m n(n+1)$$

$$\Rightarrow m\,(m+1)\,r^m = r^m\,n(n+1)$$

$$\Rightarrow (m^2 + m - n^2 - n) = 0$$

$$\Rightarrow (m-n)\,(m+n+1) = 0$$

$$\Rightarrow m = n \text{ or } m = -(n+1)$$

Therefore, solution of (5) can be written as

$$R(r) = A_n\,r^n + B_n r^{-(n+1)} \tag{7}$$

To solve (6), we put

$$\cos\theta = \mu$$

$$\Rightarrow \frac{d}{d\theta} \equiv \frac{d\mu}{d\theta}\frac{d}{d\mu} \equiv -\sin\theta\frac{d}{d\mu}$$

Therefore, equation (6) becomes.

$$-\sin\theta\,\frac{d}{d\mu}\left[\sin\theta(-\sin\theta)\frac{d\Theta}{d\mu}\right] + n\,(n+1)\,\Theta\sin\theta = 0$$

$$\Rightarrow \frac{d}{d\mu}\left(\sin^2\theta\frac{d\Theta}{d\mu}\right) + n(n+1)\Theta = 0$$

$$\Rightarrow \frac{d}{d\mu}\left[(1-\cos^2\theta)\frac{d\Theta}{d\mu}\right] + n(n+1)\Theta = 0$$

$$\Rightarrow \frac{d}{d\mu}\left[(1-\mu^2)\frac{d\Theta}{d\mu}\right] + n(n+1)\Theta = 0 \tag{8}$$

Equation (8) is a Legendre's Equation and possesses a solution known as Legendre Function of the first kind $P_n(\mu)$

Therefore,

$$\Theta = P_n(\mu)$$

Hence the general solution of (3) is of the form

$$\begin{aligned}\phi(r,\theta) &= -R(r)\,\Theta(\theta)\\ &= -[A_n\,r^n + B_n\,r^{-(n+1)}]\,P_n(\cos\theta) \qquad (9)\end{aligned}$$

(complete solution is the sum of all such solutions i.e. $\sum_{n=0}^{\infty}$ …….)

3.1. Uniform Flow. Consider the flow which corresponds to a potential given by (9) with

$$A_n = US_{1n},\ B_n\ 0, \quad (n = 0, 1, 2, \ldots\ldots) \qquad | \ S_{ij} \text{ is knonecker delta}$$

$S_{ii} = 1\ S_{ij} = 0$ for $i \neq j$

Where U is a constant.
Since P_1 $(\cos\theta) = \cos\theta$, equation (9) becomes

$$\phi(r, \theta) = -Ur \cos \theta \equiv -Uz \qquad | \ z = r \cos\theta$$

Thus

$$\bar{q} = -\nabla\phi = -\frac{\partial\phi}{\partial z}\hat{k} = U\hat{k}$$

which is a uniform streaming motion of the fluid with speed U along the direction of z-axis or the axis $\theta = 0$.

3.2. Sphere at Rest in a Uniform Stream. Consider an impermeable solid sphere of radius 'a' at rest with its centre at the pole of a system of spherical polar co-ordinates (r, θ, ψ). The sphere is immersed in an infinite homogeneous liquid with constant density ρ, which, in the absence of the sphere, would be flowing as a uniform stream with speed U along the direction $\theta = 0$.

The presence of the sphere will produce a local perturbation of the uniform streaming motion such that the disturbance diminishes with increasing distance r from centre of sphere. We say that the perturbation of the uniform stream tends to zero as $r \to \infty$.

In this problem z-axis is the axis of symmetry. Undisturbed velocity of incompressible fluid is $U\hat{k}$ i.e. $\bar{q} = U\hat{k}$

$\Rightarrow$ the velocity potential ϕ_0 due to such a uniform flow would be

$$\phi_0 = -Uz = -Ur \cos\theta$$

When the sphere is inserted, the undisturbed potential $-Ur \cos\theta$ of uniform stream has to be modified by "perturbation potential" due to the presence of the sphere. This must have the following properties.

(i) It must satisfy Laplace equation for the case of axial symmetry

(ii) It must tend to zero at large distances from the sphere

So, we write $\phi(r, \theta) = -U r \cos \theta + \phi_1 (r, \theta)\ (r \geq a)$
where ϕ_1 satisfies the Laplace equation together with boundary conditions

$$\frac{\partial\phi}{\partial r} = -U\cos\theta + \frac{\partial\phi_1}{\partial r} \qquad \left| \frac{\partial\phi}{\partial r} = 0 \right.$$ i.e. velocity normal to sphere is zero at r = a

$$\Rightarrow \frac{\partial\phi_1}{\partial r} = + U\cos\theta \ (r = a,\ a \le \theta \le \pi)$$

and

$$|\nabla\phi_1| \to 0 \text{ as } r\to\infty.$$

Hence a suitable form of function ϕ_1 is

$$\phi_1 = -B\,\bar{r}^2\ \cos\theta$$

So, we assume (in view of (9)) that

$$\phi(r, \theta) = -Ur\cos\theta - \frac{B}{r^2}\cos\theta \tag{1}$$

The constant B is to be determined from the fact that there is no flow normal to the surface r = a i.e. $\left(\frac{\partial\phi}{\partial r}\right)_{r=a} = 0$

$$\Rightarrow -U\cos\theta + \frac{2B}{a^3}\cos\theta = 0 \ \Rightarrow B = \frac{1}{2}Ua^3$$

Thus (1) becomes

$$\phi(r, \theta) = -Ur\cos\theta - \frac{Ua^3}{2r^2}\cos\theta$$

$$= -U\left(r + \frac{a^3}{2r^2}\right)\cos\theta \tag{2}$$

Now, the uniqueness theorem II infer that the velocity potential in (2) is unique.

The velocity components at P(r, θ, ψ), (r ≥ a), are

$$q_r = -\frac{\partial\phi}{\partial r} = U\left(1 - \frac{a^3}{r^3}\right)\cos\theta$$

$$q_\theta = -\frac{1}{r}\frac{\partial\phi}{\partial\theta} = -U\left(1 + \frac{a^3}{2r^3}\right)\sin\theta \tag{3}$$

$$q_\psi = -\frac{1}{r\sin\theta}\frac{\partial\phi}{\partial\psi} = 0\,.$$

Different terms related to motion are obtained as follows.

(i) Stagnation Points : Stagnation points are those points in the flow where the velocity vanishes i.e. $\bar{q} = \bar{0}$. Thus these points are obtained by solving the equations

and
$$\left.\begin{aligned} U\left(1-\frac{a^3}{r^3}\right)\cos\theta = 0 \\ U\left(1+\frac{a^3}{2r^3}\right)\sin\theta = 0 \end{aligned}\right\} \qquad (4)$$

which are satisfied only by r = a, sinθ = 0. i.e. r = a, θ = 0, π Thus the stagnation points are (r = a, θ = a) and (r = a, θ = π) on the sphere. These are referred to respectively as the **rear** and **forward** stagnation points.

(ii) Streamlines : The equations of streamlines

$$\frac{dr}{q_r} = \frac{rd\theta}{q_\theta} = \frac{r\sin\theta d\psi}{q_\psi}$$

for the present case, become

$$\frac{dr}{U\left(1-\frac{a^3}{r^3}\right)\cos\theta} = \frac{rd\theta}{-U\left(1+\frac{a^3}{2r^3}\right)\sin\theta} = \frac{r\sin\theta d\psi}{0},\ r \ge a$$

$$\Rightarrow \quad d\psi = 0 \quad \Rightarrow \psi = \text{constant.}$$

and

$$r\left(1-\frac{a^3}{r^3}\right)\cos\theta d\theta = -\left(1+\frac{a^3}{2r^3}\right)\sin\theta\ dr$$

$$\Rightarrow \quad \frac{1}{r}\left(\frac{2r^3+a^3}{r^3-a^3}\right)dr = -2\cot\theta\, d\theta$$

$$\Rightarrow \left(\frac{2r+a^3 r^{-2}}{r^2-a^3 r^{-1}}\right)dr = -2\cot\theta\, d\theta$$

Integrating, we get

$$\log\,(r^2 - a^3\, r^{-1}) = -2\log\sin\theta + \log C$$

$$\Rightarrow \log\left(\frac{r^3 - a^3}{r}\right) = -\log \sin^2\theta + \log C$$

$$\Rightarrow \quad \sin^2\theta = \frac{Cr}{r^3 - a^3}, \text{ where } C \geq 0$$

For each value of C, above equation gives a streamline in the plane ψ = constant. The choice of c = 0 corresponds to the sphere and the axis of symmetry.

(iii) Pressure at Any Point : The pressure at any point of the fluid is obtained by applying Bernoulli's equation along the streamline through that point, taking the pressure at ∞ to be of constant value p_∞. Thus, in the absence of body force, the Bernoulli's equation for homogeneous steady flow is

$$\frac{p}{\rho} + \frac{1}{2}(-\nabla\phi)^2 = C$$

At infinity, $p = p_\infty$ and $-\nabla\phi = U\hat{k}$, we get

$$C = \frac{p_\infty}{\rho} + \frac{1}{2}U^2$$

Thus

$$p = p_\infty + \frac{1}{2}\rho U^2 - \frac{1}{2}\rho(\nabla\phi)^2$$

$$\Rightarrow \qquad p = p_\infty +$$

$$\frac{1}{2}\rho U^2 - \frac{1}{2}\rho\left[U^2\left(1 - \frac{a^3}{r^3}\right)^2 \cos^2\theta + U^2\left(1 + \frac{a^3}{2r^3}\right)^2 \sin^2\theta\right]$$

$$| \nabla\phi = -\bar{q}$$

$$\Rightarrow \qquad p = p_\infty - \frac{1}{2}\rho U^2\left[\left(1 - \frac{a^3}{r^3}\right)^2 \cos^2\theta + \left(1 + \frac{a^3}{2r^3}\right)^2 \sin^2\theta - 1\right] \qquad (5)$$

which gives the pressure at any point of the fluid. Of particular interest is the distribution of pressure on the boundary of the sphere. It is obtained by putting r = a in (5) and thus

$$p = p_\infty - \frac{1}{2}\rho U^2\left[\left(1 + \frac{a^3}{2r^3}\right)^2 \sin^2\theta - 1\right]$$

$$= p_\infty - \frac{1}{2}\rho U^2\left(\frac{9}{4}\sin^2\theta - 1\right) = p_\infty + \frac{1}{8}\rho U^2 \,(4 - 9\sin^2\theta)$$

$$= p_\infty + \frac{1}{8}\rho U^2\,(9\cos^2\theta - 5)$$

The maximum pressure occurs at the stagnation points, where θ = 0 or π. Thus

$$p_{max} = p_\infty + \frac{1}{2}\rho U^2$$

(p_{max}. is also called stagnation pressure)
The minimum pressure occurs along the equatorial circle of the sphere where θ =π/2
Therefore,

$$p_{min.} = p_\infty - \frac{5}{8}\rho U^2$$

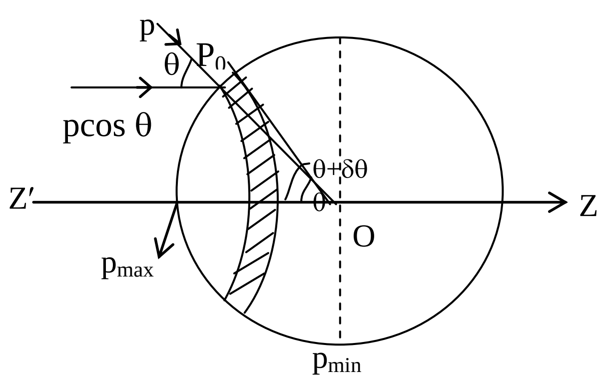
A fluid is presumed to be incapable of sustaining a negative pressure, thus

$$p_{min.} = 0 \;\Rightarrow U = \sqrt{\frac{8p_\infty}{5\rho}}$$

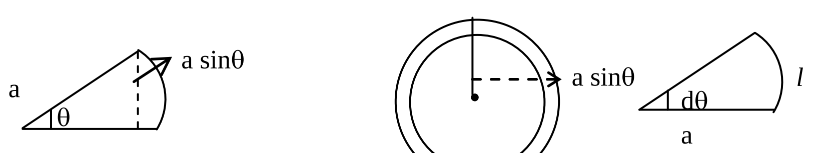
At this stage the fluid will tend to break away from the surface of the sphere and **cavitation** is said to occur. i.e. a vacuum is formed.

(iv) Thrust on the Hemisphere : Now, we find the **thrust** (force) on the hemisphere on which the liquid impinges, r = a, 0 ≤ θ ≤ π/2.

Let δS be a small element at P_0 (a, θ, ψ) of the hemisphere bounded by circles at r = a and at angular distances θ and θ + δθ from axis of symmetry (i.e. z-axis)



The component of thrust on δS is p cosθ δS. Hence the total thrust on the hemisphere is along Z′O and is given by



$$l = r\,\theta \equiv a d\theta$$

$$dS = (2\pi a \sin\theta)(a d\theta)$$

$$F = \int_{\text{hemisphere}} p \quad \cos\theta \, dS$$

$$= \int_0^{\pi/2} p \cos\theta \, (2\pi a \sin\theta) \, (a d\theta)$$

$$= \int_0^{\pi/2} (2\pi a^2) \sin\theta \cos\theta \left[p_\infty + \frac{1}{8}\rho U^2 (9\cos^2\theta - 5) \right] d\theta$$

(using value of p at boundary)

$$= \pi a^2 \left[p_\infty - \frac{1}{16}\rho U^2 \right].$$

3.3. Sphere in Motion in Fluid at Rest at Infinity. Let a solid sphere of radius 'a' centred at 0 be moving with uniform velocity $-U\hat{k}$ in incompressible fluid of infinite extent, which is at rest at infinity. Z-axis is the axis of symmetry and $\hat{k}$ is unit vector in this direction. (As the sphere is moving with velocity $-U\hat{k}$ $\Rightarrow$ the relative velocity of fluid if the sphere be considered to be at rest is $U\hat{k}$.)

The boundary value problem for ϕ is now to solve

$$\nabla^2\phi = 0 \qquad (1)$$

such that

$$\frac{-\partial\phi}{\partial r} = -U\cos\theta, \; (r = a) \qquad (2)$$

and

$$|\nabla\phi) \to 0, \;\; (r\to\infty) \qquad (3)$$

The present case is also a problem with axial symmetry about the axis $\theta = 0, \pi$, so

$$\phi = \phi(r, \theta)$$

Also, since $P_1(\cos\theta) = \cos\theta$ | Legendre's function

and the boundary condition (2) implies that the dependence of ϕ on θ must be like $\cos\theta$, therefore ϕ has the form

$$\phi = -\left(Ar + \frac{B}{r^2}\right)P_1(\cos\theta) = -\left(Ar + \frac{B}{r^2}\right)\cos\theta$$

However, to satisfy (3), it is necessary that A = 0, and then from (2), we get B $= \frac{1}{2}Ua^3$.

Thus the solution for ϕ is

$$\phi = \frac{-Ua^3}{2r^2}\cos\theta \qquad (4)$$

From here, the velocity components are obtained to be

$$q_r = -\frac{\partial\phi}{\partial r} = \frac{-Ua^3}{r^3}\cos\theta,\ q_\theta = \frac{-1}{r}\frac{\partial\phi}{\partial\theta} = \frac{-Ua^3}{2r^3}\sin\theta,\ q_\psi = 0,$$

where (r, θ, ψ) are spherical polar co-ordinates. The various terms of particular importance related to this motion are obtained as follows.

(i) Streamlines : The differential equations for streamlines are

$$\frac{dr}{q_r} = \frac{rd\theta}{q_\theta} = \frac{r\sin\theta d\psi}{q_\psi}$$

i.e. $$\frac{dr}{\frac{-Ua^3}{r^3}\cos\theta} = \frac{rd\theta}{-\frac{Ua^3}{2r^3}\sin\theta} = \frac{r\sin\theta d\psi}{0}$$

$\Rightarrow$ $\qquad d\psi = 0 \Rightarrow \psi = \text{constant.}$

and

$$\frac{dr}{r} = 2\cot\theta\, d\theta \Rightarrow \log r = 2\log\sin\theta + \log C$$

$\Rightarrow$ $\qquad r = C\sin^2\theta$

Therefore, streamline lines are given by $r = C\sin^2\theta$, ψ = constant

(ii) K.E. of the Liquid : Let S be the surface of sphere and ρ be the density of liquid, then K.E. is given by

$$T_1 = \frac{\rho}{2}\int_S \phi\frac{\partial\phi}{\partial n}dS \qquad (5)$$

Where $\hat{n}$ is the outwards unit normal. But for the sphere $\hat{n}$ is along radius vector

Therefore, $$\left(\phi\frac{\partial\phi}{\partial n}\right)_S = \left(-\phi\frac{\partial\phi}{\partial r}\right)_{r=a}$$

$$= \left(\frac{1}{2}Ua\cos\theta\right)(U\cos\theta)$$

$$= \frac{1}{2} U^2 a \cos^2\theta$$

Therefore,

$$T_1 = \frac{\rho}{2}\int_S \frac{1}{2} U^2 a \cos^2\theta dS = \frac{\rho a U^2}{4}\int_0^\pi \cos^2\theta\,(2\pi a \sin\theta)\,(a d\theta)$$

$$= \frac{\pi\rho a^3 U^2}{2}\int_0^\pi \cos^2\theta \sin\theta\, d\theta$$

$$\left|\begin{array}{l} 0 \le \theta \le \pi \\ 0 \le \psi \le 2\pi \end{array}\right.$$

$$= \frac{\pi\rho a^3 U^2}{2}\left[-\frac{\cos^3\theta}{3}\right]_0^\pi$$

$$= \frac{1}{3}\pi\rho a^2 U^2 = \left(\frac{4}{3}\pi\rho a^3\right)\left(\frac{U^2}{4}\right).$$

$$= \frac{1}{4} M^1 U^2 \qquad (6)$$

where $M^1 = \frac{4}{3}\pi\,\rho a^3$ is the mass of the liquid displaced by the sphere.

Also, K.E. of the sphere moving with speed U is given by

$$T_2 = \frac{1}{2} M U^2 \qquad (7)$$

where $M = \frac{4}{3}\pi\sigma a^3$ is the mass of the sphere, σ being the density of the material of the sphere.

Therefore, from (6) and (7), total K.E. T is given by

$$T = T_1 + T_2 = \frac{1}{2}\left(M + \frac{M'}{2}\right)U^2 \qquad (8)$$

The quantity $M + \frac{M'}{2}$ is called the virtual mass of the sphere.

3.4. Accelerating Sphere Moving in a Fluid at Rest at Infinity. The solution derived above for ϕ is applicable when the sphere translates unsteadily along a straight line. In the present case, we take U = U(t) and get the velocity potential as

$$\phi = \phi(r, \theta, t) = \frac{-U(t)a^3}{2r^2}\cos\theta \qquad (1)$$

The instantaneous values of velocity components and K.E. at time t are given by

$q_r = \dfrac{-U(t)a^3}{r^3}\cos\theta, q_\theta = \dfrac{-U(t)a^3}{2r^3}\sin\theta$, $q\psi = 0$ | similar to steady case

and $$T = \frac{1}{2}\left(M + \frac{1}{2}M'\right)U^2(t) \qquad (2)$$

The pressure at any point of the fluid is obtained by using Bernoulli's equation for unsteady flow of a homogeneous liquid, in the absence of body force, as

$$\frac{p}{\rho} + \frac{1}{2}\overline{U}^2 - \frac{\partial\phi}{\partial t} = f(t) \qquad (3)$$

where f(t) is a function of time t only.

Let p_∞ be the pressure at infinity where the fluid is at rest, then from (3), we get

$f(t) = \dfrac{p_\infty}{\rho}$ and thus

$$\frac{p}{\rho} = \frac{p_\infty}{\rho} - \frac{1}{2}\overline{U}^2 + \frac{\partial\phi}{\partial t} \qquad (4)$$

To find $\dfrac{\partial\phi}{\partial t}$, we proceed as follows :

Since $\overline{U} = -U\hat{k} = -U(t)\hat{k}$ is the velocity of the sphere, the velocity potential given in (1) can be expressed in the form

$$\phi = \frac{1}{2}\frac{a^3(\overline{U}\cdot\bar{r})}{r^3} \qquad (5)$$

since $\bar{r}$ is the position vector of a fixed point P of the fluid relative to the moving centre 0 of the sphere, it follows that

$$\overline{U} = \frac{\partial}{\partial t}(-\bar{r}) \qquad (6)$$

Also, since $r^2 = \bar{r}\cdot\bar{r} \Rightarrow r\dfrac{\partial r}{\partial t} = \bar{r}\cdot\dfrac{\partial\bar{r}}{\partial t} = -\bar{r}\cdot\overline{U}$ |using (6)

$= (-\bar{r})\cdot(-U\hat{k})$

$= rU(\hat{r}\cdot\hat{k})$ | $\bar{r} = r\,\hat{r}$

$= r\,U\cos\theta$

$$\Rightarrow \frac{\partial r}{\partial t} = U\cos\theta \qquad (7)$$

Differentiating (5) w.r.t. time t and using (6) & (7), we get

$$\frac{\partial \phi}{\partial t} = \frac{1}{2}a^3\left[-\frac{U^2}{r^3} - \frac{\cos\theta}{r^2}\frac{\partial U}{\partial t} + \frac{3U^2}{r^3}\cos^2\theta\right]$$

$$= -\frac{a^3}{2}\left[\frac{U\cos\theta}{r^2} + \frac{U^2}{r^3} - \frac{3U^2\cos^2\theta}{r^3}\right] \qquad \left|\ \dot{U} = \frac{\partial U}{\partial t}\right.$$

Also,

$$\overline{U}^2 = q_r^2 + q_\theta^2 = \frac{U^2a^6}{r^6}\cos^2\theta + \frac{U^2a^6}{4r^6}\sin^2\theta$$

$$= \frac{U^2a^6}{r^6}\left(\cos^2\theta + \frac{1}{4}\sin^2\theta\right)$$

The pressure at any point of the fluid can be obtained from equation (4).
In particular, at a point on the sphere r = a

$$\frac{\partial \phi}{\partial t} = \frac{-1}{2}\left[\dot{U}a\cos\theta + U^2 - 3U^2\cos^2\theta\right]$$

and $$\overline{U} = \frac{U^2}{4}(4\cos^2\theta + \sin^2\theta)$$

and the corresponding pressure is given by

$$\frac{p}{\rho} = \frac{p_\infty}{\rho} - \frac{1}{2}\dot{U}a\cos\theta + \frac{1}{8}U^2(9\cos^2\theta - 5) \qquad (8)$$

The force (thrust) acting on the sphere is given by

$$\overline{F} = \int_0^\pi p\cos\theta(2\pi a\sin\theta)(a d\theta)\hat{k}$$

$$= 2\pi a^2\hat{k}\int_0^\pi\left[p_\infty - \frac{1}{2}\rho\dot{U}a\cos\theta + \frac{1}{8}\rho U^2(9\cos^2\theta - 5)\right]\cos\theta\sin\theta\, d\theta$$

$$= \frac{2}{3}\pi\rho a^3\dot{U}\hat{k} = \frac{1}{2}\left(\frac{4}{3}\pi a^3\rho\right)\dot{U}\hat{k} = \frac{1}{2}M'\dot{U}\hat{k}$$

where $M' = \frac{4}{3}\pi a^3\rho$ is mass of the liquid displaced. This shows that the force acts in the direction oppositing the sphere's motion.

3.5. Equation of Motion of the Sphere. Let R be the external force per unit mass in the direction of motion of the sphere. Let us use the result that the rate of doing work is equal to the rate of increase in K.E.

Thus $$RU = \frac{d\tau}{dt} = \frac{1}{2}\frac{d}{dt}\left[\left(M + \frac{M'}{2}\right)U^2(t)\right]$$

| From (2)

$$= \left(M + \frac{M'}{2}\right)U\frac{dU}{dt}$$

$$\Rightarrow \qquad M\frac{dU}{dt} = R - \frac{1}{2}M'\frac{dU}{dt} \qquad (9)$$

If the liquid is not there, then $M' = 0$ and the equation of motion of the sphere is

$$M\frac{dU}{dt} = R \qquad (10)$$

Comparing equation (9) & (10), we note that the presence of the liquid offers a resistance of the amount $\frac{1}{2}M'\frac{dU}{dt}$ to the motion of the sphere

Let R' be the external force per unit mass on the sphere when there is no liquid, then

MR = external force on the sphere in the presence of the liquid.

$$= MR' - M'R' = (M - M')R'$$

Since, $$M = \frac{4\pi\sigma a^3}{3}, M' = \frac{4\pi\rho a^3}{3}$$

$$\therefore \qquad R = \left(\frac{\sigma - \rho}{\sigma}\right)R' \qquad (11)$$

From equations (9) & (11), we find

$$M\frac{dU}{dt} = \left(\frac{\sigma-\rho}{\sigma}\right)R' - \frac{1}{2}M'\frac{dU}{dt}$$

or $$\left(M + \frac{M'}{2}\right)\frac{dU}{dt} = \left(\frac{\sigma-\rho}{\sigma}\right)R' = \left(\frac{M - M'}{M}\right)R'$$

$$\therefore \qquad M\frac{dU}{dt} = \left(\frac{M-M'}{M+\frac{M'}{2}}\right)R' = \left(\frac{\sigma-\rho}{\sigma+\frac{1}{2}}\right)R' \qquad (12)$$

This is the required equation of motion of a sphere in a liquid at rest at infinity. From equations (10) & (12), we note that the effect of the presence of the liquid reduces the external force in the ration $\sigma - \rho : \sigma + \frac{\rho}{2}$.

3.6. Remark. We have already studied the impulsive actions in Unit-I, where, we had derived the relation between the impulsive pressure P and the velocity potential ϕ as $P = \rho\phi$. Here, we derive the expression for K.E. generated due to impulsive action.

3.7. Kinetic Energy Generated by Impulsive Motion : Let us consider incompressible fluid, initially at rest, which is set in motion by the application of impulse $\bar{I}_1, \bar{I}_2, ..., \bar{I}_m$ to rigid boundaries $S_1, S_2, \ldots, S_m$ respectively. The fluid may be of finite or infinite extent. We know that the K.E. of the irrotational motion generated in the fluid is given by

$$T = \frac{\rho}{2}\int_S \phi\frac{\partial\phi}{\partial n}dS \qquad (1)$$

where $S = S_1 + S_2 + \ldots + S_m$, $\hat{n}$ is outwards unit normal on each S_i

Let the velocity given to S_i be $\overline{U}_i$ (i = 1, 2,…, m), then on S_i, we have

$$-\frac{\partial\phi}{\partial n} = \hat{n}.\overline{U}_i \qquad (2)$$

$$|\overline{q} = -\nabla\phi$$

using (2) in (1), we get

$$T = -\frac{\rho}{2}\sum_{i=1}^{m}\overline{U}_i.\int_{S_i}\hat{n}\phi dS \qquad (3)$$

But the impulsive force exerted by the fluid on S_i is $\overline{R}_i$, where

$$\overline{R}_i = \int_{S_i}\hat{n}P\,dS = \rho\int_{S_i}\hat{n}\,\phi\,dS \qquad |\, P = \rho\phi$$

(4)

Thus from (3) & (4), we get

$$T = -\frac{1}{2}\sum_{i=1}^{m}\overline{U}_i \cdot \overline{R}_i \qquad (5)$$

3.8. Example. Incompressible liquid of constant density ρ is contained within a region bounded by two concentric rigid spherical surfaces of radii a, b (a < b). The fluid is initially at rest. If the inner boundary is suddenly given a velocity $U\hat{k}$, where $\hat{k}$ is a constant vector, show that the outer surface experiences the impulsive force

$$\frac{2\pi\rho U a^3 b^3}{b^3 - a^3}\hat{k}$$

Also calculate the corresponding K.E. generated by the impulsive motion.

Solution. The motion generated in the fluid is irrotational $\Rightarrow$ $\bar{q} = -\nabla\phi \;\Rightarrow \nabla^2\phi = 0$ which is the equation of continuity. The boundary conditions which ϕ must satisfy, are

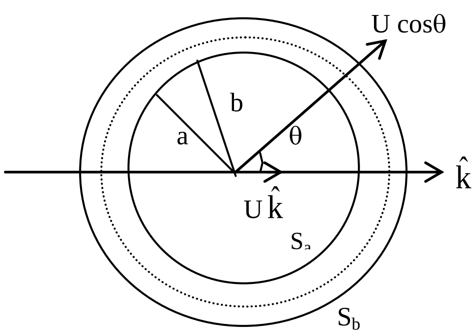
$$-\frac{\partial\phi}{\partial r} = U\cos\theta \;\; (r = a) \qquad (1)$$

$$-\frac{\partial\phi}{\partial r} = 0 \;\; (r = b) \qquad (2)$$

with (r, θ, ψ) spherical polar co-ordinates and with $\theta = 0$ along the direction of $\hat{k}$.

The form of boundary conditions suggest a solution of the form

$$\phi = -(Ar + B r^{-2})\cos\theta \qquad (3)$$



which satisfy (1) & (2) if

$$A - \frac{2B}{a^3} = U, \;\; A - \frac{2B}{b^3} = 0$$

$$\Rightarrow \qquad A = \frac{-Ua^3}{b^3 - a^3}, \;\; B = \frac{-Ua^3b^3}{2(b^3 - a^3)}$$

Thus, the solution of the problem is

$$\phi = \frac{Ua^3}{b^3 - a^3}\left(r + \frac{b^3}{2r^2}\right)\cos\theta$$

Impulsive force acting on the outer boundary in the direction of $\hat{k}$ is

$$\overline{F} = \left[\int_{S_b} (P)_{r=b} \cos\theta \, dS\right]\hat{k}$$

where $(P)_{r=b} = (\rho\phi)_{r=b}$

$$= \frac{\rho Ua^3}{b^3 - a^3}\left(b + \frac{b^3}{2b^2}\right)\cos\theta$$

$$= \frac{3}{2}\frac{2Ua^3 b\cos\theta}{b^3 - a^3}$$

and for the outer sphere r = b,

$$dS = 2\pi\,(b\sin\theta)\,(bd\theta),\ 0 \le \theta \le \pi$$

Thus, impulsive force, $\overline{F} = \int_0^\pi \frac{3}{2}\frac{\rho Ua^3 b}{b^3 - a^3}\cos^2\theta(2\pi b^2 \sin\theta)\,d\theta\,\hat{k}$

$$= \frac{3\pi\rho Ua^3 b^3 \hat{k}}{b^3 - a^3}\int_0^\pi \cos^2\theta \sin\theta \, d\theta$$

$$= \frac{2\pi\rho Ua^3 b^3}{b^3 - a^3}\hat{k}$$

Hence the result

Now, if $\overline{U}_1, \overline{U}_2$ denote the velocity of spheres of radii a & b respectively and $\overline{R}_1, \overline{R}_2$ be the corresponding impulsive forces exerted by the fluid, then

$$\overline{U}_1 = U\hat{k},\ \overline{U}_2 = \overline{0},\ \overline{R}_2 = \overline{F} = \frac{2\pi\rho Ua^3 b^3}{b^3 - a^3}\hat{k}$$

$\therefore$ K.E., $T = -\frac{1}{2}\Sigma\overline{U}_i.\overline{R}_i$

$$= -\frac{1}{2}\left[\overline{U}_1 \cdot \overline{R}_1 + \overline{U}_2 \cdot \overline{R}_2\right] = -\frac{1}{2}\overline{U}_1 \cdot \overline{R}_1 = -\frac{1}{2}U\hat{k} \cdot \overline{R}_1 \quad (4)$$

Also,

$$\overline{R}_i = \rho\int_{S_i} \hat{n}\,\phi\, dS \Rightarrow \overline{R}_1 = \rho\int_{S_1} \hat{n}\,\phi\, dS$$

$$\Rightarrow \quad \overline{R}_1 \cdot \hat{k} = \rho\int_{S_a} \hat{n} \cdot \hat{k}(\phi)_{r=a}\, dS$$

$$= -\rho \int_{S_a} \cos\theta \left[\frac{Ua^3}{b^3 - a^3} \left(a + \frac{b^3}{2a^2} \right) \cos\theta \right] 2\pi \, (a \sin\theta) \, (a \, d\theta)$$

(negative sign due to inwards normal i.e. on the inner sphere, pressure is inwards)

$$= -\frac{\rho U a^3}{b^3 - a^3} \frac{2a^3 + b^3}{2a^2} . 2\pi a^2 \int_0^{\pi} \cos^2\theta \sin\theta \, d\theta$$

$$= \frac{-2}{3} \frac{\pi\rho U a^3 (2a^3 + b^3)}{b^3 - a^3}$$

Thus, from equation (4), we get

$$T = \frac{1}{3} \frac{\pi\rho U^2 a^3 (2a^3 + b^3)}{b^3 - a^3}$$

3.9. Deduction : If we let $b \to \infty$, then it becomes the case of a sphere of radius 'a' moving in an infinite liquid at rest at infinity and we get

$$T = \underset{b \to \infty}{\mathrm{Lt}} \frac{1}{3} \frac{\pi\rho U^2 a^3 \left(1 + \frac{2a^3}{b^3} \right)}{1 - \frac{a^3}{b^3}} = \frac{1}{3} \pi\rho U^2 a^3$$

$$= \frac{1}{4} \left(\frac{4}{3} \pi\rho a^3 \right) U^2 = \frac{1}{4} M_1' U^2$$

where $M_1' = \frac{4}{3} \pi\rho a^3$ is the mass of liquid displaced by the sphere r = a

3.10. Example. (Motion of Two Concentric Spheres) : The space between two spheres is filled with incompressible fluid. The spheres have radii a, b (a < b) and move with constant speeds U, V respectively along the line of centres. Show that at the instant when the spheres are concentric, the velocity potential is given by

$$\phi = \frac{\left[(a^3 U - b^3 V) r + \frac{1}{2} (U - V) a^3 b^3 r^{-2} \right] \cos\theta}{b^3 - a^3}$$

Also determine the impulse which is required to produce the velocity U to the inner sphere, when outer sphere is at rest.

Solution. Let ρ be the density of the liquid.
We are to solve $\nabla^2 \phi = 0$ under the boundary conditions

$$-\frac{\partial\phi}{\partial r} = U\cos\theta,\ r = a \qquad (1)$$

and
$$-\frac{\partial\phi}{\partial r} = V\cos\theta,\ r = b \qquad (2)$$

where U & V are taken in the same direction.

The solution of the Laplace equation is of the form

$$\phi = -(Ar + B\bar{r}^{2})\cos\theta$$

$$\Rightarrow \qquad -\frac{\partial\phi}{\partial r} = \left(A - \frac{2B}{r^3}\right)\cos\theta$$

and thus the boundary conditions give

$$A - \frac{2B}{a^3} = U,\ A - \frac{2B}{b^3} = V$$

Solving for A & B, we find

$$B = \frac{1}{2}\frac{(U-V)a^3b^3}{a^3-b^3},\ A = \frac{a^3U - b^3V}{a^3-b^3}$$

Thus the velocity potential for this motion is

$$\phi = -\left[\left(\frac{a^3U-b^3V}{a^3-b^3}\right)r + \frac{1}{2}\frac{(U-V)a^3b^3}{a^3-b^3}\frac{1}{r^2}\right]\cos\theta$$

$$= \frac{\left[(a^3U-b^3V)r + \frac{1}{2}(U-V)a^3b^3\bar{r}^{2}\right]\cos\theta}{b^3-a^3}$$

Hence the result

Impulse :- When outer sphere is at rest, then V = 0 and from equation (3), we get

$$\phi = \frac{Ua^3}{b^3-a^3}\left(r + \frac{b^3}{2r^2}\right)\cos\theta \qquad (4)$$

Let $\quad M = \frac{4}{3}\pi a^2\sigma$ be the mass of inner sphere

and $\quad M' = \frac{4}{3}\pi a^3\rho$ is the mass of liquid displaced by the inner sphere.

If I be the impulse, then by the principle of linear momentum, we have

$$I = MU + \text{Total impulsive Pressure}$$

i.e.
$$I = MU + \int (P)_{r=a} \cos\theta \, dS$$

$$= MU + \rho \int_0^{\pi} \frac{Ua^3}{b^3 - a^3}\left(a + \frac{b^3}{2a^2}\right)\cos^2\theta \; 2\pi \,(a \sin\theta)\,(a\, d\theta)$$

$$| \; P = \rho\phi$$

i.e.
$$I = MU + \frac{\pi\rho U a^3 (2a^3 + b^3)}{b^3 - a^3}\int_0^{\pi} \cos^2\theta \sin\theta \, d\theta$$

$$= MU + \frac{2}{3}\frac{\pi\rho a^3 U(2a^3 + b^3)}{b^3 - a^3}$$

$$= MU + \frac{1}{2}\frac{M' U(2a^3 + b^3)}{b^3 - a^3}$$

3.11. Deduction :- If $b \to \infty$, then it will be the case of a solid sphere moving in an infinite liquid and

$$I = MU + \frac{M'}{2}U = \left(M + \frac{M'}{2}\right)U$$

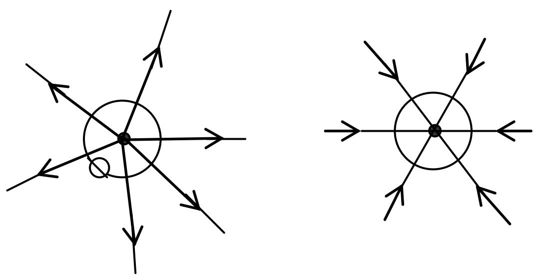
3.12. Remark. The problem in which we solve the Laplace equation $\nabla^2\phi = 0$ when the normal derivative of ϕ i.e. $\frac{\partial\phi}{\partial n}$ is given on the boundary, then such type of problem is called a **Neumann problem** whereas the solution of $\nabla^2\phi = 0$ when the value of ϕ is given on the boundary, is termed as **Dirichlet** problem.

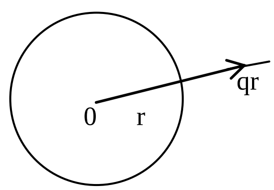
4. Sources, Sinks and Doublets (Three-dimensional Hydrodynamical Singularities)

4.1. Source : An outward symmetrical radial flow of fluid in all directions is termed as a three dimensional source or a **point source** or a **simple source.**

Thus, a source is a point at which fluid is continuously created and distributed e.g. an expanding bubble of gas pushing away the surrounding fluid. If the volume of fluid per unit time which is emitted from a simple source at 0 is constant and equal to $4\pi m$, then m is termed as strength of the source.

4.2. Sink : A negative source is called a **sink**. At such points, the fluid is constantly moving radically inwards from all directions. Thus a simple sink of strength m is a simple source of strength –m.





4.3. Velocity Potential due to a Simple Source of Strength m. Let there be a source of strength m at a point 0. With 0 as the centre, we draw a sphere of radius r around 0.

The flow across the sphere per unit volume is given by

$$\int_S \bar{q}.\hat{n}\, dS$$

In case of a source there is only the radial velocity i.e. $\bar{q}$ has only radial component q_r .

Therefore, the flow is

$$= \int_S q_r \, dS \qquad | \bar{q}.\hat{n} = q_r, \text{ since } \bar{q} \text{ and } \hat{n} \text{ have same directions i.e.}$$

radial direction.

$$= q_r (4\pi r^2).$$

Thus, we get

$$4\pi m = q_r (4\pi r^2)$$

$$\Rightarrow \qquad q_r = \frac{m}{r^2} = -\frac{\partial}{\partial r}\left(\frac{m}{r}\right) \qquad (1)$$

It is observed that curl $\bar{q} = \bar{0}$ (except at r = 0), therefore for irrotational flow,

$$q_r = -\frac{\partial \phi}{\partial r} \qquad | \bar{q} = -\nabla\phi \qquad (2)$$

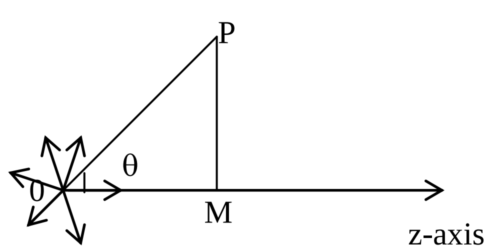
From (1) & (2), we find

$$\phi = \frac{m}{r}$$

which is the required expression for the velocity potential for a source.

4.4. Remarks. (i) For a simple sink of strength m, the velocity potential is $\phi = -\frac{m}{r}$

(ii) A source or sink implies the creation or annihilation of liquid at a point. Both are points at which the velocity potential (and stream function for two dimensional case) become infinite and therefore, they require special analysis.



4.5. A simple Source in Uniform Stream. Let us consider a simple source of strength m at 0 in a uniform stream having undisturbed velocity $U\hat{k}$, $\hat{k}$ be the unit vector along z-axis which is taken as the axis of symmetry of the flow.

We shall find the velocity potential at any point P(z, θ, ψ). From P, draw ⊥ on OZ. Let OP = r, $\angle POZ = \theta$; OM = z

We observe that the velocity potential of the uniform stream in the absence of source is

$$\left| \begin{array}{l} \bar{q} = -\nabla\phi \qquad \Rightarrow U\hat{k} = -\dfrac{\partial\phi}{\partial z}\hat{k} \\ \Rightarrow \dfrac{\partial\phi}{\partial z} = -U \Rightarrow \phi = -Uz \end{array} \right.$$

$$\phi_1 = -Uz = -Ur\cos\theta \tag{1}$$

and the velocity potential of the simple source is

$$\phi_2 = \frac{m}{r} \tag{2}$$

Thus, the velocity potential of the combination is

$$\phi = \phi_1 + \phi_2 = -Ur\cos\theta + \frac{m}{r}$$

$$= -\left(Ur\cos\theta - \frac{m}{r}\right) \tag{3}$$

From here, the velocity components at P(r, θ, ψ) are

$$q_r = -\frac{\partial\phi}{\partial r} = U\cos\theta + \frac{m}{r^2}$$

$$q_\theta = -\frac{1}{r}\frac{\partial\phi}{\partial\theta} = -U\sin\theta \qquad \left| \begin{array}{l} 0 \le \theta \le \pi \\ 0 \le \psi \le 2\pi \\ \dfrac{\partial}{\partial\psi} \equiv 0 \end{array} \right.$$

$$q_\psi = -\frac{1}{r\sin\theta}\frac{\partial\phi}{\partial\psi} = 0$$

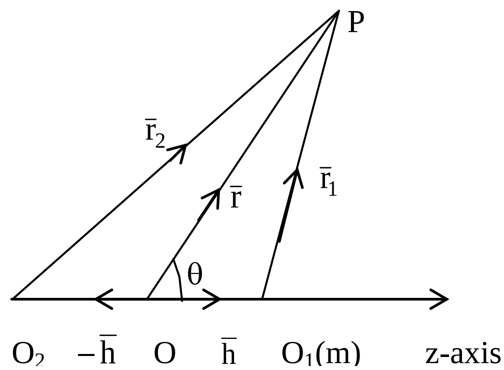
The stagnation points $(\bar{q} = 0)$ are given by $U\cos\theta + \dfrac{m}{r^2} = 0,\ \sin\theta = 0 \ \Rightarrow \theta = 0$ or π

But $\theta = 0$ gives r to be imaginary $\Rightarrow \theta = \pi$ and $r = \sqrt{\dfrac{m}{U}}$

Thus there is only one stagnation point $\left(\sqrt{\dfrac{m}{U}}, \pi, 0\right)$

4.6. Doublet (Dipole). The combination of a source and a sink of equal strength, at a small distance apart, is called a doublet.

4.7. To Find the Velocity Potential of Doublet. Suppose that there is a simple source of strength m at O_1 and a simple sink of strength m at O_2. Origin O is taken as the mid point. of $O_1 O_2$. It is also assumed that there is no other source or sink. Let P be a fixed point within the fluid and



$$\overline{OP} = \bar{r}, \quad \overline{O_1P} = \bar{r}_1, \quad \overline{O_2P} = \bar{r}_2, \quad \underline{|POO_1} = \theta,$$

$$\overline{OO_1} = \bar{h}, \quad \overline{OO_2} = -\bar{h}, \quad h = |\bar{h}|$$

The velocity potential at P due to the combination of source and sink at O_1 and O_2 is

$$\phi = \frac{m}{r_1} - \frac{m}{r_2} = \frac{mr_2 - mr_1}{r_1 r_2}$$

$$= \frac{m(r_2 - r_1)}{r_1 r_2} = \frac{m(r_2^2 - r_1^2)}{r_1 r_2 (r_1 + r_2)}$$

$$= \frac{m(\bar{r}_2 - \bar{r}_1).(\bar{r}_2 + \bar{r}_1)}{r_1 r_2 (r_1 + r_2)}$$

But $\quad \bar{r}_2 - \bar{r}_1 = 2\bar{h}$ and $\bar{r}_2 + \bar{r}_1 = 2\bar{r}$

$$\left| \begin{array}{l} \bar{r}_2 = \bar{h} + \bar{r} \\ \bar{r}_1 = -h + \bar{r} \end{array} \right.$$

Thus $$\phi = \frac{m(2\bar{h}).(2\bar{r})}{r\,r_2(r_1+r_2)} = \frac{4m\bar{h}.\bar{r}}{r_1 r_2 (r_1+r_2)}$$

$$= \frac{2\bar{\mu}.\bar{r}}{r_1 r_2 (r_1+r_2)}, \text{ where } \bar{\mu} = 2m\bar{h} \qquad (1)$$

In equation (1), let us first keep $\bar{\mu}$ a finite constant and non-zero vector, so that $\mu = |\bar{\mu}|$ is a finite constant and non-zero scalar. Let $\bar{h} \to \bar{0}$ along $\overline{O_1O}$.

Then $m\to\infty$ in such a way that $\bar{\mu}$ remains the same finite non-zero constant vector. In that case, both r_1, $r_2 \to r$ and thus under this limiting process, (1) results in

$$\phi = \frac{2\bar{\mu}.\bar{r}}{2r^3} = \frac{\mu r\cos\theta}{r^3} = \frac{\mu\cos\theta}{r^2} \qquad (2)$$

The limiting source sink combination obtained at 0 when we keep the direction of $\bar{h}$ fixed but let $h\to 0$ and $m\to\infty$ with $\mu = 2mh$ remaining a finite non-zero constant, is called a three-dimensional doublet (or dipole). The scalar quantity μ is called the moment or strength of the doublet. The vector quantity $\bar{\mu} = \mu\hat{\mu}$ is called the vector moment of the doublet & $\hat{\mu}$ (unit vector from 0_2 to 0_1) determines the direction of the axis of the doublet from sink to source.

From (2), the velocity components are given by

$$q_r = -\frac{\partial\phi}{\partial r} = \frac{2\mu\cos\theta}{r^3}$$

$$q_\theta = -\frac{1}{r}\frac{\partial\phi}{\partial\theta} = \frac{\mu\sin\theta}{r^3}$$

$$q_\psi = 0$$

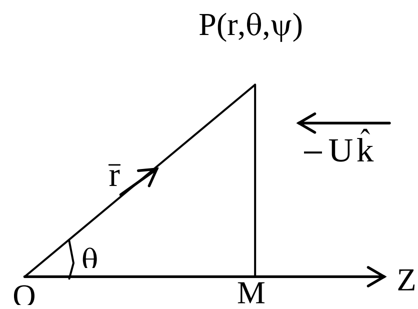
The streamlines due to the doublet are given by

$$\frac{dr}{\frac{2\mu\cos\theta}{r^3}} = \frac{rd\theta}{\frac{\mu\sin\theta}{r^3}} = \frac{r\sin\theta d\psi}{0}$$

$\Rightarrow$ $$d\psi = 0 \quad \Rightarrow \psi = \text{constant and } \frac{dr}{r} = 2\cot\theta\, d\theta$$

$\Rightarrow$ $$r = A\sin^2\theta$$

4.8. Doublet in a Uniform Stream. Let there be a doublet of vector moment $\bar{\mu} = \mu\hat{k}$ at O in a uniform stream whose velocity in the absence of the doublet is $-U\hat{k}$ (U = constant).



Let P be a point in the fluid having spherical polar co-ordinates (r, θ, ψ), the direction $\overline{OZ}$ of the doublets axis being the line θ = 0. We shall find the resultant velocity potential due to the combination of the uniform stream and the doublet. We know that the velocity potential due to the uniform stream is

$$\phi_1 = Uz = Ur\cos\theta \qquad (1)$$

and the velocity potential due to a doublet at O, is

$$\phi_2 = \frac{\mu\cos\theta}{r^2} \qquad (2)$$

Thus, the resultant velocity potential at P. due to the combination, is

$$\phi = \phi_1 + \phi_2 = (Ur + \mu\bar{r}^2)\cos\theta$$

From here, the velocity component are

$$q_r = -\frac{\partial\phi}{\partial r} = -\left(U - \frac{2\mu}{r^3}\right)\cos\theta$$

$$q_\theta = -\frac{1}{r}\frac{\partial\phi}{\partial\theta} = \left(U + \frac{\mu}{r^3}\right)\sin\theta$$

$$q_\psi = \frac{1}{r\sin\theta}\frac{\partial\phi}{\partial\psi} = 0$$

Stagnation points are determined by solving.

$$\left(U - \frac{2\mu}{r^3}\right)\cos\theta = 0, \left(U + \frac{\mu}{r^3}\right)\sin\theta = 0 \qquad |\,\bar{q} = \bar{0}$$

which are satisfied when $\sin\theta = 0$ and $r = \left(\frac{2\mu}{U}\right)^{1/3}$

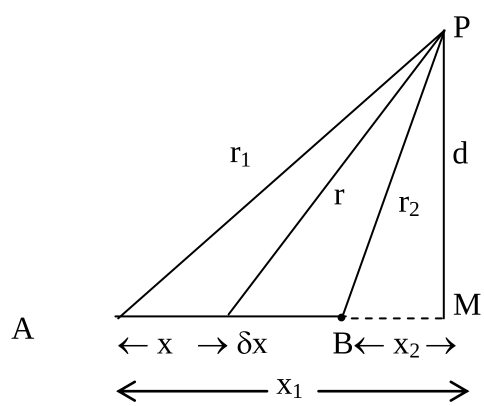
Thus, we have the two stagnation points.

$$\left(\left(\frac{2\mu}{U}\right)^{1/3}, 0\right) \text{ and } \left(\left(\frac{2\mu}{U}\right)^{1/3}, \pi\right)$$

which lie on the axis of symmetry.

If we write r = a i.e. $a = \left(\frac{2\mu}{U}\right)^{1/3}$ i.e. $\mu = \frac{1}{2} U a^3$, then for the region $r \geq a$, we obtain the same velocity potential as for a uniform flow past a fixed impermeable sphere of radius a and centre 0. Thus, for $r \geq a$, the effect of the sphere is that of a doublet of strength $\mu = \frac{1}{2} Ua^3$ situated at its centre, its axis pointing upstream. So the sphere can be represented by a suitably chosen singularity at its centre.

4.9. Line Distribution of Sources. Let us consider a uniform line source AB of strength m per unit length. This means that the elemental section of AB at a distance. x from A and of length δx is a point source of strength $m\delta x$.



Let P be a point in the fluid at a distance r from this element, then the velocity potential at P due to the point source is $\frac{m\delta x}{r}$.

The total velocity potential at P due to the entire line distribution AB (= 2*l*) is

$$\phi = m \int_0^{2l} \frac{dx}{r} \qquad (1)$$

Let AM = x_1, BM = x_2, where AM is the orthogonal projection of AP on AB. Also, let PM = d, AP = r_1, BP = r_2 . Since $r^2 = (x_1 - x)^2 + d^2 = (x_1 - x)^2 + r_1^2 - x_1^2$, therefore from (1), we get

$$\phi = m \int_0^{2l} \frac{dx}{\sqrt{(x_1 - x)^2 + (r_1^2 - x_1^2)}}$$

$$= m\left[\frac{\log\left\{(x_1 - x) + \sqrt{(x_1 - x)^2 + (r_1^2 - x_1^2)}\right\}}{-1}\right]_0^{2l} \qquad \left| \because \int_\alpha^\beta \frac{1}{\sqrt{x^2 + a^2}}\,dx = \left[\log\left(x + \sqrt{x^2 + a^2}\right)\right]_\alpha^\beta \right.$$

$$= m\left[\log\left\{(x_1 - x) + \sqrt{(x_1 - x)^2 + (r_1^2 - x_1^2)}\right\}\right]_{2l}^0$$

$$= m\left[\log(x_1 + r_1) - \log\left\{x_2 + \sqrt{x_2^2 + r_1^2 - x_1^2}\right\}\right] \qquad | \because x_1 - 2l = x_1 - AB = x_2$$

$$= m \log\left(\frac{x_1 + r_1}{x_2 + r_2}\right), \text{ where } r_1^2 - x_1^2 = d^2 = r_2^2 - x_2^2.$$

Again, the relation $r_1^2 - x_1^2 = r_2^2 - x_2^2$

$$\Rightarrow \qquad \frac{r_1 + x_1}{r_2 + x_2} = \frac{r_2 - x_2}{r_1 - x_1} = \frac{r_1 + r_2 + x_1 - x_2}{r_1 + r_2 + x_2 - x_1}$$

$$= \frac{r_1 + r_2 + 2l}{r_1 + r_2 - 2l}$$

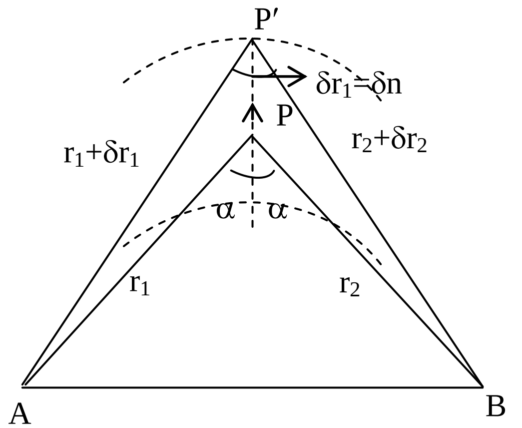
Thus, $$\phi = m \log\left(\frac{r_1 + r_2 + 2l}{r_1 + r_2 - 2l}\right)$$

$$= m \log\left(\frac{a + l}{a - l}\right) \qquad (2)$$

where 2a is the length of major axis of the ellipsoid of revolution through P having A and B as foci since for such an ellipsoid $r_1 + r_2$ = constant. It follows from here that the equipotential surfaces ϕ = constant are precisely the family of confocal ellipsoid $r_1 + r_2 = 2a$ obtained when a is allowed to vary.

Expression for Velocity :- The velocity at P is given by $\bar{q} = -\nabla\phi = -\left(\frac{\partial\phi}{\partial n}\right)\hat{n}$

(3)



Let P be any point on the ellipsoid specified by parameter a and P′ the neighbouring point on the ellipsoid specified by parameter a + δa, where $\overline{PP'} = \delta n\,\hat{n}$

Thus $\bar{q} = -m\frac{\partial}{\partial n}\left[\log\frac{a+l}{a-l}\right]\hat{n} = -m\left[\frac{1}{a+l} - \frac{1}{a-l}\right]\frac{\partial a}{\partial n}\hat{n} = \frac{2lm}{a^2 - l^2}\frac{\partial a}{\partial n}\hat{n}$ (4)

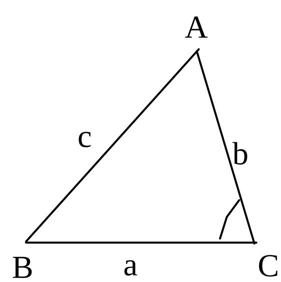
The normal at P to the a-surface bisects the angle 2α between the focal radii AP, BP.

Now,

$$(r_1 + \delta r_1)^2 = r_1^2 + (\delta n)^2 - 2r_1\,\delta n\cos(180-\alpha)$$

$$= r_1^2 + (\delta n)^2 + 2r_1\,\delta n\cos\alpha$$

$$\left|\begin{array}{l} \cos C = \dfrac{a^2 + b^2 - c^2}{2ab} \\ \Rightarrow c^2 = a^2 + b^2 - 2ab\cos C \end{array}\right.$$



$\Rightarrow$ $2r_1\,\delta r_1 = 2r_1\,\delta n\cos\alpha + (\delta n)^2 - (\delta r_1)^2$

$\Rightarrow$ $\delta r_1 = \delta n\cos\alpha$ $\quad | (\delta r_1)^2 = (\delta n)^2$

$\Rightarrow$ $\frac{\partial r_1}{\partial n} = \cos\alpha$

Similarly, $\frac{\partial r_2}{\partial n} = \cos\alpha$

Since, $2a = r_1 + r_2$

$$\Rightarrow \quad 2\frac{\partial a}{\partial n} = \frac{\partial r_1}{\partial n} + \frac{\partial r_2}{\partial n}. = \cos\alpha + \cos\alpha = 2\cos\alpha$$

$$\Rightarrow \quad \frac{\partial a}{\partial n} = \cos\alpha$$

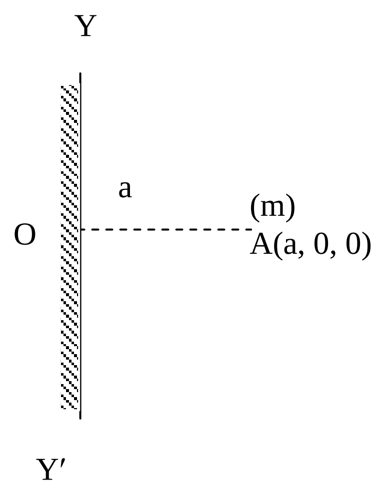
and thus from equation (4), the velocity of fluid at P is given by $\bar{q} = \left[\frac{2lm\cos\alpha}{a^2 - l^2}\right]\hat{n}$

5. Hydrodynamical Images for Three Dimensional Flows

Let us consider a fluid containing a distribution of sources, sinks and doublets. If a surface S can be drawn in the fluid across which there is no flow, then any system of sources, sinks and doublets on opposite sides of this surface S may be said to be images of one another w.r.t. to the surface. Further, if the surface S be considered as a rigid boundary and the liquid removed from one side of it, the motion on the other side will remain unaltered.

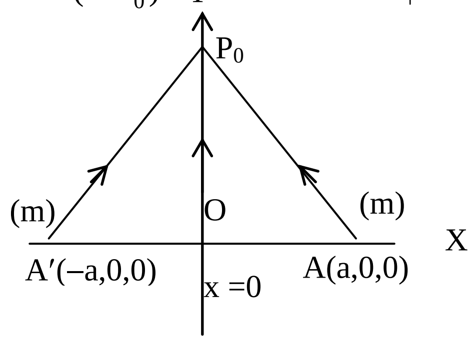
5.1. Images in a Rigid Impermeable Infinite Plane. (i) Image of a source in a plane : consider a simple source of strength m situated at A(a, 0, 0) at a distance a from an infinite plane YY′.

We shall show that the appropriate image system for this is an equal source of strength m at A′(–a, 0, 0), the reflection of A in the plane.



To prove this, we consider two equal sources f strength m at A(a, 0, 0) & A′ (–a, 0, 0) with no rigid boundary. Let P_0 be any point on the plane YY′. Then the fluid velocity at P_0 due to the two sources is

$$\bar{q} = \frac{m}{(AP_0)^3}\overline{AP_0} + \frac{m}{(A'P_0)^3}\overline{A'P_0} \qquad \left|\ \bar{q} = \frac{m}{r^2}\hat{r} = \frac{m}{r^3}\bar{r}\right.$$



$$\Rightarrow \quad \bar{q} = \frac{m}{(AP_0)^3}(\overline{AP_0} + \overline{A'P_0})$$

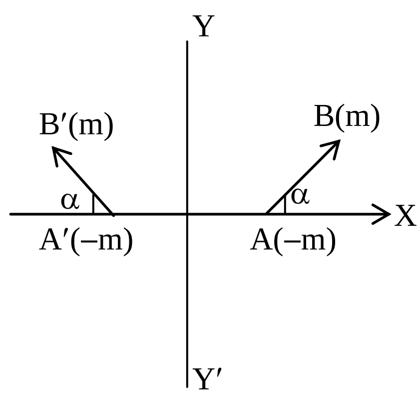
$$= \frac{m}{(AP_0)^3}(2\overline{OP_0}) = \frac{2m}{(AP_0)^3}(\overline{OP_0}) \qquad \left|\ \begin{aligned} &\because \overline{AP_0} + \overline{A'P_0} \\ &= (\overline{AO} + \overline{oP_0}) + (\overline{A'_0} + \overline{OP_0}) \\ &= 2\overline{OP_0} \end{aligned}\right.$$

This shows that at any point P_0 of the plane YY′, the fluid flows tangentially to the plane x = 0 and so there is no transport of fluid across this plane.

Let ϕ denotes the velocity potential then, at all points P_0 on the plane YY′, the normal component of velocity is zero

$\Rightarrow \dfrac{\partial\phi}{\partial n} = 0$. Hence, the image of a source at A in the rigid plane YY′ is a source at A′, as required.

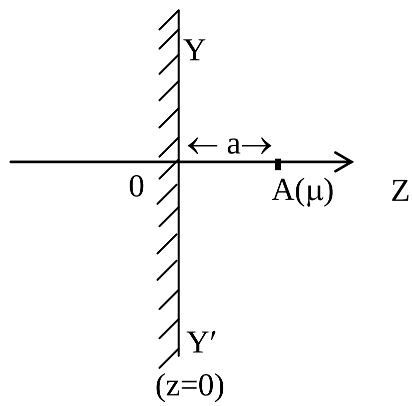
(ii) Image of Doublet in a Plane : Consider a pair of sources –m at A and m at B, taken close together and on one side of the rigid plane YY'. The image system is –m at A', m at B′, where A′ & B′ are respectively the reflections of A and B in the plane YY′. In the limiting case, when B→A along $\overline{BA}$ in such a way as to form a doublet at A, we find that the image of



a doublet in an infinite impermeable rigid plane is a doublet of equal strength and symmetrically disposed to the other w.r.t the plane.

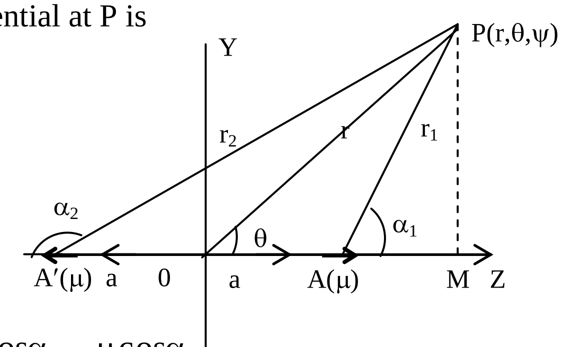
5.2. Example. A three dimensional doublet of strength μ whose axis is in the direction $\overline{OZ}$ is distant a from the rigid plane z = 0 which is the sole boundary of liquid of constant density ρ, infinite in extent. If p_∞ be the pressure at ∞, show that the pressure on the plane is least at a distance $\frac{a\sqrt{5}}{2}$ from the doublet

Solution. Let there be a doublet of strength μ at the point A with OA = a and YY′ (i.e. z = 0) be the infinite plane. Then the image system is an equal doublet of strength μ at A′, the reflection of A in the plane z = 0, and the axis along $\overline{ZO}$. The line OZ is taken as the initial line θ = 0 and plane z = 0 is θ = π/2.



so that P(r, θ, ψ) is confined to the region 0 ≤ θ ≤ π/2. Let AP = r_1, A′P = r_2 and α_1, α_2 be the angles which these lines make with the axis of the doublets as shown in the figure.

Then, the velocity potential at P is



$$\phi = \frac{\mu\cos\alpha_1}{r_1^2} + \frac{\mu\cos\alpha_2}{r_2^2} \tag{1}$$

where
$$\left.\begin{aligned} r_1^2 &= r^2 + a^2 - 2ra\cos\theta \\ r_2^2 &= r^2 + a^2 + 2ra\cos\theta \end{aligned}\right\} \tag{2}$$

(By cosine formulae in Δ POA, POA′)

But $$\cos\alpha_1 = \frac{AM}{r_1} = \frac{OM - OA}{r_1} = \frac{r\cos\theta - a}{r_1}$$

and $$\cos(180 - \alpha_2) = \frac{A'M}{r_2} = \frac{A'O + OM}{r_2} = \frac{a + r\cos\theta}{r_2}$$

$\Rightarrow$ $$\cos\alpha_2 = -\frac{(a + r\cos\theta)}{r_2}$$

Using these relations in (1), we get

$$\phi = \frac{\mu}{r_1^2}\left(\frac{r\cos\theta - a}{r_1}\right) + \frac{\mu}{r_2^2}\left[\frac{-(a + r\cos\theta)}{r_2}\right]$$

$$= \mu\left[\frac{r\cos\theta - a}{r_1^3} - \frac{r\cos\theta + a}{r_2^3}\right] \qquad (3)$$

Further from (2), we have

$$2\,r_1\frac{\partial r_1}{\partial r} = 2r - 2a\cos\theta \quad \Rightarrow \frac{\partial r_1}{\partial r} = \frac{r - a\cos\theta}{r_1}$$

Similarly, $\dfrac{\partial r_2}{\partial r} = \dfrac{r + a\cos\theta}{r_2}, \dfrac{\partial r_1}{\partial \theta} = \dfrac{ra\sin\theta}{r_1}$

$$\frac{\partial r_2}{\partial \theta} = -\frac{ra\sin\theta}{r_2}.$$

Thus from (3), the velocity components are given by

$$q_r = -\frac{\partial\phi}{\partial r} = \mu\left[\frac{\cos\theta}{r_2^3} - 3\left(\frac{\partial r_2}{\partial r}\right)\frac{1}{r_2^4}(r\cos\theta + a) - \frac{\cos\theta}{r_1^3} + 3\left(\frac{\partial r_1}{\partial r}\right)\frac{1}{r_1^4}(r\cos\theta - a)\right]$$

$$= \mu\left[\frac{\cos\theta}{r_2^3} - 3\frac{(r + a\cos\theta)(r\cos\theta + a)}{r_2^5} - \frac{\cos\theta}{r_1^3} + \frac{3(r - a\cos\theta)(r\cos\theta - a)}{r_1^5}\right]$$

$$q_\theta = -\frac{1}{r}\frac{\partial\phi}{\partial\theta} = \frac{\mu}{r}\left[\frac{r\sin\theta}{r_1^3} + 3\frac{(r\cos\theta - a)\left(\dfrac{\partial r_1}{\partial\theta}\right)}{r_1^4} - \frac{r\sin\theta}{r_2^3} - 3\frac{(r\cos\theta + a)\left(\dfrac{\partial r_2}{\partial\theta}\right)}{r_2^4}\right]$$

$$= \frac{\mu}{r}\left[\frac{r\sin\theta}{r_1^3} + \frac{3ra\sin\theta(r\cos\theta - a)}{r_1^5} - \frac{r\sin\theta}{r_2^3} + 3\frac{ra\sin\theta(r\cos\theta + a)}{r_2^5}\right]$$

$q_\psi = 0$

When the point P lies on the plane YY′ or $\theta = \pi/2$, we have $r_1^2 = r_2^2 = r^2 + a^2$ and so at $(r, \pi/2, \psi)$, the velocity components are

$$q_r = -6\mu\, ra/(r^2 + a^2)^{5/2},\ q_\theta = 0,\ q_\psi = 0\ .$$

Along the streamline through this point, Bernoulli's equation is

$$\frac{p}{\rho} + \frac{1}{2}\bar{q}^2 = \text{const} = \frac{p_\infty}{\rho},$$

where $\bar{q} = \bar{0}$ at infinity.

Thus, the pressure at any point on the plane YY′ is given by

$$p = p_\infty - \frac{1}{2}\rho\left[36\mu^2 a^2 r^2 / (r^2 + a^2)^5\right]$$

i.e. $$p(r) = p_\infty - \frac{18\rho\mu^2 a^2 r^2}{(r^2 + a^2)^5}$$

Now,

$$p'(r) = \frac{dp}{dr} = 36\rho\mu^2 a^2 r(4r^2 - a^2)/(r^2 + a^2)^6$$

which gives $p'(r) = 0$ when $r = \frac{1}{2}a$

Also

$$p'\left(\frac{a}{2} -\right) < 0,\ p'\left(\frac{a}{2} +\right) > 0$$

i.e. $p'(r)$ changes sign from negative to positive when r passes through $\frac{a}{2}$

$\Rightarrow$ p is minimum at $r = \frac{a}{2}$ $\theta = \pi/2$

i.e. at the point $P_0\left(\frac{a}{2}, \pi/2, \psi\right)$

The distance P_0A is given by

$$\sqrt{\left(\frac{a}{2}\right)^2 + a^2} = \frac{\sqrt{5}}{2}a$$

Hence p is least at a distance $\frac{\sqrt{5}}{2}a$ from the doublet and the minimum value is

$$p_{min.} = p_\infty - \frac{9}{2}\rho\mu^2\left(\frac{4}{5}\right)^5\frac{1}{a^6}$$

5.3. Images in Impermeable Spherical Surfaces. We have already studied the effect of placing a solid impermeable sphere in a uniform stream of incompressible fluid, taking the case of axial symmetry. Here, we discuss the disturbance produced when a sphere is placed in more general flow.

We shall make use of Weiss's Sphere Theorem which states as follows :

"Let $\phi(r, \theta, \psi)$ be the velocity potential at a point P having spherical polar co-ordinates (r, θ, ψ) in an incompressible fluid having irrotational motion and no rigid boundaries. Also suppose that ϕ has no singularities within the region $r \leq a$. Then if a solid impermeable sphere of radius a is introduced into the flow with its centre at the origin of co-ordinates, the new velocity potential at P in the fluid is

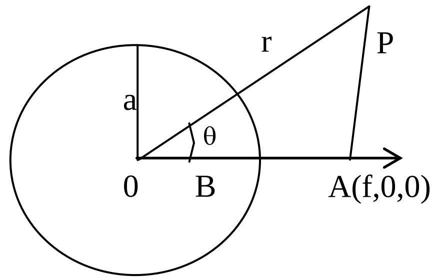
$$\phi(r, \theta, \psi) + \frac{a}{r}\phi\left(\frac{a^2}{r}, \theta, \psi\right) - \frac{1}{a}\int_0^{a^2/r} \phi(R, \theta, \psi)\, dR, \ (r > a)$$

where r and $\frac{a^2}{r}$ are the inverse points w.r.t the sphere of radius a."

Here, the last two terms refer to perturbation potential due to the presence of the sphere.

(i) **Image of a Source in a Sphere :** Suppose a source of strength m is situated at point A at a distance f(> a) from the centre of the sphere of radius a.

Let B be the inverse point of A w.r.t. the sphere, then OB = a^2/f



The velocity potential at P(r, θ, ψ) in the fluid due to a simple source of strength m at A(f, 0, 0) is

$$\phi(r, \theta) = \frac{m}{AP}$$

From Δ OAP, $\cos\theta = \dfrac{(OP)^2 + (OA)^2 - (AP)^2}{2(OP)(OA)} = \dfrac{r^2 + f^2 - (AP)^2}{2rf}$

$\Rightarrow$ $AP = \sqrt{r^2 + f^2 - 2rf\cos\theta}$

Thus, the velocity potential is

$$\phi(r, \theta) = m(r^2 + f^2 - 2rf\cos\theta)^{-1/2} \qquad (1)$$

Introducing a solid sphere in the region $r \le a$, where $a < f$, we obtain on using Weiss's sphere theorem, a perturbation potential

$$\frac{a}{r}\phi\left(\frac{a^2}{r}, \theta\right) - \frac{1}{a}\int_0^{a^2/r} \phi(R, \theta)\, dR$$

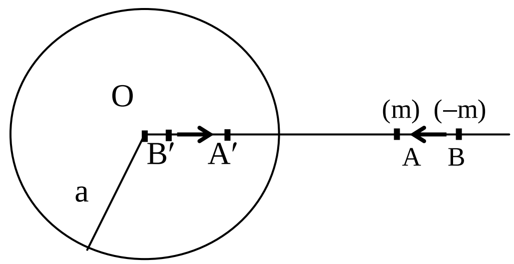
i.e. $$\frac{am}{r}\left[\frac{a^4}{r^2} + f^2 - 2\frac{a^2}{r} f\cos\theta\right]^{-1/2} - \frac{m}{a}\int_0^{a^2/r} [R^2 + f^2 - 2Rf\cos\theta]^{-1/2}\, dR$$

i.e. $$\frac{(ma/f)}{\sqrt{r^2 - 2r(a^2/f)\cos\theta + (a^2/f)^2}} - \frac{m}{a}\int_0^{a^2/r} \frac{dR}{\sqrt{R^2 - 2Rf\cos\theta + f^2}}$$

This shows that the image system of a point source of strength m placed at distance f(> a) from the centre of solid sphere consists of a source of strength

$\frac{ma}{f}$ at the inverse point $\frac{a^2}{f}$ in the sphere, together with a continuous line distribution of sinks of uniform strength $\frac{m}{a}$ per unit length extending from the centre to the inverse point.

(ii) Image of a doublet in a sphere when the axis of the doublet passes through the centre of the sphere :- Let us consider a doublet AB with its axis $\overline{BA}$ pointing towards the centre 0 of a sphere of radius a. Let OA = f, OB = f + δf. Let A′, B′ be the inverse points of A & B in the sphere so that



$$OA' = a^2/f,\ OB' = a^2/(f+\delta f).$$

At A, B we associate simple sources of strengths m and –m so that the strength of the doublet is μ = mδf, where μ is to remain a finite non-zero constant as $m\to\infty$ and $\delta f\to 0$ simultaneously.

$$B'A' = OA' - OB' = \frac{a^2}{f} - \frac{a^2}{f+\delta f} = \frac{a^2}{f} - \frac{a^2}{f}\left(1+\frac{\delta f}{f}\right)^{-1}$$

$$= \frac{a^2}{f} - \frac{a^2}{f} + \frac{a^2}{f}\frac{\delta f}{f} \text{ to the first order}$$

$$= \frac{a^2}{f^2}\delta f \text{ to the first order}$$

Now, from the case of "Image of source in a sphere", the image of m at A consists of $\frac{ma}{f}$ at A′ together with a continuous line distribution from O to A′ of sinks of strength $\frac{m}{a}$ per unit length and the image of –m at B consists of $\frac{-ma}{(f+\delta f)}$ at B′ together with a continuous line distribution from O to B′ of sources of strength $\frac{m}{a}$ per unit length.

The line distribution of sinks and sources from 0 to B′ cancel each other leaving behind a line distribution of sinks of strength $\frac{m}{a}$ per unit length from B′ to A′ i.e. sink of strength $\frac{m}{a}$ B′A′ = $\frac{m}{a}\left(\frac{a^2}{f^2}\delta f\right) = \frac{a}{f^2}(m\delta f) = \frac{\mu a}{f^2}$ at B′. The source at B′ is of strength

$$\frac{-ma}{f+\delta f} = \frac{-ma}{f}\left(1+\frac{\delta f}{f}\right)^{-1} = -\frac{ma}{f}\left(1-\frac{\delta f}{f}\right) \quad , \qquad \text{to the first}$$

order terms

$$= \frac{-ma}{f} + \frac{ma}{f^2}\delta f = \frac{-ma}{f} + \frac{\mu a}{f^2}$$

which is equivalent to a sink $\frac{ma}{f}$ at B' and a source $\frac{\mu a}{f^2}$ at B′.

As there is already a sink $\frac{\mu a}{f^2}$ at B′, therefore source and sink at B′ neutralize.

Finally, we are left with source $\frac{ma}{f}$ at A′ and a sink. $\frac{ma}{f}$ at B′. Thus, to the first order, we obtain a doublet at A′ of strength

$$\frac{ma}{f}\,(B'A') = \frac{ma}{f}\,\frac{a^2}{f^2}\delta f$$

$$= \frac{ma^3}{f^3}\delta f = \frac{\mu a^3}{f^3}.$$

Hence in the limiting case as $\delta f \to 0$, $m \to \infty$, we obtain a doublet at A of strength μ with its axis towards O, together with a doublet at the inverse point A′ of strength $\frac{\mu a^3}{f^3}$ with its axis away from O.

6. Stream Function for an Axi-Symmetric Flow (Stoke's Stream Function)

If the streamlines in all the planes passing through a given axis are the same, the fluid motion is said to be axi-symmetric. We have already considered such flow for irrotational motion in spherical polar co-ordinates. (r, θ, ψ) in which the line $\theta = 0$ is the axis of symmetry.

Suppose the z-axis be taken as axis of symmetry, then $q_\theta = 0$ and the fluid motion is the same in every plane θ = constant (meridian plane) and suppose that a point P in the fluid may be specified by cylindrical polar co-ordinates (r, θ, z). Thus, all the quantities associated with the flow are independent of θ. The equation of continuity in cylindrical co-ordinates, becomes

$$\frac{\partial}{\partial r}(rq_r)+\frac{\partial}{\partial z}(rq_z)=0$$

i.e. $$\frac{\partial}{\partial r}(rq_r)=-\frac{\partial}{\partial z}(rq_z) \qquad (1)$$

This is the condition of exactness of the differential equation

$$rq_r dz - r\, q_z\, dr = 0 \qquad (2)$$

This means that (2) is an exact differential equation and let L.H.S. be an exact differential $d\Psi$(say)

Therefore,

$$rq_r\, dz - rq_z\, dr = d\Psi = \frac{\partial \Psi}{\partial r}dr+\frac{\partial \Psi}{\partial z}dz$$

which gives

$$\frac{\partial \Psi}{\partial r}=-rq_z, \frac{\partial \Psi}{\partial z}=rq_r \qquad (3)$$

The function Ψ in (3) is called **Stoke's stream function.**

The equation of streamlines in the meridian plane θ = constant at a fixed time t is

$$\frac{dr}{q_r}=\frac{dz}{q_z}$$

$\Rightarrow$ $$q_z\, dr = q_r\, dz$$

Using (3), we get

$$-\frac{1}{r}\frac{\partial \Psi}{\partial r}dr=\frac{1}{r}\frac{\partial \Psi}{\partial z}dz$$

$\Rightarrow$ $$\frac{\partial \Psi}{\partial r}dr+\frac{\partial \Psi}{\partial z}dz=0$$

$\Rightarrow$ $$d\Psi = 0$$

$\Rightarrow$ Ψ = constant = C

which represent the streamlines.

6.1. Stoke's Stream Function in Spherical Polar Co-ordinates (r, θ ψ) : We consider the axi-symmetric motion in r, θ plane such that q_ψ = 0. The equation of continuity in spherical polar co-ordinates becomes

$$\frac{1}{r^2}\frac{\partial}{\partial r}(r^2 q_r) + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial \theta}(r \sin\theta q_\theta) = 0$$

i.e. $$\frac{\partial}{\partial r}(r^2 \sin\theta q_r) = \frac{\partial}{\partial \theta}(-r \sin\theta q_\theta) \qquad (1)$$

This is condition of exactness for the different equation

$$r \sin\theta \, q_\theta \, dr - r^2 \sin\theta \, q_r \, d\theta = 0 \qquad (2)$$

Thus the expression on L.H.S. of (2) is equal to an exact differential function Ψ such that

$$r \sin\theta \, q_\theta \, d_r - q_r \, r^2 \sin\theta \, d\theta = d\Psi = \frac{\partial \Psi}{\partial r} dr + \frac{\partial \Psi}{\partial \theta} d\theta$$

$$\Rightarrow \quad \frac{\partial \Psi}{\partial r} = q_\theta r \sin\theta, \frac{\partial \Psi}{\partial \theta} = -q_r \, r^2 \sin\theta.$$

6.2. Remark. In the above cases, the motion need not be irrotational i.e. velocity potential may not exist. In case of irrotational motion, it can easily be shown that the velocity potential ϕ and the Stoke's stream function Ψ do not satisfy C–R equations due to the fact that Ψ is not harmonic.

6.3. Stoke's Stream Function for a Uniform Stream : Let a uniform stream with velocity U be in the direction of z-axis such that $\bar{q} = U\hat{k}$. Then, from the relations

$$q_3 = -\frac{1}{r}\frac{\partial \Psi}{\partial r}, q_r = \frac{1}{r}\frac{\partial \Psi}{\partial z},$$

we get $$U = -\frac{1}{r}\frac{\partial \Psi}{\partial r}, 0 = \frac{1}{r}\frac{\partial \Psi}{\partial z}$$

$$\Rightarrow \quad \frac{\partial \Psi}{\partial r} = -Ur, \frac{\partial \Psi}{\partial z} = 0$$

$\Rightarrow$ $\Psi = -U\,\dfrac{r^2}{2}$, where the constant of integration is found to be zero.

In spherical polar co-ordinates we have

$$\Psi = -\frac{U}{2}(r\sin\theta)^2 = -\frac{U}{2}r^2\sin^2\theta.$$

6.4. Stoke's Stream Function for a Simple Source at Origin : In case of simple source

$$\bar{q} = f(r)\,\hat{r}$$

But we have already calculated that for a source of strength m at origin.

$$\bar{q} = \frac{m}{r^2}\hat{r}(r > 0) \text{ in spherical polar co-ordinates.}$$

i.e. $$(q_r, q_\theta) = \frac{m}{r^2}\hat{r} \qquad (1)$$

Also, we know that in spherical polar co-ordinates,

$$q_r = -\frac{1}{r^2\sin\theta}\frac{\partial\Psi}{\partial\theta},\; q_\theta = \frac{1}{r\sin\theta}\frac{\partial\Psi}{\partial r} \qquad (2)$$

From (1) & (2), we get

$$\frac{m}{r^2} = -\frac{1}{r^2\sin\theta}\frac{\partial\Psi}{\partial\theta}, \frac{\partial\Psi}{\partial r} = 0$$

$$\Rightarrow \qquad \frac{\partial\Psi}{\partial\theta} = -m\sin\theta, \frac{\partial\Psi}{\partial r} = 0$$

$$\Rightarrow \qquad \Psi = m\cos\theta\,.$$

A constant may be added to this solution and this is usually done to make Ψ = 0 along the axis of symmetry $\theta = 0$. In such case,

$$\Psi = m(\cos\theta - 1)$$

For a sink of strength m at origin, the Stoke's stream function is

$$\Psi = m(1 - \cos\theta)$$

6.5. Stoke's Stream Function for a Doublet at Origin : We assume that the flow is due to only a doublet at origin 0 of strength μ. Taking the axis $\theta = 0$ of the system of spherical co-ordinates to coincide with the axis of the doublet, we find that the velocity potential at P(r, θ, ψ) is

$$\phi = \frac{\mu\cos\theta}{r^2} \quad (r > 0) \tag{1}$$

$$\Rightarrow \qquad q_r = -\frac{\partial\phi}{\partial r} = \frac{2\mu\cos\theta}{r^3}, q_\theta = -\frac{1}{r}\frac{\partial\phi}{\partial\theta} = \frac{\mu\sin\theta}{r^3} \quad q_\psi = 0 \tag{2}$$

But the relations between the velocity components and the Stoke's stream function Ψ are

$$q_r = -\frac{1}{r^2\sin\theta}\frac{\partial\Psi}{\partial\theta},\ q_\theta = \frac{1}{r\sin\theta}\frac{\partial\Psi}{\partial r} \tag{3}$$

From (2) and (3), we get

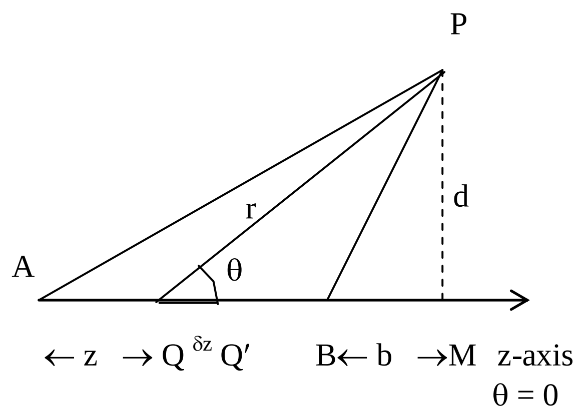
$$\frac{\partial\Psi}{\partial\theta} = -\frac{2\mu\sin\theta\cos\theta}{r}, \frac{\partial\Psi}{\partial r} = \frac{\mu\sin^2\theta}{r^2}$$

Integrating, we get

$$\Psi = \frac{-\mu\sin^2\theta}{r}$$

6.6. Stoke's Stream Function due to a Uniform Line Source : Let a uniform line source of fluid extends along the streamline segment AB of length l. Consider an element QQ′ of length δz at a distance z (= AQ) from A. Thus we have a simple source of strength m δz, where m is the constant source strength per unit length of the distribution along AB.

Let QP = r, $\angle$PQB = Q, PM = d



The Stoke's stream function $\delta\Psi$ at P for the simple source of strength $m\delta z$ at Q is $m\delta z(\cos\theta - 1)$. Then, the value of the Stoke's stream function Ψ at P due to entire line source AB is given by

$$\Psi = m\int_0^l(\cos\theta - 1)dz = m\int_0^l\cos\theta dz - m\int_0^l dz$$

$$= m\int_0^{l} \frac{l+b-z}{\sqrt{d^2+(l+b-z)^2}}\,dz - ml$$

In ΔPQM, $\cos\theta = \dfrac{QM}{PQ} = \dfrac{QB+BM}{PQ}$

$= \dfrac{l-z+b}{r} = \dfrac{l-z+b}{\sqrt{d^2+(l+b-z)^2}}$

Putting $l + b - z = x \Rightarrow dz = -dx$

When $z = 0$, $x = l + b$,

when $z = l$, $x = b$

Therefore,

$$\Psi = m \int_{l+b}^{b} \frac{x(-dx)}{\sqrt{d^2+x^2}} - ml$$

or $$\Psi = \frac{m}{2}\int_{b}^{l+b} (d^2 + x^2)^{-1/2}\,(2x)\,dx - ml$$

$$= \frac{m}{2}\left[\frac{\sqrt{d^2+x^2}}{1/2}\right]_{b}^{l+b} - ml$$

$$= m\left[\sqrt{d^2+(l+b)^2} - \sqrt{d^2+b^2}\right] - ml$$

$$= m[AP - BP] - mAB$$

$$= m[AP - BP - AB]\ .$$

As p is the only variable point, the simpler form m (AP–BP) can be taken for evaluating velocity components at P. The stream surfaces are

$$\Psi = \text{constant i.e. } AP - BP = \text{constant.}$$

These are confocal hyperboloids of revolution about AB, with A and B as foci.

We have shown earlier that the equipotentials were confocal ellipsoids of revolution about AB with the same foci. Also it is well known result that two families of confocals intersect orthogonally.

6.7. Stoke's Stream Function for a Doublet in a Uniform Stream : Let a doublet of vector moment $\mu\hat{k}$ is situated at origin 0 in a uniform stream whose undisturbed velocity is $-U\hat{k}$.

In spherical polar co-ordinates (r, θ, ψ), the Stoke's stream functions for each separate distribution are

$$\Psi_1 = \frac{1}{2}Ur^2\sin^2\theta \qquad \text{(for uniform stream, } \bar{q} = -U\hat{k}\text{)}$$

$$\Psi_2 = -\frac{\mu}{r}\sin^2\theta \qquad \text{(for doublet at origin)}$$

Hence the stream function for the combination is

$$\Psi\,(r, \theta) = \left(\frac{1}{2}Ur^2 - \mu/r\right)\sin^2\theta$$

The equation of the stream surfaces are $\Psi(r, \theta)$ = constant.

In particular, the stream surfaces for which $\Psi = 0$ are given by

$$\left(\frac{1}{2}Ur^2 - \mu/r\right)\sin^2\theta = 0$$

$\Rightarrow$ $\qquad \sin\theta = 0$ or $\frac{1}{2}Ur^2 - \frac{\mu}{r} = 0$

$\Rightarrow$ $\qquad \theta = 0, \pi$ i.e. the z-axis or $r = \left(\frac{2\mu}{U}\right)^{1/3}$, the surface of the sphere with centre 0 and radius $\left(\frac{2\mu}{U}\right)^{1/3}$.

7. Irrotational Motion in Two-dimensions

Suppose that a fluid moves in such a way that at any given instant, the flow pattern in a certain plane within the fluid is the same as that in all other parallel planes within the fluid. Then at the considered instant, the flow is said to be two-dimensional flow or plane flow. Any one of the parallel planes is then termed as flame of flow.

If we take the plane of flow as the plane z = 0, then at any point in the fluid having cartesian co-ordinates (x, y, z), all physical quantities i.e. velocity, density, pressure etc, associated with the fluid are independent of z.

Thus $\bar{q} = \bar{q}\,(x, y, t)$ $\rho = \rho(x, y, t)$ etc

Plane flows, as described above, cannot be achieved in reality, but in certain important cases, close approximation to planarity of flow may occur.

We have already considered such flow when defining Lagrange's stream function. We consider here some special methods for treating two-dimensional irrotational motion.

7.1. Use of Cylindrical Polar Co-ordinates. For an incompressible irrotational flow of uniform density, the equation of continuity $\nabla^2\phi = 0$ for the velocity potential $\phi(r, \theta, z)$ in cylindrical polar co-ordinates (r, θ, z) is

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\phi}{\partial r}\right)+\frac{1}{r^2}\frac{\partial^2\phi}{\partial\theta^2}+\frac{\partial^2\phi}{\partial z^2}=0 \qquad (1)$$

If the flow is two dimensional and the co-ordinate axes are so chosen that all physical quantities associated with the fluid are independent of z, then $\phi = \phi(r, \theta)$ and (1) simplifies to

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\phi}{\partial r}\right)+\frac{1}{r^2}\frac{\partial^2\phi}{\partial\theta^2}=0 \qquad (2)$$

Let us seek solutions of (2) by putting

$$\phi(r, \theta) = -f(r)\, g(\theta) \qquad (3)$$

in (2) for separation of variables. Thus, we get

$$g(\theta)\ \frac{1}{r}\frac{d}{dr}\left[rf'(r)\right]+\frac{1}{r^2}f(r)g''(\theta) = 0$$

i.e. $$\frac{r\frac{d}{dr}\left[rf'(r)\right]}{f(r)} = -\frac{g''(\theta)}{g(\theta)} \qquad (4)$$

Thus, L.H.S. of (4) is a function of r only and R.H.S. is a function of θ only. As r and θ are independent variables, so each side of (4) is a constant λ(say). Thus, we have

$$\frac{r^2 f''(r) + rf'(r)}{f(r)} = \lambda; \quad \frac{g''(\theta)}{g(\theta)} = -\lambda$$

i.e. $$r^2 f''(r) + r f'(r) - \lambda f(r) = 0 \qquad (5)$$

and $$g''(\theta) + \lambda g(\theta) = 0 \qquad (6)$$

Equation (6) has periodic solutions when $\lambda > 0$. Normally the physical problem requires that $g(\theta + 2\pi) = g(\theta)$ and this is satisfied when $\lambda = n^2$ for $n = 1, 2, 3, \ldots$.

Thus, the basic solution of (6) are

$$g(\theta) = c_1 \cos n\theta + c_2 \sin n\theta \qquad (7)$$

Now, (5) is of Euler-homogeneous type and it is reduced to a linear different equation of constant co-efficients by putting $r = e^t$ i.e. $t = \log r \Rightarrow \frac{dt}{dr} = \frac{1}{r}$

Also, $$f'(r) = \frac{df}{dr} = \frac{df}{dt}.\frac{dt}{dr} = \frac{1}{r}\frac{df}{dt}$$

and $$f''(r) = \frac{d^2f}{dr^2} = \frac{d}{dr}\left(\frac{df}{dr}\right) = \frac{d}{dr}\left(\frac{1}{r}\frac{df}{dt}\right)$$

$$= \frac{1}{r}\frac{d}{dr}\left(\frac{df}{dt}\right) + \frac{df}{dt}\left(-\frac{1}{r^2}\right)$$

$$= \frac{1}{r}\left[\frac{d}{dt}\left(\frac{df}{dt}\right)\frac{dt}{dr}\right] - \frac{1}{r^2}\frac{df}{dt}$$

$$= \frac{1}{r^2}\frac{d^2f}{dt^2} - \frac{1}{r^2}\frac{df}{dt}$$

$$\Rightarrow \qquad r^2 f''(r) = \frac{d^2 f}{dt^2} - \frac{df}{dt}$$

Therefore, equation (5) reduces to

$$\frac{d^2 f}{dt^2} - \frac{df}{dt} + \frac{df}{dt} - n^2 f = 0$$

$$\Rightarrow \qquad \frac{d^2 f}{dt^2} - n^2 f = 0$$

It's solution is

$$f = \exp(\pm nt) = e^{\pm nt} = \left(e^t\right)^{\pm n} = r^{\pm n}$$

i.e. $$f = c_3 r^n + c_4 r^{-n} \qquad (8)$$

A special solution of (2) is obtained by linear superposition of the forms (7) & (8) to give

$$\phi(r, \theta) = -f(r)\, g(\theta)$$

$$= -(A_n r^n + B_n r^{-n})(C_n \cos n\theta + D_n \sin n\theta) \qquad (9)$$

The most general solution is of the form

$$\phi(r, \theta) = -\sum_{n=1}^{\infty} (A_n r^n + B_n r^{-n})(C_n \cos n\theta + D_n \sin n\theta) \qquad (10)$$

7.2. Particular cases. (i) for n = 0, we have

$$f = k_1 + k_2 t = k_1 + k_2 \log r$$

and $$g = k_3 + k_4 \theta$$

so that another solution of (2) is

$$\phi(r, \theta) = -(k_1 + k_2 \log r)(k_3 + k_4 \theta)$$

(ii) for n = 1, we get a special solution as

$$\phi = -r \cos\theta,\ \phi = -r \sin\theta,\ \phi = -r^{-1} \cos\theta,\ \phi = -r^{-1} \sin\theta$$

7.3. Example. Discuss the uniform flow past an infinitely long circular cylinder.

Solution. Let P be a point with cylindrical polar co-ordinates (r, θ, z) in the flow region of an unbounded incompressible fluid of uniform density moving irrotationally with uniform velocity $-U\hat{i}$ at infinity past the fixed solid cylinder $r \le a$

When the cylinder r = a is introduced, it will produce a perturbation which is such as to satisfy Laplace equation and to become vanishingly small for large r. This suggests taking the velocity potential for $r > a$, $0 \le \theta \le 2\pi$ in the form

$$\phi(r, \theta) = Ur\cos\theta - Ar^{-1}\cos\theta\,, \qquad (1)$$

where the velocity potential of the uniform stream is $Ux = Ur\cos\theta$ and due to perturbation, it is $-Ar^{-1}\cos\theta$ which $\to 0$ as $r\to\infty$ and gives rise to a velocity pattern which is symmetrical about θ = 0, π. (the term $r^{-1}\sin\theta$ is not there since it does not give symmetric flow)

As there is no flow across r = a, so the boundary condition on the surface is

$$\frac{\partial\phi}{\partial r} = 0, \text{ when } r = a \qquad (2)$$

Applying (2) in (1), we get $A = -Ua^2$ for all θ satisfying $\theta \le \theta \le 2\pi$.

Thus, the velocity potential for a uniform flow past a fixed infinite cylinder is

$$\phi(r, \theta) = U\cos\theta\left(r + \frac{a^2}{r}\right), \; r > a, \; 0 \le \theta \le 2\pi \qquad (3)$$

From here, the cylindrical components of velocity are ($\bar{q} = -\nabla\phi$)

$$q_r = -\frac{\partial\phi}{\partial r} = -U\cos\theta\left(1 - \frac{a^2}{r^2}\right)$$

$$q_\theta = -\frac{1}{r}\frac{\partial\phi}{\partial\theta} = \frac{1}{r}U\sin\theta\left(r + \frac{a^2}{r}\right) = U\sin\theta\left(1 + a^2/r^2\right)$$

$$q_z = -\frac{\partial\phi}{\partial z} = 0$$

We note that as $r\to\infty$, $q_r = -U\cos\theta$, $q_\theta = U\sin\theta$ which are consistent with the velocity at infinity $-U\hat{i}$ of the uniform stream.

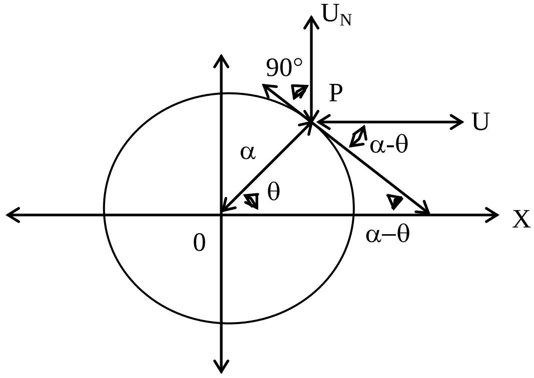
7.4. Example. A cylinder of infinite length and nearly circular section moves through an infinite volume of liquid with velocity U at right-angles to its axis and in the direction of positive x-axis. If the section is specified by the equation.

$$r = a(1 + \in \cos n\theta)$$

where n is positive integer and $\in$ is small, show that the approximate value of the velocity potential of the fluid is

$$Ua\left[\frac{a}{r}\cos\theta + \in\left(\frac{a}{r}\right)^{n+1}\cos(n+1)\theta - \in\left(\frac{a}{r}\right)^{n-1}\cos(n-1)\theta\right]$$

Solution. Let the tangent at a point P on the plane of cylinder makes angles α, $(\pi-\alpha)$ with the radial line OP drawn from 0 as shown in the figure



At large radial distances r from OZ, the fluid velocity becomes vanishingly small.

Let us assume the velocity potential $\phi(r, \theta)$ of the form $-r^{-k} \begin{smallmatrix}\cos\\ \sin\end{smallmatrix} k\theta$ (k = 1, 2,…..).

Thus, we seek a solution of the form

$$\phi(r, \theta) = -\sum_{k=1}^{\infty} r^{-k}(A_k \cos k\theta + B_k \sin k\theta) \qquad (1)$$

(If we take k = 0, this would add on to ϕ an arbitrary constant A_0).

At $\theta = 0$ and $\theta = \pi$ on the boundary, $q_\theta = 0$ which is satisfied by taking $B_k = 0$ (k = 1, 2,…..)

Thus, the velocity potential simplifies to the form

$$\phi(r, \theta) = -\sum_{k=1}^{\infty} A_k r^{-k} \cos k\theta \qquad (2)$$

which approximately remains unaltered on replacing θ by $2\pi-\theta$.

At any point (r, θ, z) of the fluid, the cylindrical polar velocity components are $(q = -\nabla\phi)$

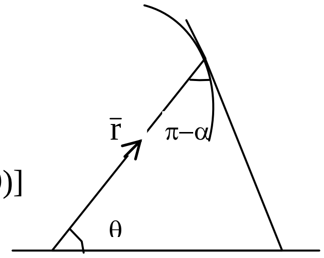
$$q_r = -\frac{\partial\phi}{\partial r} = -\sum_{k=1}^{\infty} k.\, A_k.\, r^{-(k+1)} \cos k\theta$$

$$q_\theta = -\frac{1}{r}\frac{\partial\phi}{\partial\theta} = -\sum_{k=1}^{\infty} k.\, A_k.\, r^{-(k+1)} \sin k\theta$$

$$q_z = -\frac{\partial\phi}{\partial z} = 0$$

At P on the boundary, since $(\pi - \alpha)$ is the angle between the tangent and the radius vector OP, therefore

$$\cot(\pi-\alpha) = \frac{1}{r}\frac{dr}{d\theta} = \frac{d}{d\theta}(\log r)$$



$$\Rightarrow \qquad -\cot\alpha = \frac{d}{d\theta}[\log a(1+\in\cos n\theta)]$$

$$= \frac{1}{a(1+\in\cos n\theta)}(-a\in n \sin n\theta)$$

$$\Rightarrow \qquad \cot\alpha = \frac{\in n \sin n\theta}{1+\in\cos n\theta} \qquad \sin(\pi-\alpha) = r\frac{d\theta}{ds} \quad (3)$$

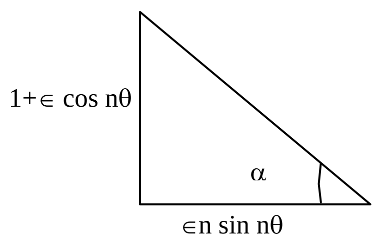
The normal component of velocity U_N of the boundary at P is $\cos(\pi-\alpha) = \dfrac{dr}{ds}$

$$U_N = U\sin(\alpha-\theta)$$

$$= U(\sin\alpha\cos\theta - \cos\alpha\sin\theta)$$

i.e. $$U_N = \frac{U[\cos\theta(1+\in\cos n\theta) - \sin\theta \in n \sin n\theta]}{\sqrt{(1+\in\cos n\theta)^2 + \in^2 n^2 \sin^2 n\theta}} \qquad (4)$$

As there is $n\theta$ transport of fluid across the surface and $n\theta$ breakaway from it, so U_N is also the normal velocity component of the fluid.



Thus,

$U_N = q_r \sin\alpha + q_\theta \cos\alpha.$

$$= \frac{\left[-\sum_{k=1}^{\infty} k A_k r^{-(k+1)} \cos k\theta\right](1+\in \cos n\theta)}{\sqrt{(1+\in \cos n\theta)^2 + (\in n \sin n\theta)^2}} + \frac{\left[-\sum_{k=1}^{\infty} k A_k r^{-(k+1)} \sin k\theta\right](\in n \sin n\theta)}{\sqrt{(1+\in \cos n\theta)^2 + (\in n \sin n\theta)^2}}$$

$$= \frac{-\sum_{k=1}^{\infty} k A_k a^{-(k+1)} (1+\in \cos n\theta)^{-(k+1)} [\cos k\theta (1+\in \cos n\theta) + \sin k\theta \in n \sin n\theta]}{\sqrt{(1+\in \cos n\theta)^2 + (\in n \sin n\theta)^2}} \tag{5}$$

Equating the two forms for U_n, we get

$$-\sum_{k=1}^{\infty} k A_k a^{-(k+1)} (1+\in \cos n\theta)^{-(k+1)} [\cos k\theta (1+\in \cos n\theta) + \in n \sin k\theta \sin n\theta]$$

$$= U[\cos\theta(1+\in \cos n\theta) - \in n \sin\theta \sin n\theta] \tag{6}$$

We further simplify (6) for the terms upto 1st order in $\in$.

L.H.S. of (6)

$$= -\sum_{k=1}^{\infty} k A_k a^{-(k+1)} [1 - \in (k+1)\cos n\theta][\cos k\theta + \in \cos k\theta \cos n\theta + \in n \sin k\theta \sin n\theta]$$

$$= -\sum_{k=1}^{\infty} k A_k a^{-(k+1)} [\cos k\theta - \in (k+1) \cos k\theta \cos n\theta$$

$$+ \in \cos k\theta \cos n\theta + \in n \sin k\theta \sin n\theta]$$

$$= -\sum_{k=1}^{\infty} k A_k a^{-(k+1)} [\cos k\theta - \in k \cos k\theta \cos n\theta + \in n \sin k\theta \sin n\theta]$$

$$= -\sum_{k=1}^{\infty} k A_k a^{-(k+1)} \left[\cos k\theta - \frac{\in k}{2}\{\cos(n+k)\theta + \cos(n-k)\theta\}\right.$$

$$\left. + \frac{\in n}{2}\{\cos(n-k)\theta - \cos(n+k)\theta\}\right]$$

$$= -\sum_{k=1}^{\infty} k A_k a^{-(k+1)} \left[\cos k\theta - \frac{\in}{2}(n+k)\cos(n+k)\theta + \frac{\in}{2}(n-k)\cos(n-k)\theta\right] \tag{7}$$

R.H.S. of (6)

$$= U\left[\cos\theta + \frac{\in}{2}\{\cos(n+1)\theta + \cos(n-1)\theta\} - \frac{\in n}{2}\{\cos(n-1)\theta - \cos(n+1)\theta\}\right]$$

$$= U\left[\cos\theta + \frac{\in}{2}\{(1+n)\cos(n+1)\theta + (1-n)\cos(n-1)\theta\}\right] \qquad (8)$$

Correct to the first order of approximation, from (6), (7) & (8), comparing coefficients of cos θ, cos (n–1)θ, cos(n+1)θ, we get

$$U = \frac{-A_1}{a^2} \quad \Rightarrow \quad A_1 = -Ua^2 \qquad (9) \qquad | \; n + k \neq 1$$

$$-\left[(n-1)A_{n-1}a^{-n} + \frac{1}{2}A_1 a^{-2} \in (n-1)\right] = \frac{1}{2}U \in (n-1)$$

| In (7) cos kθ → cos(n–1)θ
cos (n–k)θ → cos (n–1)θ
similarly for n+1

and $$-\left[(n+1)A_{n+1}a^{-(n+2)} - \frac{1}{2}A_1 a^{-2} \in (n+1)\right] = -\frac{1}{2}U \in (n+1)$$

$$\Rightarrow \qquad A_{n-1} = U\in a^n, \quad A_{n+1} = -U\in a^{n+2}$$

All A_k other than A_1, A_{n-1}, A_{n+1} are zero. Putting the value of these three non-zero co-efficients in (2), we get

$$\phi(r, \theta) = -[A_1 r^{-1}\cos\theta + A_{n-1}\, r^{-(n-1)}\cos(n-1)\,\theta + A_{n+1}\, r^{-(n+1)}\cos(n+1)\theta]$$

$$= Ua\left[\frac{a}{r}\cos\theta + \in\left(\frac{a}{r}\right)^{n+1}\cos(n+1)\theta - \in\left(\frac{a}{r}\right)^{n-1}\cos(n-1)\theta\right].$$

Hence the result.

8. The Complex Potential

Here, we confine our attention to irrotational plane flows of incompressible fluid of uniform density for which the velocity potential $\phi(x, y)$ and the stream function $\psi(x, y)$ exist. Here (x, y) specify two dimensional Cartesian co-ordinates in a plane of flow. Let us write

$$W = \phi(x, y) + i\psi(x, y) \qquad (1)$$

We suppose that all four first-order partial derivatives of ϕ & ψ with respect to x, y exist and are continuous throughout the plane of flow. Now, the velocity $\bar{q} = (u, v)$ has components satisfying $\bar{q} = -\nabla\phi$.

$$u = -\frac{\partial\phi}{\partial x} = -\frac{\partial\psi}{\partial y},\ v = -\frac{\partial\phi}{\partial y} = \frac{\partial\psi}{\partial x} \qquad (2)$$

Thus ϕ and ψ satisfy the C-R equations and so W must be an analytic function of $z = x + iy$

Therefore, we can write (1) as

$$W = f(z) = \phi + i\psi \qquad (3)$$

The function $W = f(z)$ is called the **complex potential** of the plane flow.

8.1. Complex Velocity. We have

$$W = \phi + i\psi \text{ and } z = x + iy$$

Differentiating partially w.r.t. x, we get

$$\frac{\partial W}{\partial x} = \frac{\partial\phi}{\partial x} + i\frac{\partial\psi}{\partial x} = \frac{\partial\phi}{\partial x} - i\frac{\partial\phi}{\partial y} = -u + iv$$

But $$\frac{\partial W}{\partial x} = \frac{dW}{dz}.\frac{\partial z}{\partial x} = \frac{dW}{dz} \qquad \left|\ \because \frac{\partial z}{\partial x} = 1\right.$$

Thus. $$\frac{dW}{dz} = -u + iv$$

$$\Rightarrow -\frac{dW}{dz} = u - iv = q\cos\theta - iq\sin\theta$$

$$= q(\cos\theta - i\sin\theta) = q\,e^{-i\theta}$$

The combination $u - iv$ is known as **complex velocity**

Thus, speed $q = \left|-\frac{dW}{dz}\right| = \sqrt{u^2 + v^2}$

and for stagnation points, $\frac{dW}{dz} = 0$

8.2. Example. Discuss the flow for which complex potential is

$$W = z^2$$

Solution. We have

$$W = \phi + i\psi = z^2 = (x + iy)^2 = x^2 - y^2 + 2i\,xy$$

$$\Rightarrow \phi(x, y) = x^2 - y^2, \quad \psi(x, y) = 2xy$$

The equipotentials, ϕ = constant, are the rectangular hyperbolae $x^2 - y^2$ = constant having asymptotes $y = \pm x$.

The streamlines, ψ = constant, are the rectangular hyperbolae xy = constant having the axes $x = 0$, $y = 0$ as asymptotes Also $\frac{dW}{dz} = 2z$, therefore the only stagnation point is the origin. The two families of the hyperbolae cut orthogonally in accordance with general theory.

8.3. Complex Potential for a Uniform Stream. Let the uniform stream advance with a velocity having magnitude U and being inclined at angle α to the positive direction of the x-axis.

Then, we have $u = U\cos\alpha$, $V = U\sin\alpha$ and thus

$$-\frac{dW}{dz} = u - iv = U\,e^{-i\alpha}$$

The simplest form of W, ignoring the constant of integration, is

$$W = -Uz\,e^{-i\alpha}$$

i.e. $\phi + i\psi = -U(x + iy)(\cos\alpha - i\sin\alpha)$

$$= -U(x\cos\alpha + y\sin\alpha) - U\hat{i}\,(y\cos\alpha - x\sin\alpha)$$

Equating real and imaginary parts, we get

$$\phi = -U(x\cos\alpha + y\sin\alpha)$$

$$\psi = -U(y\cos\alpha - x\sin\alpha)$$

Thus, the equations of equipotentials are

$$x\cos\alpha + y\sin\alpha = \text{constant} \qquad (1)$$

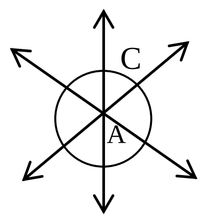
These equations represent a family of parallel streamlines. The equations of the streamlines are

$$y\cos\alpha - x\sin\alpha = \text{constant} \qquad (2)$$

These equations represent another family of parallel streamlines inclined at angle α to the positive x-direction. The two family of streamlines intersect orthogonally in accordance with general theory.

8.4. Line Source and Line Sink. Line source and line sink are the two-dimensional analogues of the three-dimensional simple source and sink. Let A be any point of the considered plane of flow and C be any closed curve surrounding A. We construct a cylinder having its generators through the points of C and normal to the plane of flow. Suppose that in each plane of flow, fluid is emitted radically and symmetrically from all points on the infinite line through A normal to the plane of flow and such that the rate of emission from all such points as A is the same. Then the line through A is called a **line source.** We may take the closed curve C to be a circle having centre A and radius r.

Suppose the line source emits fluid at the rate $2\pi m\rho$ units of mass per unit length of the source per unit time, in all directions in the plane of flow (say, xy-plane). We define the strength of the line source to be m. A line source of strength $-m$ is called a line sink.



An example of a line source is a long straight hose with perforations along its length, commonly used for watering lawns for long periods of time.

8.5. Complex Potential for a Line Source. Let there be a line source of strength m per unit length at z = 0. Since the flow is radial, the velocity has the radial component q_r only. Then the flow across a circle of radius r is (by law of conservation of mass)

$$(2\pi\, r\, q_r)\rho = 2\pi m\rho$$

$$\Rightarrow \qquad q_r = \frac{m}{r}$$

The complex potential is obtained from the relation

$$-\frac{dW}{dz} = u - iv = q_r \cos\theta - i\, q_r \sin\theta$$

$$= q_r\,(\cos\theta - i\sin\theta) = \frac{m}{r}\,e^{-i\theta}$$

$\Rightarrow \qquad \dfrac{dW}{dz} = -\dfrac{m}{r} e^{-i\theta} = \dfrac{-m}{re^{i\theta}} = \dfrac{-m}{z}$

Integrating, we get

$$W = -m \log z$$

where we have ignored the constant of integration.

We can write it as

$$\phi + i\psi = -m \log (r\, e^{i\theta})$$

$$= -m \log r - i\, m\theta$$

$\Rightarrow \qquad \phi = -\ m \log r,\ \psi = -m\theta$

Thus, the equipotentials and streamlines have the respective forms

$$r = \text{constant},\ \theta = \text{constant}$$

i.e. $\qquad x^2 + y^2 = \text{constant},\ \tan^{-1} \dfrac{y}{x} = \text{constant}$

i.e. $\qquad x^2 + y^2 = C_1,\ y = C_2 x\ .$

Thus the equipotentials are circles and streamlines are straight lines passing through origin.

If the line source is at $z = z_0$ instead of $z = 0$, then the complex potential is

$$W = -m \log (z - z_0)$$

For a line sink of strength m per unit length at $z = z_0$, the complex potential is

$$W = m \log (z - z_0).$$

If there are a number of line sources at $z = z_1, z_2, \ldots, z_n$ of respective strengths $m_1, m_2, \ldots, m_n$ per unit length, then the complex potential is

$$W = -m_1 \log(z - z_1) - m_2 \log (z - z_2) \ldots\ldots -m_n \log (z - z_n).$$

8.6. Complex Potential for a Line Doublet. The combination of a line source and a line sink of equal strength when placed close to each other gives a line doublet. Let us take a line source of strength m per unit length at $z = a\, e^{i\alpha}$ and a line sink of strength m per unit length at $z = 0$

Therefore, the complex potential due to the combination is

$$W = -m \log(z - ae^{i\alpha}) + m \log(z-0)$$

$$= -m \log \frac{z - ae^{i\alpha}}{z} = -m \log\left(1 - \frac{ae^{i\alpha}}{z}\right)$$

$$= m\left[\frac{ae^{i\alpha}}{z} + \frac{a^2 e^{2i\alpha}}{2z^2} + \frac{a^3 e^{3i\alpha}}{3z^3} \ldots\right]$$

In the figure,
OP = a = δs where a is the distance between the source and sink.
As $a \to 0$, $m \to \infty$ so that $ma \to \mu$ and thus, we get

$$W = \frac{\mu e^{i\alpha}}{z}$$

If the line sink is situated at $z = z_0$, then the complex potential is

$$W = \frac{\mu e^{i\alpha}}{z - z_0}$$

If $\alpha = 0$, then the line source is on x-axis and thus.

$$W = \frac{\mu}{z - z_0}$$

If there are number of line doublets of strengths $\mu_1, \mu_2 \ldots\ldots \mu_n$ per unit length with line sinks at points $z_1, z_2, \ldots\ldots, z_n$ and their axis being inclined at angles $\alpha_1, \alpha_2, \ldots.., \alpha_n$ with the positive direction of x-axis, then the complex potential is given by

$$W = \mu_1 \frac{e^{i\alpha_1}}{z - z_1} + \mu_2 \frac{e^{i\alpha_2}}{z - z_2} + \ldots\ldots.. + \mu_n \frac{e^{i\alpha_n}}{z - z_n}$$

8.7. Example. Discuss the flow due to a uniform line doublet at origin of strength μ per unit length and its axis being along the x-axis.
Solution. We know that the complex potential for a doublet is

$$W = \frac{\mu e^{i\alpha}}{z - z_0}$$

and when the doublet is at origin having its axis along x-axis, then $\alpha = 0$, $z_0 = 0$

$$\therefore \qquad W = \frac{\mu}{z} = \frac{\mu}{x + iy} = \frac{\mu(x - iy)}{x^2 + y^2}$$

$$\Rightarrow \qquad \phi + i\psi = \frac{\mu x}{x^2 + y^2} - i\frac{\mu y}{x^2 + y^2}$$

$$\Rightarrow \qquad \phi = \frac{\mu x}{x^2 + y^2}, \qquad \psi = \frac{-\mu y}{x^2 + y^2}$$

Thus the equipotentials, ϕ = constant, are the coaxial circles

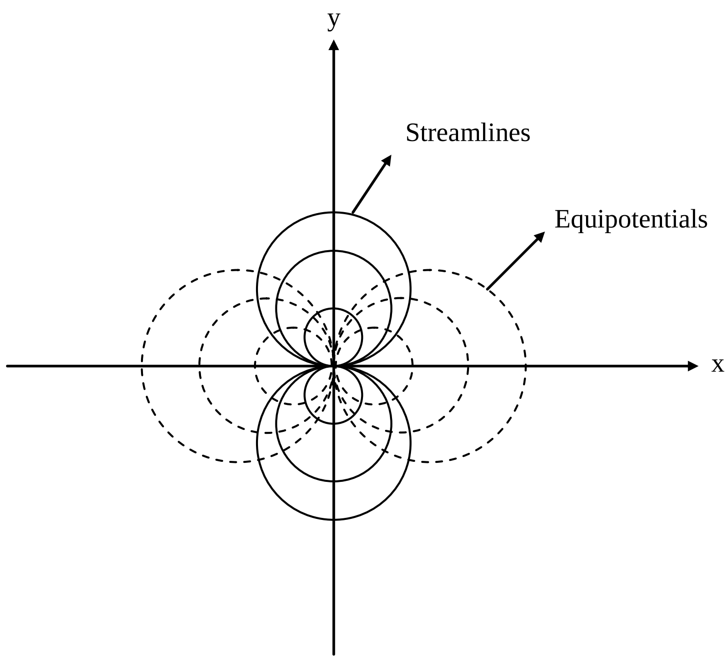
$$x^2 + y^2 = 2k_1 x \qquad (1)$$

and the streamlines, ψ = constant, are the coaxial circles

$$x^2 + y^2 = 2k_2 y \qquad (2)$$

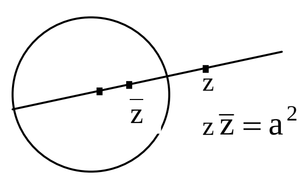
Family (1) have centres $(k_1, 0)$ and radii k_1 and family (2) have centres $(0, k_2)$ and radii k_2

The two families are orthogonal



8.8. Milne-Thomson Circle Theorem : Let f(z) be the complex potential for a flow having no rigid boundaries and such that there are no singularities within the circle $|z| = a$. Then on introducing the solid circular cylinder $|z| = a$, with impermeable boundary, into the flow, the new complex potential for the fluid outside the cylinder is given by

$$W = f(z) + \bar{f}\,(a^2/z),\ |z| \geq a$$



Proof. Let C be he **cross-section** of the cylinder with equation $|z| = 1$.

Therefore, on the circle C, $|z| = a \;\Rightarrow z\,\bar{z} = a^2 \;\Rightarrow \bar{z} = a^2/z$

where $\bar{z}$ is the image of the point z w.r.t. the circle. If z is outside the circle, then $\bar{z} = a^2/z$ is inside the circle. Further, all the singularities of f(z) lie outside C and the singularities of $f(a^2/z)$ and therefore those of $\bar{f}\,(a^2/z)$ lie inside C. Therefore $\bar{f}(a^2/z)$ introduces no singularity outside the cylinder. Thus, the functions f(z) and $f(z) + \bar{f}\,(a^2/z)$ both have the same singularities outside C. Therefore the conditions satisfied by f(z) in the absence of the cylinder are satisfied by $f(z) + \bar{f}\,(a^2/z)$ in the presence of the cylinder. Further, the complex potential, after insertion of the cylinder $|z| = a$, is

$$W = f(z) + \bar{f}\,(a^2/z) = f(z) + \bar{f}\,(\bar{z})$$

$$= f(z) + \overline{f(z)}$$

$$= \text{a purely real quantity}$$

But we know that $W = \phi + i\psi$

It follows that $\psi = 0$

This proves that the circular cylinder $|z| = a$ is a streamline i.e. C is a streamline. Therefore, the new complex potential justifies the fluid motion and hence the circle theorem.

8.9. Uniform Flow Past a Fixed Infinite Circular Cylinder. We have already dealt with this problem using cylindrical polar co-ordinates. Here, we use the concept of complex potential.

The velocity potential due to an undisturbed uniform stream having velocity $-U\hat{i}$ (U is real) is $Ux = U\,\mathrm{Re}(z)$.

Since z is an analytic function, the corresponding complex potential is

$$f(z) = Uz$$

Thus

$$\bar{f}(\bar{z}) = \overline{f(z)} = \overline{Uz} = \overline{U}\,\bar{z} = U\bar{z}$$

and so

$$\bar{f}\,(a^2/z) = Ua^2/z\ .$$

With the cylinder $|z| = a$ present, by circle theorem, the complex potential, for the liquid region $|z| \geq a$, is

$$W = f(z) + \bar{f}\,(a^2/z)$$

i.e. $$\phi + i\psi = U\left(z + \frac{a^2}{z}\right)$$

Taking $z = re^{i\theta}$, where $r \geq a$, equating real and imaginary parts, we get

$$\phi = \mathrm{Re}(W) = U\cos\theta\left(r + \frac{a^2}{r}\right)$$ | Same expression as derived earlier

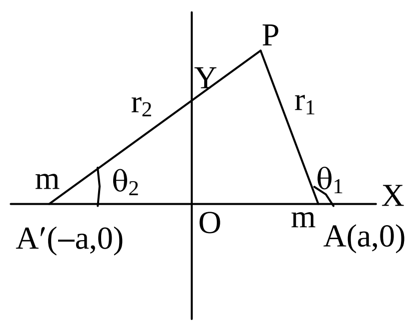
$$\psi = \text{Im}(W) = U \sin\theta \left(r - \frac{a^2}{r} \right)$$

The perturbation term $\bar{f}(a^2/z) = U\dfrac{a^2}{z}$ gives the image of the flow in the cylinder. This image represents a uniform line doublet of strength Ua^2 per unit length and axis in the direction $\hat{i}$.

9. Images in Two Dimensions

In a two dimensional fluid motion, if the flow across a curve C is zero, then the system of line sources, sinks, doublets etc on one side of the curve C is said to form the images of line sources, sinks, doublets etc on the other side of C. To discuss the images in two dimensions, we use complex potential.

9.1. Image of a Line Source in a Plane. Without loss of generality we take the rigid impermeable plane to be x = 0 and perpendicular to the plane of flow (xy-plane). Thus we are to determine the image of a line source of strength m per unit length at A(a, 0) w.r.t. the streamline OY. Let us place a line source per unit length at A′(–a, 0).



The complex potential of strength at a point P due to the system of line sources, is given by

$$W = -m \log z\text{–}a) - m \log(z + a)$$

$$= -m \log [(z-a)(z+a)]$$

$$= -m \log \left[r_1 e^{i\theta_1} r_2 e^{i\theta_2}\right] = -m \log\left[r_1 r_2 e^{i(\theta_1+\theta_2)}\right]$$

$\Rightarrow$ $$\phi + i\psi = -m \log (r_1 r_2) - im (\theta_1 + \theta_2)$$

$\Rightarrow$ $$\psi = -m (\theta_1 + \theta_2)$$

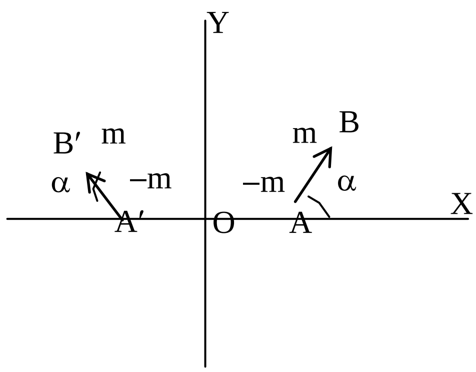
If P lies on y-axis, then PA = PB $\Rightarrow \angle PAB = \angle PBA$

i.e. $\pi - \theta_1 = \theta_2 \quad \Rightarrow \theta_1 + \theta_2 = \pi$

Thus $\psi = -m\pi$ = constant

which shows that y-axis is a streamline. Hence the image of a line source of strength m per unit length at A(a, 0) is a source of strength m per unit length at A′(–a, 0). In other words, image of a line source w.r.t. a plane (a stream line) is a line source of equal strength situated on opposite side of the plane (stream line) at an equal distance.

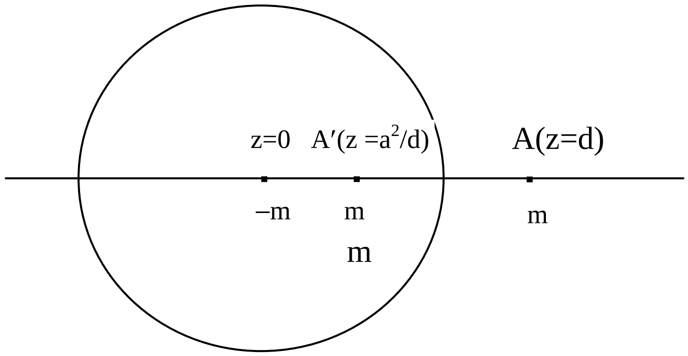
9.2. Image of a Line Doublet in a Plane. Let us consider the rigid impermeable plane to be x = 0 and perpendicular to the plane of flow (xy-plane). Thus we are to determine the image of a line doublet w.r.t.



the stream line OY. Let there be line sources at the points A and B, taken very close together, of strengths –m and m per unit length. Their respective images in OY are –m at A′, m at B′, where A′, B′ are the reflections of A, B in OY. The line $\overrightarrow{AB}$ makes angle α with $\overrightarrow{OX}$. Thus $\overrightarrow{A'B'}$ makes angle $(\pi-\alpha)$ with $\overrightarrow{OX}$. In the limiting case, as $m\to\infty$, $AB\to 0$, we have equal line doublets at A and A′ with their axes inclined at α, $(\pi-\alpha)$ to $\overrightarrow{OX}$. Hence, either of the line doublet is the hydrodynamical image of the other in the infinite rigid impermeable plane (stream line) x = 0

9.3. Image of Line Source in a Circular Cylinder (or in a circle). Let a line source of strength m per unit length be present at a point z = d in the fluid; d > a. Let us then insert a circular cylinder |z| = a in the fluid. The complex potential in the absence of cylinder is –m log (z–d) and after the insertion of cylinder, by circle theorem, we get

$$\phi + i\psi = W = -m \log (z-d) - m \log(a^2/z) - d$$



$$= -m \log (z-d) - m \log\left[\left(\frac{-d}{z}\right)\left(-\frac{a^2}{d}+z\right)\right]$$

$$= -m \log (z-d) - m \log (z-a^2/d) + m \log z + \text{constant} \quad (1)$$

Ignoring the constant term, we observe from (1) that the complex potential represents a line source at $z = d$, another line source at the inverse point $z = a^2/d$ and an equal line sink at the centre of the circle. Thus the image of a line source of strength m per unit length at $z = d$ in a cylinder is an equal line source at the inverse point $z = a^2/d$ together with an equal line sink at the centre $z = 0$ of the circle. Further, (1) can be written as

$$\phi + i\psi = -m\left[\log\left\{(x-d)^2+y^2\right\}^{1/2} + i\tan^{-1}\left(\frac{y}{x-d}\right)\right]$$

$$-m\left[\log\left\{\left(x-\frac{a^2}{d}\right)^2+y^2\right\}^{1/2} + i\tan^{-1}\left(\frac{y}{x-a^2/d}\right)\right]$$

$$+ m\left[\log(x^2+y^2)^{1/2} + i\tan^{-1}\frac{y}{x}\right] \quad .|\ \log z = \log + i\theta\ r = \sqrt{x^2+y^2}\ ,\ \theta$$

$$= \tan^{-1}\frac{y}{x}$$

$$\Rightarrow \qquad \psi = -m\tan^{-1}\left(\frac{y}{x-d}\right) - m\tan^{-1}\left(\frac{y}{x-a^2/d}\right) + m\tan^{-1}\frac{y}{x}\ .$$

$$= -m \tan^{-1}\left[\frac{\dfrac{y}{x-d}+\dfrac{y}{x-a^2/d}}{1-\dfrac{y}{x-d}\,\dfrac{y}{x-a^2/d}}\right] + m\tan^{-1}\frac{y}{x}$$

$$\left|\begin{array}{l} x^2+y^2=a^2 \\ \Rightarrow a^2-y^2=x^2 \end{array}\right.$$

$$= -m\tan^{-1}\frac{y}{x} + m\tan^{-1}\frac{y}{x} = 0\,.$$

Thus, the circular cylinder is a streamline i.e. there is no flow of fluid across the cylinder.

9.4. Image of a Line Doublet in a Circular Cylinder (or in a Circle). Let there be a line doublet of strength μ per unit length at the point z = d, its axis being inclined at an angle α with the x-axis. The line doublet is assumed to be perpendicular to the plane of flow i.e. parallel to the axis of cylinder. The complex potential in the absence of the cylinder, is

$$\frac{\mu e^{ie}}{z-d}$$

When the cylinder |z| = a is inserted, the complex potential, by circle theorem, becomes

$$W = \frac{\mu e^{i\alpha}}{z-d} + \frac{\mu e^{-i\alpha}}{(a^2/z)-d}$$

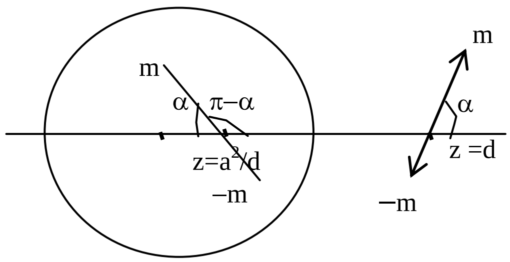
$$= \frac{\mu e^{i\alpha}}{z-d} - \frac{\mu e^{-i\alpha} z}{d\left(z - a^2/d\right)}$$

$$= \frac{\mu e^{i\alpha}}{z-d} + \frac{\mu z\, e^{i(\pi-\alpha)}}{d\left(z - a^2/d\right)}$$

$$= \frac{\mu e^{i\alpha}}{z-d} + \frac{\mu e^{i(\pi-\alpha)}}{d} + \frac{\mu a^2}{d^2}\,\frac{e^{i(\pi-\alpha)}}{z - a^2/d} \qquad (1)$$

If the constant term (second term) in (1) is neglected, then the complex potential in (1) is due to a line doublet of strength μ per unit length at z = d, inclined at an angle α with x-axis and another line doublet of strength $\frac{\mu a^2}{d^2}$ per unit length at the inverse point $z = a^2/d$ inclined at an angle $\pi-\alpha$ with x-axis.

Thus the image of a line doublet of strength μ per unit length $z = d$ inclined at angle α with x-axis is a line doublet of strength $\frac{\mu a^2}{d^2}$ per unit length at the inverse point a^2/d which is inclined at an angle $\pi-\alpha$ with x-axis.



9.5. Remark. The above two cases i.e. (iii) and (iv) alongwith 'uniform flow past a fixed infinite circular cylinder' are applications of Milne-Thomson circle theorem.

9.6. Example. What arrangement of sources and sinks will give rise to the function $W = \log\left(z - \frac{a^2}{z}\right)$?

Also prove that two of the streamlines are a circle $r = a$ and $x = 0$

Solution. We have $W = \log\left(z - \frac{a^2}{z}\right) = \log\left(\frac{z^2 - a^2}{z}\right)$

i.e. $\phi + i\psi = \log(z^2-a^2) - \log z$

$$= \log(z-a) + \log(z+a) - \log z \qquad (1)$$

This represents a line source at $z = 0$ and two line sinks at $z = \pm a$, each of strength unity per unit length. We can write

$$\phi + i\psi = \log(x-a + iy) + \log(x + a + iy) - \log(x + iy)$$

$$\Rightarrow \quad \psi = \tan^{-1}\frac{y}{x-a} + \tan^{-1}\frac{y}{x+a} - \tan^{-1}\frac{y}{x}$$

$$= \tan^{-1}\left(\frac{\dfrac{y}{x-a}+\dfrac{y}{x+a}}{1-\dfrac{y^2}{x^2-a^2}}\right)-\tan^{-1}\frac{y}{x}$$

$$= \tan^{-1}\left(\frac{2xy}{x^2-y^2-a^2}\right)-\tan^{-1}\frac{y}{x}$$

$$= \tan^{-1}\left[\left(\frac{x^2+y^2+a^2}{x^2+y^2-a^2}\right)\frac{y}{x}\right] \qquad (2)$$

Since ψ = constant is the equation of the streamlines, therefore equations for streamlines are

$$y\,(x^2 + y^2 + a^2) = (x^2 + y^2 - a^2)x \tan \alpha$$

where α is a constant.

In particular, if we take $\alpha = \pi/2$, then we get the streamlines as

$$(x^2 + y^2 - a^2)\, x = 0$$

i.e. $x^2 + y^2 - a^2 = 0,\ x = 0$

i.e. $x^2 + y^2 = a^2 \qquad x = 0$

i.e. $r = a,\ x = 0$.

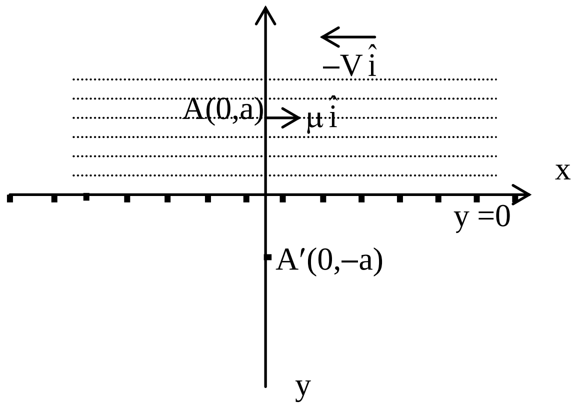
Hence the result.

9.7. Example. A two dimensional doublet of strength $\mu\hat{i}$ per unit length is at a point $z = ia$ in a stream of velocity $-V\hat{i}$ in a semi-infinite liquid of constant density occupying the half plane $y > 0$ and having $y = 0$ as a rigid impermeable boundary, $\hat{i}$ being the unit vector in the positive x-axis. Show that the complex potential of the motion is

$$W = Vz + 2\mu z/(z^2 + a^2)$$

Also show that for $0 < \mu < 4a^2V$, there are no stagnation points on the boundary and that the pressure on it is a minimum at the origin and maximum at the points $(\pm a\sqrt{3}, 0)$.

Solution. We know that the image of the line doublet $\mu\hat{i}$ at point A(0, a) is a line doublet $\mu\hat{i}$ at point A′(0, –a)



Therefore, the complex potential of the system is

$$W = Vz + \frac{\mu}{z-ia} + \frac{\mu}{z+ia}$$

$$= Vz + \frac{2\mu z}{z^2+a^2} = Vz + 2\mu z\,(z^2 + a^2)^{-1}$$

From here, we get

$$\frac{dW}{dz} = V + 2\mu(a^2 - z^2)\,(a^2 + z^2)^{-2}$$

On the boundary y = 0 and thus z = x, therefore,

$$q = \left|-\frac{dW}{dz}\right| = \left|-\frac{dW}{dx}\right| = V + 2\mu\,(a^2 - x^2)\,(a^2 + x^2)^{-2}$$

For stagnation points $\frac{dW}{dx} = 0$

$$\Rightarrow \quad Vx^4 + 2x^2\,(Va^2 - \mu) + Va^4 + 2\mu a^2 = 0 \qquad (1)$$

which is a quadratic in x^2 whose discriminant is

$$\Delta = 4[(Va^2 - \mu)^2 - V\,(Va^4 + 2\mu a^2)]$$

$$= 4\mu\,(\mu - 4a^2V)$$

From here, $\Delta < 0$ if $0 < \mu < 4a^2V$, showing that the quadratic equation (1) has no real root. Therefore there is no stagnation points on the boundary y = 0.

Applying Bernoullis equation along the streamline y = 0, we have

$$\frac{p}{e}+\frac{1}{2}\left[V+2\mu\frac{a^2-x^2}{(a^2+x^2)^2}\right]^2 = \text{constant}$$

$$\left|\frac{p}{e}+\frac{1}{2}q^2 = \text{constant.}\right.$$

P is maximum when X = $\left[V+\frac{2\mu(a^2-x^2)}{(a^2+x^2)^2}\right]^2$ is minimum and conversely.

From here, we get

$$X^{1/2} = V + 2\mu\,(a^2-x^2)\,(a^2+x^2)^{-2}$$

Differentiating w.r.t. x, we get

$$\frac{1}{2}X^{-1/2}.\,X' = -4\mu x\,(3a^2-x^2)\,(a^2+x^2)^{-3}\;X' = \frac{dX}{dx}$$

For extreme values of X, we have X′ = 0 which gives

$$x = 0, \pm a\sqrt{3}.$$

We observe that X′ changes sign from positive to –ve when x passes through zero and thus X is maximum at x = 0 $\Rightarrow$ p is minimum at x = 0 i.e. at (0, 0) i.e. the origin.

Similarly X′ changes sign from negative to positive as x passes through $\pm a\sqrt{3}$ showing that X is minimum at x = $\pm a\sqrt{3}$ and thus p is maximum at ($\pm a\sqrt{3}$,0) .

10. Blasius Theorem

In a steady two dimensional irrotational flow given by the complex potential W = f(z), if the pressure forces on the fixed cylindrical surface C are represented by a force (X, Y) and a couple of moment M about the origin of co-ordinates, then neglecting the external forces,

$$X - iY = \frac{i\rho}{2}\int_C \left(\frac{dW}{dz}\right)^2 dz$$

$$M = \text{Real part of}\left[-\frac{\rho}{2}\int_C z\left(\frac{dW}{dz}\right)^2 dz\right]$$

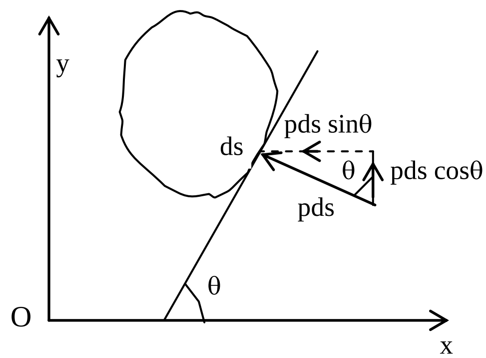
where ρ is the density of the fluid

Proof. Let ds be an element of arc at a point P(x, y) and the tangent at p makes an angle θ with the x-axis. The pressure at P(x, y) is pds, p is the pressure per unit length. pds acts along the inward normal to the cylindrical surface and its components along the co-ordinate axes are

$$pds\cos(90+\theta), \quad pds\cos\theta$$

i.e. $\quad -pds\sin\theta, \quad pds\cos\theta$

The pressure at the element ds is



$$dF = dX + idY$$

$$= -p\sin\theta\, ds + ip\cos\theta\, ds$$

$$= ip\,(\cos\theta + i\sin\theta)\, ds$$

$$\left|\begin{array}{l} pds\sin\theta \text{ along negative } x-\text{axis} \\ \Rightarrow -pds\sin\theta \text{ along positive } x-\text{axis} \end{array}\right.$$

$$= ip\left(\frac{dx}{ds} + i\frac{dy}{ds}\right)ds \qquad \left| \cos\theta = \frac{dx}{ds}, \quad \sin\theta = \frac{dy}{ds} \right.$$

$$= ip\,(dx + idy) = ip\, dz \qquad (1)$$

The pressure equation, in the absence of external forces, is

$$\frac{p}{\rho} + \frac{1}{2}q^2 = \text{constant}$$

or $\qquad p = -\frac{1}{2}\rho q^2 + k \qquad (2)$

Further $\qquad \frac{dW}{dz} = -u + iv = -q\cos\theta + iq\sin\theta$

$$= -q\ (\cos\theta - i\ \sin\theta) = -q\ e^{-i\theta} \qquad (3)$$

and $dz = dx + idy = \left(\frac{dx}{ds} + i\frac{dy}{ds}\right) ds = (\cos\theta + i\ \sin\theta)\ ds = e^{i\theta}\ ds \qquad (4)$

The pressure on the cylinder is obtained by integrating (1). Therefore,

$$F = X + iY = \int_C ipdz = \int_C i\ (k - 1/2\ \rho q^2)\ dz$$

$$= -\frac{i\rho}{2}\int_C q^2\ dz \qquad \mid \because \int_C dz = 0$$

$$= -\frac{i\rho}{2}\int_C q^2\ e^{i\theta} ds$$

From here ;

$$X - iY = \frac{i\rho}{2}\int_C q^2\ e^{-i\theta}\ ds$$

$$= \frac{i\rho}{2}\int_C (q^2\ e^{-2i\theta})\ e^{i\theta}\ ds$$

$$= \frac{i\rho}{2}\int_C \left(\frac{dW}{dz}\right)^2 dz \qquad \mid \text{using (3) \& (4)}$$

The moment M is given by

$$M = \int_C \left|\bar{r} \times d\bar{F}\right| = \int_C [(pds\ \sin\theta)\ y + (pds\ \cos\theta)\ x]$$

$$= \int_C \left[p\left(\frac{dy}{ds}\right) y\, ds + p\left(\frac{dx}{ds}\right) x\, ds\right]$$

$$\left|\begin{array}{l} \vec{r} \times d\vec{F} \\ = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ x & y & 0 \\ -pds\sin\theta & pds\cos\theta & \end{vmatrix} \end{array}\right.$$

$$= \int_C p(x\ dx + ydy]$$

$$= \int_C \left(k - \frac{1}{2}\rho q^2\right)(xdx + ydy)$$

$$= k\int_C d\left[\frac{1}{2}(x^2+y^2)\right] - \frac{\rho}{2}\int_C q^2\,(xdx + ydy)$$

$$= -\frac{\rho}{2}\int_C q^2(xdx + ydy) \qquad | \because \ 1^{st} \ \text{integral}$$

vanishes.

$$= \frac{-\rho}{2}\int_C q^2\,(x\cos\theta + y\sin\theta)\,ds \qquad \left|\begin{array}{l} dx = \cos\theta ds \\ dy = \sin\theta ds \end{array}\right.$$

$$= \text{R.P. of } \left[\frac{-\rho}{2}\int_C q^2(x+iy)(\cos\theta - i\sin\theta)ds\right]$$

$$= \text{R.P of } \left[\frac{-\rho}{2}\int_C q^2 z e^{-i\theta} ds\right]$$

$$= \text{R.P of}\left[\frac{-e}{2}\int_C z(q^2 e^{-2i\theta})e^{i\theta} ds\right]$$

$$= \text{R.P. of } \left[-\frac{e}{2}\int_C z\left(\frac{dW}{dz}\right)^2 dz\right].$$

Hence the theorem.

11. Two-dimensional Irrotational Motion Produced by Motion of Cylinders

Here, we discuss two-dimensional irrotational motion produced by the motion of cylinders in an infinite mass of liquid at rest at infinity (the local fluid moves with the cylinder). The cylinders move at right angles to their generators which are taken parallel to z-axis. Thus we get the xy-plane as the plane of flow. For the sake of simplicity, we take the cylinders of unit length. For such motion, the stream function ψ or velocity potential ϕ) is determined in the light of the following conditions.

(i) ψ satisfies Laplace equation i.e. $\nabla^2\psi = 0$ at every point of the liquid.

(ii) Since the liquid is at rest at infinity, so

$$\frac{\partial\psi}{\partial x} = 0 \text{ and } \frac{\partial\psi}{\partial y} = 0 \text{ at infinity.}$$

(iii) Along any fixed boundary, the normal component of velocity must be zero so that $\frac{\partial\psi}{\partial s} = 0$ i.e.

ψ = constant, which means that the boundary must coincide with a streamline.

(iv) On the boundary of the moving cylinder, the normal component of the velocity of the liquid must be equal to normal component of velocity of the cylinder.

Further, we observe that the two-dimensional solution of the Laplace equation $\nabla^2\psi = 0$, in polar co-ordinates (r, θ), is

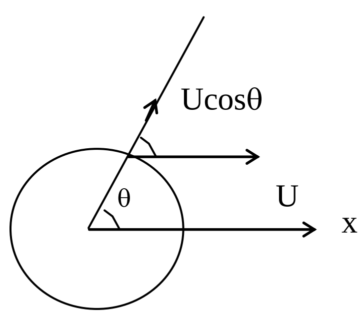
$$\psi = A_n r^n \cos n\theta + B_n r^n \sin \theta$$

where n is any integer, A_n and B_n being constants. Also, all the observations made for ψ, are valid for velocity potential ϕ, where ϕ and ψ satisfy C–R equations.

11.1. Motion of a Circular Cylinder. Let us consider a circular cylinder of radius a moving with velocity U along x-axis in an infinite mass of liquid at rest at infinity. The velocity potential ϕ which is the solution of $\nabla^2\phi = 0$, must satisfy the following conditions.

(i) $\left(-\frac{\partial\phi}{\partial r}\right)_{r=a} = U\cos\theta$

(ii) $-\frac{\partial\phi}{\partial r}$ and $-\frac{1}{r}\frac{\partial\phi}{\partial\theta} \to 0$ as $r\to\infty$



A suitable form of ϕ is

$$\phi(r, \theta) = \left(Ar + \frac{B}{r}\right)\cos\theta \tag{1}$$

$$\Rightarrow \qquad -\frac{\partial\phi}{\partial r} = \left(-A + \frac{B}{r^2}\right)\cos\theta \tag{2}$$

Applying conditions (i) and (ii) in (2), we get

$$\left(-A + \frac{B}{a^2}\right)\cos\theta = U\cos\theta, (-A + 0.B) = 0 \text{ for all } \theta.$$

$$\Rightarrow \quad -A + \frac{B}{a^2} = U, A = 0$$

$$\Rightarrow \quad A = 0, B = U a^2$$

Thus $$\phi(r, \theta) = \frac{Ua^2}{r}\cos\theta \qquad (3)$$

The second condition of (ii) is evidently satisfied by ϕ in (3)

But $$\frac{\partial\phi}{\partial r} = \frac{1}{r}\frac{\partial\psi}{\partial\theta} \qquad \text{(C–R equation)}$$

so, $$\frac{1}{r}\frac{\partial\psi}{\partial\theta} = -\frac{Ua^2}{r^2}\cos\theta$$

i.e. $$\frac{\partial\psi}{\partial\theta} = -\frac{Ua^2}{r}\cos\theta$$

Neglecting constant of integration, we get

$$\psi = -\frac{Ua^2}{r}\sin\theta \qquad (4)$$

Thus $$W = \phi + i\psi = \frac{Ua^2}{r}(\cos\theta - i\sin\theta)$$

$$= \frac{Ua^2}{re^{i\theta}} = \frac{Ua^2}{z}$$

which gives the complex potential for the flow.

11.2. Remarks. (i) For the case of 'Uniform flow past a fixed circular cylinder', using circle theorem, we have obtained the complex potential as

$$W = f(z) + f(a^2/z)$$

$$= Uz + U\frac{a^2}{z}$$

where the cylinder moves with velocity U along positive direction of x-axis. If we give a velocity U to the complete system, along the positive direction of x-axis, then the stream comes to rest and the cylinder moves with velocity U in x-direction.

Thus, we get

$$W = Uz + U\frac{a^2}{z} - Uz = \frac{Ua^2}{z}$$

(ii) Similarly, if we impose a velocity U in the negative direction of x-axis to the complete system in the present problem, then the cylinder comes to rest and the liquid flows past the fixed cylinder with velocity U in negative x-axis direction and thus we get

$$W = \frac{Ua^2}{z} + Uz.$$

(iii) If we put $Ua^2 = \mu$, then we get

$$W = \frac{\mu}{z} = \frac{\mu e^{i\theta}}{z-0}$$

which shows that the complex potential due to a circular cylinder with velocity U along x-axis in an infinite mass of liquid is the same as the complex potential due to a line doublet of strength $\mu = Ua^2$ pre unit length situated at the centre with its axis along x-axis.

11.3. Example. A circular cylinder of radius a is moving in the fluid with velocity U along the axis of x. Show that the motion produced by the cylinder in a mass of fluid at rest at infinity is given by the complex potential

$$W = \phi + i\psi = \frac{Ua^2}{z - Ut}$$

Find the magnitude and direction of the velocity in the fluid and deduce that for a marked particle of fluid whose polar co-ordinates are (r, θ) referred to the centre of the cylinder as origin,

$$\frac{1}{r}\frac{dr}{dt} + i\frac{d\theta}{dt} = \frac{U}{r}\left(\frac{a^2}{r^2}e^{i\theta} - e^{-i\theta}\right) \text{ and } \left(r - \frac{a^2}{r}\right)\sin\theta = \text{constant}$$

Solution. The cylinder is given to be moving along x-axis. At time t, it has moved through a distance Ut. Taking z = Ut as the origin, the complex potential is

$$W = \phi + i\psi = \frac{Ua^2}{z - Ut}$$

Therefore $$-\frac{dW}{dz} = \frac{Ua^2}{(z-Ut)^2} = \frac{Ua^2}{r^2}e^{-2i\theta} \;, \quad z - Ut = re^{i\theta}$$

i.e. $$u - iv = \frac{Ua^2}{r^2}(\cos 2\theta - i \sin 2\theta)$$

$\Rightarrow$ $$u = \frac{Ua^2}{r^2}\cos 2\theta, \quad v = \frac{Ua^2}{r^2}\sin 2\theta$$

Therefore, $$q = \sqrt{u^2 + v^2} = \frac{Ua^2}{r^2}$$

The direction of velocity is $\tan\alpha = \dfrac{v}{u} = \tan 2\theta \;\Rightarrow \alpha = 2\theta$

When the cylinder is fixed and its centre is at 0, then

$$W = Uz + \frac{Ua^2}{z} = U(x+iy) + \frac{Ua^2}{r^2}(x - iy)$$

i.e. $$\phi + i\psi = Ur(\cos\theta + i\sin\theta) + \frac{Ua^2}{r}(\cos\theta - i\sin\theta)$$

$\Rightarrow$ $$\phi = Ur\cos\theta + \frac{Ua^2\cos\theta}{r}, \quad \psi = U\left(r - \frac{a^2}{r}\right)\sin\theta$$

The streamlines are given by ψ = constant

$\Rightarrow$ $$\left(r - \frac{a^2}{r}\right)\sin\theta = \text{constant}$$

Further,

$$\frac{dr}{dt} = -\frac{\partial\phi}{\partial r} = -U\cos\theta + \frac{Ua^2}{r^2}\cos\theta \qquad |\bar{q} = -\nabla\phi$$

$$r\frac{d\theta}{dt} = -\frac{1}{r}\frac{\partial\phi}{\partial\theta} = U\sin\theta + \frac{Ua^2\sin\theta}{r^2}$$

$\Rightarrow$ $$\frac{1}{r}\frac{dr}{dt} + i\frac{d\theta}{dt} = -\frac{U\cos\theta}{r} + \frac{Ua^2\cos\theta}{r^3} + \frac{iU\cos\theta}{r} + i\frac{Ua^2\sin\theta}{r^3}$$

$$= \frac{U}{r}\left(\frac{a^2}{r^2}e^{i\theta} - e^{-i\theta}\right)$$

Hence the result.

11.4. Equation of Motion of a Circular Cylinder. Let a circular cylinder of radius a move with a uniform velocity U along x-axis in a liquid at rest at infinity. The complex potential for the resulting motion, is $\phi + i\psi = W = \frac{Ua^2}{z}$, where origin is taken at the centre of the cylinder.

Thus, $$\phi = \frac{Ua^2}{r}\cos\theta, \quad \psi = -\frac{Ua^2}{r}\sin\theta$$

so $$\left(\frac{\partial\phi}{\partial r}\right)_{r=a} = -U\cos\theta$$

Let T_1 be the K.E. of the liquid on the boundary of the cylinder and T_2 that of the cylinder. Let σ and ρ be the densities of material of the cylinder and the liquid respectively. Then

$$T_1 = -\frac{\rho}{2}\int_C \phi\frac{\partial\phi}{\partial n}ds$$

$$= -\frac{\rho}{2}\int_0^{2\pi}\left(\phi\frac{\partial\phi}{\partial r}\right)_{r=a} a\,d\theta, \quad s = a\theta \;\Rightarrow ds = a\,d\theta \qquad | \; l = r\theta$$

$$= \frac{\rho}{2}\int_0^{2\pi}\left(\frac{Ua^2}{a}\cos\theta\right)(U\cos\theta)\,a\,d\theta$$

$$= \frac{\rho U^2 a^2}{2}\int_0^{2\pi}\cos^2\theta\,d\theta$$

$$= \frac{\pi\rho U^2 a^2}{2} = (\pi a^2\rho)\frac{U^2}{2} = M'\frac{U^2}{2},$$

where $M' = \pi a^2\rho$ = mass of the liquid displaced by the cylinder of unit length.

K.E. of the cylinder, $T_2 = \frac{1}{2}MU^2$, $M = \pi a^2\sigma$

Thus, total K.E. of the liquid and cylinder is

$$T = T_1 + T_2 = \frac{1}{2}(M + M')\,U^2 \qquad (1)$$

Let R be the external force on the cylinder in the direction of motion. We use the fact that rate of change of total energy is equal to the rate at which work is being done by external forces at the boundary.

$$\therefore \qquad RU = \frac{1}{2}\frac{d}{dt}(M + M')\,U^2$$

$$\left|\begin{array}{l} \dfrac{\text{work done}}{\text{time}} = \dfrac{\text{force.distance}}{\text{time}} \\ = \text{force.velocity} \end{array}\right.$$

$$= \frac{M + M'}{2} 2U \frac{dU}{dt}$$

$$= (M + M')\,U \frac{dU}{dt}$$

$$\Rightarrow \qquad M\frac{dU}{dt} = R - M'\frac{dU}{dt} \qquad (2)$$

Equation (2) is the equation of motion of the cylinder. This shows that the presence of liquid offers resistance (drag force) to the motion of the cylinder, since if there is no liquid, then $M' = 0$ and we get

$$M\frac{dU}{dt} = R \qquad (3)$$

Now, if $\frac{R}{M}$ = external force on the cylinder per unit mass be constant and conservative, then by the energy equation, we get

$$\frac{1}{2}(M + M^1)\,U^2 - (M - M^1)\frac{R}{M}r = \text{constant} \qquad (4)$$

where r is the distance moved by the cylinder in the direction of R. Diff. (4) w.r.t. t, we get

$$(M + M')\,U\frac{dU}{dt} - (M - M')\frac{R}{M}U = 0$$

$$\text{or} \qquad M\frac{dU}{dt} = \frac{M - M'}{M + M'}R = \frac{\pi\sigma a^2 - \pi\rho a^2}{\pi\sigma a^2 + \pi\rho a^2}R$$

i.e. $$M\frac{dU}{dt} = \frac{\sigma-\rho}{\sigma+\rho}R \qquad (5)$$

which gives another form of equation of motion

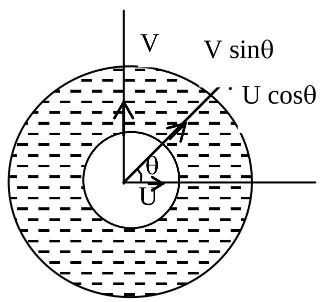
If U = (u, v) and R = (X, Y), then

$$M\frac{du}{dt} = \frac{\sigma-\rho}{\sigma+\rho}X,\ M\frac{dv}{dt} = \frac{\sigma-\rho}{\sigma+\rho}Y \qquad (6)$$

Are the equations of motion of the cylinder in Cartesian co-ordinates. Comparing (3) and (5), it can be said that the effect of the presence of the liquid is to reduce external forces in the ratio

$$\sigma-\rho : \sigma + \rho.$$

11.5. Motion of two co-axial cylinders. Let us consider two co-axial cylinders of radii a and b ($a < b$). The space between them is filled with liquid of density ρ . Let the cylinders move parallel to themselves in directions at right angles with velocities U and V respectively, as shown in the figure



The boundary conditions for the velocity potential ϕ which is the solution of $\nabla^2\phi = 0$, are $(\bar{q} = -\nabla\phi)$

(i) $$-\frac{\partial\phi}{\partial r} = U\cos\theta,\ r = a \qquad (1)$$

(ii) $$-\frac{\partial\phi}{\partial r} = V\sin\theta,\ r = b \qquad (2)$$

A suitable form of velocity potential is

$$\phi = \left(Ar + \frac{B}{r}\right)\cos\theta + \left(Cr + \frac{D}{r}\right)\sin\theta \qquad (3)$$

$$\Rightarrow \quad \frac{\partial\phi}{\partial r} = \left(A - \frac{B}{r^2}\right)\cos\theta + \left(C - \frac{D}{r^2}\right)\sin\theta \qquad (4)$$

Using (1) & (2) in (4), we get

$$-U\cos\theta = \left(A-\frac{B}{a^2}\right)\cos\theta + \left(C-\frac{D}{a^2}\right)\sin\theta$$

$$-V\sin\theta = \left(A-\frac{B}{b^2}\right)\cos\theta + \left(C-\frac{D}{b^2}\right)\sin\theta$$

Comparing co-efficients of $\cos\theta$ and $\sin\theta$, we get

$$A-\frac{B}{a^2} = -U, \quad C-\frac{D}{a^2} = 0$$

$$A-\frac{B}{b^2} = 0, \quad C-\frac{D}{b^2} = -V$$

Solving these equations, we obtain

$$A = -\frac{Ua^2}{a^2-b^2},\ B = \frac{-Ua^2b^2}{a^2-b^2},\ C = \frac{Vb^2}{a^2-b^2},\ D = \frac{Va^2b^2}{a^2-b^2}$$

Thus, (3) becomes

$$\phi = -\frac{Ua^2}{a^2-b^2}\left(r+\frac{b^2}{r}\right)\cos\theta + \frac{Vb^2}{a^2-b^2}\left(r+\frac{a^2}{r}\right)\sin\theta$$

$$= \frac{Ua^2}{b^2-a^2}\left(r+\frac{b^2}{r}\right)\cos\theta - \frac{Vb^2}{b^2-a^2}\left(r+\frac{a^2}{r}\right)\sin\theta \quad (5)$$

The expression for ψ can be obtained from

$$\frac{\partial\phi}{\partial r} = \frac{1}{r}\frac{\partial\psi}{\partial\theta}$$

i.e. $$\frac{\partial\psi}{\partial\theta} = r\frac{\partial\phi}{\partial r}$$

$$= \frac{Ua^2}{b^2-a^2}\left(r-\frac{b^2}{r}\right)\cos\theta - \frac{Vb^2}{b^2-a^2}\left(r-\frac{a^2}{r}\right)\sin\theta$$

Integrating and neglecting the constant of integration, we get

$$\psi = \frac{Ua^2}{b^2 - a^2}\left(r - \frac{b^2}{r}\right)\sin\theta + \frac{Vb^2}{b^2 - a^2}\left(r - \frac{a^2}{r}\right)\cos\theta \quad (6)$$

It should be noted that the values of ϕ and ψ given by (5) and (6), hold only at the instant when the cylinders are on starting i.e. the initial motion.

11.6. Corollary. If the cylinders move in the same direction then the boundary conditions are

(i) $-\dfrac{\partial\phi}{\partial r} = U\cos\theta,\ r = a$

(ii) $-\dfrac{\partial\phi}{\partial r} = V\cos\theta,\ r = b$

Using these conditions in (4), comparing co-efficients of $\cos\theta$ and $\sin\theta$ and then solving the resulting equations, we get

$$A = \frac{Ua^2 - Vb^2}{b^2 - a^2},\ B = \frac{-UVa^2b^2}{b^2 - a^2},\ C = 0,\ D = 0$$

So, $\phi = \dfrac{1}{b^2 - a^2}\left[\left(Ua^2 - Vb^2\right)r - \dfrac{UVa^2b^2}{r}\right]\cos\theta$

and $\psi = \dfrac{1}{b^2 - a^2}\left[\left(Ua^2 - Vb^2\right)r + \dfrac{UVa^2b^2}{r}\right]\sin\theta$

11.7. Example. An infinite cylinder of radius a and density σ is surrounded by a fixed concentric cylinder of radius b and the intervening space is filled with liquid of density ρ. Prove that the impulse per unit length necessary to start the inner cylinder with velocity V is

$$\frac{\pi a^2}{b^2 - a^2}[(\sigma+\rho)\, b^2 - (\sigma-\rho)\, a^2]V$$

Suppose that V is taken along the x-axis.

Solution. Let the velocity potential be

$$\phi = \left(Ar + \frac{B}{r}\right)\cos\theta + \left(Cr + \frac{D}{r}\right)\sin\theta \quad (1)$$

The boundary conditions are $(\bar{q} = -\nabla\phi)$

(i) $-\dfrac{\partial\phi}{\partial r} = V\cos\theta,\ r = a$

(ii) $-\dfrac{\partial \phi}{\partial r} = 0,\ r = b$

Applying these conditions in (1) and then comparing co-efficients of cosθ and sinθ, we get

$$A - \frac{B}{a^2} = -V, \quad C - \frac{D}{a^2} = 0$$

$$A - \frac{B}{b^2} = 0, \quad C - \frac{D}{b^2} = 0$$

Solving for A, B, C, D, we obtain

$$A = \frac{Va^2}{b^2 - a^2},\ B = \frac{Va^2b^2}{b^2 - a^2},\ C = D = 0$$

Thus, the potential (1) is

$$\phi = \frac{1}{b^2 - a^2}\left(Va^2 r + \frac{Va^2b^2}{r}\right)\cos\theta$$

Now, the impulsive pressure at a point on r = a (along x-axis), is

$$P = (\rho\ \phi)_{r = a} = \frac{\rho Va^2}{b^2 - a^2}\left(r + \frac{b^2}{r}\right)\cos|_{r=a}$$

$$= \frac{\rho Va}{b^2 - a^2}(a^2 + b^2)\cos\theta$$

The impulsive pressure on the mole cylinder is

$$\int_0^{2\pi} \frac{\rho Va}{b^2 - a^2}\ (a^2 + b^2)\cos\theta.\ a\cos\theta\ d\theta$$

$$= -\pi\ a^2\rho\ \left(\frac{b^2 + a^2}{b^2 - a^2}\right)V$$

Now, change in momentum = the sum of impulsive forces

Therefore, $\pi\ a^2\sigma\ (V–0) = I - \pi a^2\rho\ \left(\dfrac{b^2 + a^2}{b^2 - a^2}\right)V$

$\Rightarrow$ $I = \pi a^2\sigma\ V + \pi a^2\rho\left(\dfrac{b^2 + a^2}{b^2 - a^2}\right)V$

Thus, impulse due to external forces, is

$$I = \frac{\pi a^2 V}{b^2 - a^2}\ [\ \sigma\ (b^2 – a^2) + \rho(b^2 + a^2)]$$

$$= \frac{\pi a^2 V}{b^2 - a^2}\ [(\sigma + \rho)\ b^2 - (\sigma - \rho)\ a^2]$$

Hence the result.

UNIT – III

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1. Vortex Motion

So far we have confined our attention to the cases involving irrotational motion only. But the most general displacement of a fluid involves rotation such that the rotational vector (vortex vector or vorticity) $\bar{\xi} = \text{curl}\,\bar{q} \neq \bar{0}$. Here we consider the theory of rotational or vortex motion. First of all we revisit some elementary definitions.

Lines drawn in the fluid so as at every point to coincide with the instantaneous axis of rotation of the corresponding fluid element are called **vortex lines.** Portions of the fluid bounded by vortex lines drawn through every point of an infinity small closed curve are called **vortex filaments** or simply **vortices** and the boundary of a vortex filament is called a **vortex tube.**

If C is a closed curve, then **circulation** about C is given by

$$\Gamma = \oint_C \bar{q}.d\bar{r} = \int_S \hat{n}.\text{curl}\,\bar{q}\,dS = \int_S \hat{n}.\bar{\xi}\,dS = \int_S \bar{\xi}.d\bar{S}$$

The quantity $\left|\hat{n}\cdot\bar{\xi}\right|\delta S$ is called the strength of the vortex tube. A vortex tube with a unit strength is called a unit vortex tube.

We shall observe some important results for vortex motion which are consequences of the following theorem due to Lord Kelvin.

1.1. Kelvin's Circulation Theorem (Consistency of circulation). The circulation around a closed contour C moving with the inviscid (non-viscous) fluid is constant for all times provided that the external forces (body forces) are conservative and the density is a function of pressure only.

Proof. The circulation round a closed curve C of fluid particles is defined by

$$\Gamma = \oint_C \bar{q}.d\bar{r},$$

where $\bar{q}$ is the velocity and $\bar{r}$ is the position vector of a fluid particle at any time t.

Time derivative of Γ following the motion of fluid is

$$\frac{d\Gamma}{dt} = \frac{d}{dt}\oint_C \bar{q}\cdot d\bar{r} = \oint_C \frac{d}{dt}(\bar{q}\cdot d\bar{r})$$

$$= \oint_C \left[\frac{d\bar{q}}{dt} \cdot d\bar{r} + \bar{q} \cdot \frac{d}{dt}(d\,\bar{r}) \right]$$

$$= \oint_C \left[\frac{d\bar{q}}{dt} \cdot d\bar{r} + \bar{q} \cdot d\bar{q} \right] \qquad (1) \qquad \left| \because \frac{d}{dt}(d\bar{r}) = d\left(\frac{d\bar{r}}{dt}\right) = d\bar{q} \right.$$

Since the system of forces is conservative; therefore $\bar{F} = -\nabla\Omega$, where Ω is a potential function Euler's equation of motion is

$$\frac{d\bar{q}}{dt} = \bar{F} - \frac{1}{\rho}\nabla p = -\nabla\Omega - \frac{1}{\rho}\nabla p \qquad (2)$$

Multiplying each term of (2) scalarly by $d\bar{r}$, we get

$$d\bar{r} \cdot \frac{d\bar{q}}{dt} = -d\bar{r} \cdot \nabla\Omega - \frac{1}{\rho} d\bar{r}.\nabla p$$

i.e. $$\frac{d\bar{q}}{dt}.d\bar{r} = -d\Omega - \frac{dp}{\rho} \qquad (3) \qquad \left| \because d\bar{r}.\nabla \equiv d \right.$$

Thus from (1), we get

$$\frac{d\Gamma}{dt} = \oint_C \left(-d\Omega - \frac{dp}{\rho} + \bar{q} \cdot d\bar{q} \right)$$

$$= \oint_C \left[d\left(\frac{1}{2}\bar{q}^2 - \Omega \right) - \frac{1}{\rho} dp \right]$$

$$= \oint_C d\left(\frac{1}{2}\bar{q}^2 - \Omega \right) - \oint_C \frac{1}{\rho} dp$$

$$= \left[\frac{1}{2}\bar{q}^2 - \Omega \right]_A^A - \oint_C \frac{dp}{\rho}$$

$$= 0 - \oint_C \frac{dp}{\rho} \qquad (4)$$

where A is any point on the closed contour C. Now, if density is a function of pressure only, then the integral $\oint_C \frac{dp}{\rho}$ vanishes and hence we get

$$\frac{d\Gamma}{dt} = 0 \quad \Rightarrow \ \Gamma = \text{constant for all time}$$

Corollary (1). In a closed circuit C of fluid particles moving under the same conditions as in the theorem,

$$\int_S \text{curl } \bar{q}.d\bar{S} = \int_S \bar{\xi}.d\bar{S} = \text{constant} \tag{5}$$

where S is any open surface whose rim is C. To establish (5), we note that, by Stock's theorem,

$$\int_S \text{curl}\,\bar{q}.d\bar{S} = \oint_C \bar{q}.d\bar{r} = \Gamma = \text{constant}$$

This shows that the product of the cross-section and angular velocity at any point on a vortex filament is constant all along the vortex filament and for all times.

Corollary (2). Under the conditions of the theorem, vortex lines move with the fluid.

Proof. Let C be any closed curve drawn on the surface of a vortex tube. Let S be the portion of the vortex tube rimmed by C. By definition vortex lines lie on S. Thus

$$0 = \int_S \text{curl } \bar{q}.d\bar{S} = \oint_C \bar{q}.d\bar{r} \qquad | \because \text{ on surface circulation is zero}$$

Let C be a material curve and S be a material surface, then

$$\frac{d}{dt}\int_S (\hat{n}.\text{curl } \bar{q})dS = \int_S \frac{D}{Dt}(\hat{n}.\text{curl } \bar{q})dS = 0$$

Thus $\hat{n}.\text{curl}\,\bar{q}$ remains zero, so that S remains a surface composed of vortex lines. Consequently vortex lines and tubes move with the fluid i.e. vortex filaments are composed of the same fluid particles. This explains why smoke rings maintain their forms for long periods of time.

Corollary (3). Under the conditions of the theorem, if the flow is irrotational in a material region of the fluid at some particular time (e.g. $t = 0$ or $t = t_0$), the flow is always irrotational in that material region thereafter.

i.e. If the motion of an ideal fluid is once irrotational it remains irrotational for ever afterwards provided the external forces are conservative and density ρ is a function of pressure p only.

Proof. Suppose that at some instant ($t = t_0$), the fluid on the material surface S is irrotational

Then, $\bar{\xi} = 0$ (1)

for all points of S.

Let C be the boundary of surface S, then

$$\Gamma = \oint_C \bar{q}.d\bar{r} = \int_S (\hat{n}.\text{curl } \bar{q})\, dS = \int_S (\hat{n}.\bar{\xi}) ds = 0 \qquad | \text{ using (1)}$$

But by Kelvin's circulation theorem, Γ is constant for all times. Hence circulation Γ is zero for all subsequent times. At any later time,

$$\int_S \hat{n}.\bar{\xi}\, dS = 0$$

If we now take S to be non-zero infinitesimal element, say ΔS, then

$\hat{n}.\bar{\xi}\,\Delta S = 0 \quad \Rightarrow \bar{\xi} = 0$ at all points of S for all times and the motion is irrotational permanently. This proves the permanency of irrotational motion.

1.2. Remarks (i) The above three corollaries are properties of vortex filaments.

(ii) The Kelvin's theorem is true whether the motion be rotational or irrotational In case of irrotational motion, $\bar{\xi} = \bar{0}$ and thus $\Gamma = 0$

(iii) From the results of the theorem, we conclude that vortex filaments must either form closed curves or have their ends on the bounding surface of the fluid. A vortex in an ideal fluid is therefore permanent.

1.3. Vorticity Equation. Euler's equation of motion for an ideal fluid under the action of a conservative body force with potential Ω per unit mass is

$$\frac{D\bar{q}}{Dt} = \frac{\partial \bar{q}}{\partial t} + \nabla\left(\frac{1}{2}\bar{q}^2\right) - \bar{q} \times \bar{\xi} = -\nabla\Omega - \frac{1}{\rho}\nabla p \qquad (1)$$

where the vorticity $\bar{\xi} = \text{curl } \bar{q} = \nabla \times \bar{q}$. If the fluid has constant density, then taking curl of equation (1), we get

$$\nabla \times \frac{\partial \bar{q}}{\partial t} + \nabla \times \left[\nabla\left(\frac{1}{2}\bar{q}^2\right)\right] - \nabla \times (\bar{q} \times \bar{\xi}) = \nabla \times \left(-\nabla\Omega - \frac{1}{\rho}\nabla p\right)$$

$\Rightarrow \qquad \nabla\times\dfrac{\partial\bar{q}}{\partial t}-\nabla\times(\bar{q}\times\bar{\xi})=0$

$\Rightarrow \qquad \dfrac{\partial}{\partial t}-(\nabla\times\bar{q})-\nabla\times(\bar{q}\times\bar{\xi})=0$

$\Rightarrow \qquad \dfrac{\partial\bar{\xi}}{\partial t}=\nabla\times(\bar{q}\times\bar{\xi})$

$$=(\bar{\xi}\cdot\nabla)\bar{q}-(\bar{q}\cdot\nabla)\bar{\xi}$$

$\Rightarrow \qquad \dfrac{\partial\bar{\xi}}{\partial t}+(\bar{q}\cdot\nabla)\bar{\xi}=(\bar{\xi}\cdot\nabla)\bar{q}$

i.e. $$\frac{D\bar{\xi}}{Dt}=(\bar{\xi}\cdot\nabla)\bar{q} \qquad (2)$$

which is the required vorticity equation.

Equation (2) is called Helmholtz's vorticity equation. For two-dimensional motion, the vorticity vector $\bar{\xi}$ is perpendicular to the velocity vector $\bar{q}$ and the R.H.S. of (2) is identically zero. Thus, for two dimensional motion of an ideal fluid, vorticity is constant.

In the case, when body force is not conservative, equation (2) becomes

$$\frac{D\bar{\xi}}{Dt}=(\bar{\xi}\cdot\nabla)\bar{q}+\text{curl }\bar{F}$$

where $\bar{F}$ is body force per unit mass.

1.4. Example. A motion of in viscid incompressible fluid of uniform density is symmetrical about the axis r = 0 where (r, θ, z) are cylindrical polar co-ordinates. The cylindrical polar resolutes of velocity are [q_r(r, z), 0, q_z(r, z)]. Show that if a fluid particle has vorticity of magnitude ξ_0 when r = r_0, its vorticity when at general distance r from the axis of symmetry has magnitude $\xi = (\xi_0/r_0)r$, if any body forces acting are conservative.

Solution. The vorticity vector $\bar{\xi}$ satisfies the vorticity equation

$$\frac{D\bar{\xi}}{Dt} = (\bar{\xi} \cdot \nabla)\,\bar{q} \qquad (1)$$

Now,

$$\bar{\xi} = \operatorname{curl} \bar{q} = \frac{1}{r}\begin{vmatrix} \hat{r} & r\hat{\theta} & \hat{z} \\ \dfrac{\partial}{\partial r} & \dfrac{\partial}{\partial \theta} & \dfrac{\partial}{\partial z} \\ q_r(r,z) & 0 & q_z(r,z) \end{vmatrix}$$

$$= \frac{1}{r}\left[\hat{r}\,\frac{\partial}{\partial \theta}\{q_z(r,z)\} + r\,\hat{\theta}\left\{\frac{\partial}{\partial z}q_r(r,z) - \frac{\partial}{\partial r}q_z(r,z)\right\} + \hat{z}(0)\right]$$

$$= \left[\frac{\partial q_r}{\partial z} - \frac{\partial q_z}{\partial r}\right]\hat{\theta}\ . \qquad (2)$$

Therefore,

$$(\bar{\xi} \cdot \nabla) = \bar{\xi} \cdot \left(\hat{r}\,\frac{\partial}{\partial r} + \hat{\theta}\,\frac{1}{r}\frac{\partial}{\partial \theta} + \hat{z}\,\frac{\partial}{\partial z}\right)$$

$$= \hat{\theta}\left(\frac{\partial q_r}{\partial z} - \frac{\partial q_z}{\partial r}\right) . \left(\hat{r}\,\frac{\partial}{\partial r} + \hat{\theta}\,\frac{1}{r}\frac{\partial}{\partial \theta} + \hat{z}\,\frac{\partial}{\partial z}\right)$$

$$= \frac{1}{r}\left(\frac{\partial q_r}{\partial z} - \frac{\partial q_z}{\partial r}\right)\frac{\partial}{\partial \theta} \qquad \Big|\ \hat{\theta}.\hat{r} = 0 = \hat{\theta}.\hat{z}$$

Thus
$$(\bar{\xi}.\nabla)\bar{q} = \frac{1}{r}\left(\frac{\partial q_r}{\partial z} - \frac{\partial q_z}{\partial r}\right)\frac{\partial}{\partial \theta}(q_r\hat{r} + q_z\hat{z})$$

$$= \frac{q_r}{r}\left[\frac{\partial q_r}{\partial z} - \frac{\partial q_z}{\partial r}\right]\frac{\partial \hat{r}}{\partial \theta}$$

$$= \frac{q_r}{r}\left(\frac{\partial q_r}{\partial z} - \frac{\partial q_z}{\partial r}\right)\hat{\theta} \qquad \Big|\ \frac{\partial \hat{r}}{\partial \theta} = \hat{\theta},\ \ \frac{\partial \hat{z}}{\partial \theta} = 0 \qquad (3)$$

Hence $(\bar{\xi}.\nabla)\,\bar{q} = \frac{qr}{r}\bar{\xi}$ | using (2) (4)

$\therefore$ From (1) & (4), we get

$$\frac{D\bar{\xi}}{Dt} = \frac{qr}{r}\bar{\xi} \qquad (5)$$

Now, $q_r = \bar{q}.\hat{r}$, so equation (5) becomes

$$r\frac{D\bar{\xi}}{Dt} = \bar{q}.\hat{r}\,\bar{\xi} \qquad (6)$$

Since $\bar{r}^2 = r^2 \Rightarrow \bar{r}.\frac{D\bar{r}}{Dt} = r\frac{Dr}{Dt}$

$\Rightarrow$ $\frac{\bar{r}}{r}.\frac{D\bar{r}}{Dt} = \frac{Dr}{Dt} \Rightarrow \hat{r}.\frac{D\bar{r}}{Dt} = \frac{Dr}{Dt} \Rightarrow \hat{r}.\bar{q} = \frac{Dr}{Dt}$

Using this in (6), we get

$$r\frac{D\bar{\xi}}{Dt} = \frac{Dr}{Dt}\bar{\xi}$$

$\Rightarrow$ $r\frac{D\bar{\xi}}{Dt} - \bar{\xi}\,\frac{Dr}{Dt} = 0 \qquad \Rightarrow \frac{r\frac{D\bar{\xi}}{Dt} - \bar{\xi}\frac{Dr}{Dt}}{r^2} = 0$

$\Rightarrow$ $\frac{D}{Dt}\left(\frac{\bar{\xi}}{r}\right) = 0 \Rightarrow \frac{\bar{\xi}}{r} = \text{const} = \frac{\xi_0}{r_0}$

$$\Rightarrow \frac{r^2}{\xi} = \left(\frac{\xi_0}{r_0}\right) r$$

Hence the result.

2. Vorticity in Two-dimensions

For an incompressible fluid in the xy-plane, we have

$$\bar{q} = (u, v, 0), \quad \nabla = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, 0\right)$$

Therefore, $\bar{\xi} = \nabla \times \bar{q} = (0, 0, \dfrac{\partial v}{\partial x} - \dfrac{\partial u}{\partial y})$

$$= \hat{k}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)$$

which shows that in two-dimensional flow, the vorticity vector is perpendicular to the plane of flow.

Also, $\xi = |\bar{\xi}| = \dfrac{\partial v}{\partial x} - \dfrac{\partial u}{\partial y}$

Thus $\bar{\xi} = \hat{k}\, \xi$

Now, for this case, the Helmholtz's vorticity equation

$$\frac{d\bar{\xi}}{dt} = (\bar{\xi} \cdot \nabla)\bar{q} \text{ gives}$$

$$\frac{d\bar{\xi}}{dt} = 0 \Rightarrow \bar{\xi} = \text{constant}$$

i.e. ξ = constant.

which shows that in the two-dimensional motion of an incompressible fluid, the vorticity of any particle remains constant.

Here, we may regard ξ as a vortex strength per unit area.

Also, in terms of stream function, we have

$$u = -\frac{\partial \psi}{\partial y}, \quad v = \frac{\partial \psi}{\partial x}$$

Therefore, $\bar{\xi} = \hat{k}\left(\dfrac{\partial^2 \psi}{\partial x^2} + \dfrac{\partial^2 \psi}{\partial y^2}\right) = \hat{k}\ \nabla^2 \psi$

i.e. $\xi = \nabla^2 \psi$

This gives vorticity in terms of the stream function.

2.1. Circular Vortex. The section of a cylindrical vortex tube whose cross-section is a circle of radius a, by the plane of motion is a circle and the liquid inside such a tube is said to form a circular vortex.

If ω is the angular velocity and πa^2 the cross-sectional area of the vortex tube, then circulation

$$\Gamma = \oint_C \bar{q}\cdot d\bar{r} = \int_S \text{curl}\,\bar{q}\cdot\hat{n}\,dS = \int_S \text{curl}\,\bar{q}\cdot d\bar{S}$$

$$= \omega \int_S dS = \omega\pi a^2 = k(\text{say})$$

This product of the cross-section and angular velocity at any point of the vortex tube is constant along the vortex and is known as the strength of the circular vortex.

2.2.Rectilinear or Columnar Vortex Filament. The strength k of circular vortex is given by $k = \omega\pi a^2$. If we let $a\to 0$ and $\omega\to\infty$ such that the product $\omega\pi^2$ a remains constant, we get a rectilinear vortex filament and represent it by a point in the plane of motion. Such vortex filament may be regarded as straight gravitating rod of fluid lying perpendicular to the plane of flow. It is also termed as a uniform line vortex. The strength of a vortex filament is positive when the circulation round it is anticlockwise and negative when clockwise.

2.3. Different Types of Vortices. We may divide vortices into the following four types

(i) **Forced vortex** in which the fluid rotates as a rigid body with constant angular velocity.

(ii) **Free cylindrical vortex** for which the fluid moves along streamlines which are concentric circles in horizontal planes and there is no variation of total energy with radius.

(iii) **Free spiral vortex** which is a combination the free cylindrical vortex and a source (radial flow)

(iv) **Compound vortex** in which the fluid rotates as a forced vortex at the centre and as a free vortex outside.

2.4. Complex Potential for Circulation about a Circular Cylinder (Circular vortex). In case of a doubly connected region, the possibility of cyclic motion does exist and as such we proceed to explain it presently in the case of circle.

If the circulation in a closed circuit is $2\pi k$, then k is called the strength of the circulation.

Let us consider the complex potential

$$W = \phi + i\psi = ik \log z \qquad (1)$$

On the circular cylinder $|z| = a$, $z = a\, e^{i\theta}$

Thus, $W = ik \log (a\, e^{i\theta}) = ik (\log a + i\theta)$

i.e. $\quad \phi + i\psi = -k\theta + ik \log a$

$\Rightarrow \quad \phi = -k\theta,\ \psi = k \log a = \text{constant}.$

This shows that the circular cylinder is a streamline and thus equation (1) gives the required complex potential for circulation about a circular cylinder.

When the fluid moves once round the cylinder in the positive sense, θ increases by 2π and then

$$\phi_1 = -k(\theta + 2\pi) = -k\theta - 2\pi k$$

$$= \phi - 2\pi k$$

Therefore, circulation $= 2\pi k = \phi - \phi_1$

$\qquad$ = decrease in ϕ moving once round the circuit.

Hence there is a circulation of amount $2\pi k$ about the cylinder.

Also, $\quad \dfrac{dw}{dz} = \dfrac{ik}{z}$

$\Rightarrow \quad q = \left| -\dfrac{dW}{dz} \right| = \dfrac{k}{r}$

i.e. $k = rq$

Therefore, $\quad k = q$ when $r = 1$

Thus k is the speed at unit distance from the origin.

2.5. Complex Potential for Rectilinear Vortex (Line Vortex). Let us consider a cylindrical vortex tube whose cross-section is a circle of radius a; surrounded by infinite mass of liquid. We assume that vorticity over the area of the circle is constant and is zero outside the circle.

Let ψ be the stream function, then

$$\bar{\xi} = \nabla^2 \psi \, \hat{k}$$

i.e. $$\xi = \nabla^2 \psi = \frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2}$$

$$= \frac{\partial^2 \psi}{\partial r^2} + \frac{1}{r}\frac{\partial \psi}{\partial r} + \frac{1}{r^2}\frac{\partial^2 \psi}{\partial \theta^2}$$

Since there is a symmetry about the origin ψ is a function of r only and so $\frac{\partial^2 \psi}{\partial \theta^2} = 0$.

$\therefore$ $$\xi = \frac{1}{r}\frac{d}{dr}\left(r\frac{d\psi}{dr}\right), \text{ for } r < a$$

$$= 0, \text{ for } r > a$$

i.e. $$\frac{d}{dr}\left(r\frac{d\psi}{dr}\right) = r\xi, \text{ for } r < a$$

$$= 0, \text{ for } r > a$$

Integrating, we find

$$r\frac{d\psi}{dr} = \xi\frac{r^2}{2} + A, \text{ for } r < a$$

$$= B, \text{ for } r > a$$

We are interested in the fluid motion outside the cylinder $|z| = a$. Therefore, integrating the second of the above result, we get

$$\psi = B \log r + C, \text{ for } r > a.$$

The constant C may be chosen to be zero. Further, for $r > a$, the vorticity is zero and the fluid motion is irrotational, therefore velocity potential ϕ exists and is related to ψ as

$$-\frac{1}{r}\frac{\partial \phi}{\partial \theta} = \frac{\partial \psi}{\partial r} = \frac{B}{r}$$

$\Rightarrow$ $$\phi = -B\theta + D$$

$\Rightarrow \qquad \phi = -B\theta$, neglecting D

Let k be the circulation while moving once round the cylinder, then

$$k = \text{decrease in value of } \phi \text{ on describing the circuit once}$$

$$= -B\,[\theta-(\theta+2\pi)] = 2\pi B$$

$\Rightarrow \qquad B = k/2\pi = K(\text{say})$

Thus, $\phi = -K\theta$ and $\psi = ZK \log r$

Hence $W = \phi + i\psi = -k\theta + iK \log r$

$$= iK(\log r + i\theta)$$

$$= iK \log z = i\frac{k}{2\pi}\log z.$$

If the rectilinear vortex is situated at the point $z = z_0$, then by shifting the origin, we get

$$W = iK \log(z-z_0)$$

If there are vortices of strengths $K_1, K_2, \ldots K_n$ situated at $z_1, z_2, \ldots, z_n$ respectively, then the complex potential is

$$W = iK_1 \log(z-z_1) + iK_2 \log(z-z_2) + \ldots + iK_n \log(z-z_n).$$

2.6. Remarks (i) By a vortex, we mean a rectilinear vortex or line vortex.

(ii) $K = k/2\pi$, where K is the strength of a vortex and k that of circulation

2.7. Complex Potential for a Spiral Vortex. The combination of a source and a vortex is called a spiral vortex or a **vortex source.**

Let us consider a source of strength m and a vortex of strength K both at the origin. Then the complex potential is

$$W = -m \log z + iK \log z$$

$$= (-m + iK)\log z = (-m + iK)\log(re^{i\theta})$$

$$= (-m + iK)(\log r + i\theta)$$

$$\Rightarrow \qquad \phi + i\psi = -m \log r - K\theta + i(-m\theta + K \log r)$$

Therefore, $\qquad \phi = -(m \log r + K\theta),\ \psi = -m\theta + K \log r$

If we go once round the origin, then ϕ decrease by $2\pi K$ and ψ be $2\pi m$.

2.8. Example. Find the complex potential for the motion due to a system consisting of a coincident line-source of strength m per unit length and line-vortex of strength K per unit length in the presence of a circular cylinder of radius a, whose axis is parallel to and at a distance b(> a) from the line of the source and vortex. Show that the cylinder is attracted by a force of magnitude

$$2\pi\rho\, a^2 (m^2 + K^2)/ b(b^2-a^2)$$

per unit length.

Solution. We suppose the line-source and line-vortex to be at the origin, then the complex potential is

$$W = -m \log z + iK\log z = (iK-m) \log z \qquad (1)$$

When the circular cylinder $|z-b| = a$ $(b > a)$ is inserted, the complex potential, by circle theorem, becomes

$$W = (iK-m) \log z + (-iK-m) \log \left(\frac{a^2}{z-b} + b\right) \qquad (2)$$

where

$$|z-b| = a \Rightarrow (\bar{z} - b)(z-b) = a^2$$

$$\Rightarrow \qquad \bar{z} - b = \frac{a^2}{b} \Rightarrow \bar{z} = \frac{a^2}{z-b} + b$$

By Blasius theorem, force on the cylinder C is given by

$$X - iY = -\pi\rho\left[\text{ sum of residues of } \left(\frac{dW}{dz}\right)^2 \text{ within C }\right] \qquad (3)$$

$$\text{Now } \left(\frac{dW}{dz}\right)^2 = \left\{\frac{iK-m}{z} - \frac{iK+m}{a^2+b(z-b)}\frac{a^2}{z-b}\right\}^2 \qquad (4)$$

The only singularities of $\left(\frac{dW}{dz}\right)^2$ within C are at z = b and z = $b - \frac{a^2}{b}$ since z = 0 is not inside C.

Now,

$$\text{residue } (z = b) = -2\left(\frac{K^2+m^2}{b}\right) \qquad \Big|\text{Only product term of (4)}$$

and will contribute

$$\text{residue } \left(z = b - \frac{a^2}{b}\right) = \frac{2(K^2+m^2)b}{(b^2-a^2)}\ .$$

Therefore, from (3), we get

$$X - iY = -2\pi\rho\,(K^2+m^2)\left[\frac{b}{b^2-a^2} - \frac{1}{b}\right]$$

$$= -2\pi\rho a^2(K^2+m^2)/b(b^2-a^2)$$

Thus

$$Y = 0, \quad X = -2\pi\rho a^2\,(K^2+m^2)/b(b^2-a^2)\ .$$

The negative sign implies that the cylinder is attracted towards the origin where the spiral vortex is situated.

2.9. Complex Potential for a Vortex Doublet. Two equal and opposite vortices placed at small distance apart, form a vortex doublet.

Let us consider a vortex of strength K at $z = ae^{i\alpha}$ and another vortex of strength –K at z = 0, then the complex potential is

$$W = iK \log(z - ae^{i\alpha}) - iK \log z$$

$$= iK \log\left(\frac{z - ae^{i\alpha}}{z}\right) = iK \log\left(1 - \frac{ae^{i\alpha}}{z}\right)$$

$$= -iK\left(\frac{ae^{i\alpha}}{z} + \frac{a^2e^{2i\alpha}}{2z^2} + \ldots\ldots\right)$$

As $a \to 0$, $K \to \infty$, then $Ka \to \mu$ and we obtain

$$W = \frac{-i\mu e^{i\alpha}}{z} = \frac{\mu e^{i(\alpha-\pi/2)}}{z}$$

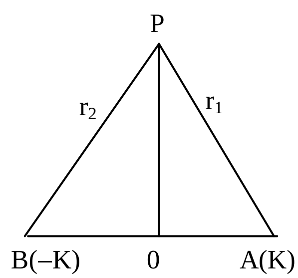
This is the required complex potential for a vortex doublet at the origin.

Also, we note that the complex potential for a doublet at the origin is $\frac{\mu e^{i\alpha}}{z}$. Thus, it follows that the complex potential of a vortex doublet is the same as that for a doublet with its axes rotated through a right angle.

2.10. Image of Vortex in a Plane. Let us consider two line vortices of strengths K and –K per unit length at $A(z = z_1)$ and $B(z = z_2)$ respectively. The complex potential due to these line vortices is

$$W = \phi + i\psi = iK \log(z-z_1) - iK \log(z-z_2)$$

$$\Rightarrow \qquad \psi = K \log \left| \frac{z-z_1}{z-z_2} \right| = K \log \frac{r_1}{r_2}$$



OA = OB

$|z-z_1| = r_1$, $|z-z_2| = r_2$

If $r_1 = r_2$, then $\psi = K \log 1 = 0$

Thus the plane boundary OP is a streamline so that there is no flow across OP. Hence the line vortex at B with strength –K per unit length is the image of the line vortex at A with strength K per unit length so that A and B are at equal distances from OP.

2.11. Remark. In case of two dimensions (as for sources, sinks and doublets), a vortex means a line vortex and strength means strength per unit length.

2.12. Image of a Vortex in a Circular Cylinder (or in a circle). Let a vortex of strength k be present at z = d, then the complex potential is iK log (z–d). When the cylinder $|z| = a$ is introduced into the fluid, the complex potential, by circle theorem, becomes

$$W = iK \log(z-d) - iK \log\left(\frac{a^2}{z} - d\right)$$

i.e. $$\phi + i\psi = iK \log(z-d) - iK \log\left(z - \frac{a^2}{d}\right) + iK \log z + \text{constant} \qquad (1)$$

$$= iK\left[\log\{(x-d)^2+y^2\}^{1/2}+i\tan^{-1}\left(\frac{y}{x-d}\right)\right]$$

$$-iK\left[\log\left\{\left(x-\frac{a^2}{d}\right)^2+y^2\right\}^{1/2}+i\tan^{-1}\left(\frac{y}{x-\frac{a^2}{d}}\right)\right]$$

$$+iK\left[\log\sqrt{x^2+y^2}+i\tan^{-1}\frac{y}{x}\right]$$

where we have ignored the constant term

$$\Rightarrow \quad \psi = K \log\left[\frac{a\{(a\cos\theta-d)^2+a^2\sin^2\theta\}^{1/2}}{\left\{\left(a\cos\theta-\frac{a^2}{d}\right)^2+a^2\sin^2\theta\right\}^{1/2}}\right]$$

$$\left| \begin{array}{l} |z|=a \quad \Rightarrow z=ae^{i\theta} \\ \Rightarrow x=a\cos\theta, \\ y=a\sin\theta \end{array} \right.$$

$$= K \log d = \text{constant}.$$

This shows that the cylinder is a streamline. Thus (1) represents the complex potential of the fluid motion. From (1), we observe that the image of a vortex of strength K at z = d is a vortex of strength –K at the inverse point $z = a^2/d$ together with a vortex of strength K at z = 0 i.e. centre of the circle.

2.13. Circulation about a Circular Cylinder in a Uniform Stream. Let a liquid be in motion with a velocity –U along the x-axis. The complex potential due to the stream is Uz. If the circular cylinder of radius a is introduced inside the liquid, then the complex potential, by circle theorem, becomes $Uz + U\frac{a^2}{z}$.

Let there be a circulation k about the cylinder. The complex potential due to circulation is ik log z. Thus the complex potential of the whole system is

$$W = Uz + U\frac{a^2}{z} + ik\log z. \qquad (1)$$

$$\Rightarrow \qquad -\bar{q} = \frac{dW}{dz} = U - U\frac{a^2}{z^2} + \frac{ik}{z}$$

At the stagnation points, $\bar{q} = 0$ i.e. $q = 0$

$$\Rightarrow \qquad U - \frac{Ua^2}{z^2} + \frac{ik}{z} = 0$$

$$\Rightarrow \qquad Uz^2 + ikz - U a^2 = 0$$

$$\Rightarrow \qquad z = -\frac{ik}{2U} \pm a\sqrt{1 - \frac{k^2}{4a^2U^2}}$$

Since a and U are constants, therefore the flow potential term depends very much on the magnitude of k. We shall consider three cases.

Case I. When $k < 2aU$ i.e. $\frac{k^2}{4a^2U^2} < 1$, we put

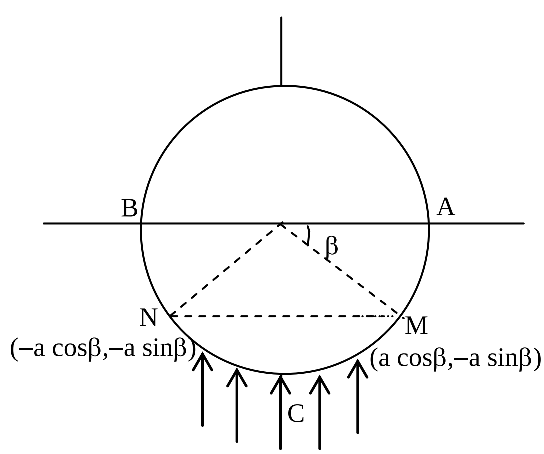
$$\frac{k^2}{4a^2U^2} = \sin^2\beta \text{ and then}$$

$$z = -ia\sin\beta \pm a\cos\beta$$

Thus the stagnation points are $(a\cos\beta, -a\sin\beta)$ and $(-a\cos\beta, -a\sin\beta)$

Further $|z| = a\,|\pm\cos\beta - i\sin\beta| = a$

$\therefore$ The stagnation points lie on the boundary of the cylinder. They lie on the line MN below the diameter AB as shown in the fig. The velocity increases above MN and decreases below MN.



Further, from Bernoull's equation,

$$\frac{p}{\rho}+\frac{1}{2}q^2=\text{constant}$$

we observe that the pressure decreases above MN and increases below MN.

Thus, there is an increase of pressure by the liquid due to circulation. If there is no circulation, then $k = 0 \Rightarrow \sin\beta=0$

$\Rightarrow \qquad \beta = 0, \pi, z = \pm a$

Therefore, MN coincides with AB and thus the stagnation points are at A and B. Therefore we conclude that the circulation brings the stagnation points downwards and put an upward thrust on the cylinder.

Case II. When $k = 2aU$ i.e. $\dfrac{k^2}{4a^2U^2}=1$, then $\sin\beta = 1$

$\Rightarrow \qquad \beta = \pi/2, \; z = -ia \;\; \Rightarrow |z| = a$

and thus the stagnation points coincide at C, the bottom of the cylinder.

Case III. When $k > 2aU$ i.e. $\dfrac{k^2}{4a^2U^2} > 1$, then we put $\dfrac{k^2}{4a^2U^2} = \cosh^2\beta$ so that

$$z = a(-i\cosh\beta \pm \sinh\beta)$$

$$= -ia\,e^{\beta}, -i\,a\,e^{-\beta}$$

$\therefore$ The stagnation points lie on y-axis.

Further $|(-iae^{\beta})(-iae^{-\beta})| = a^2$

this shows that the stagnation points are inverse points w.r.t. the circular boundary of the cylinder. One of these points lie inside and other is outside the cylinder. The point which is inside the cylinder does not belong to the motion.

$$\left|\begin{array}{l} |z_1|=|-iae^{\beta}|=ae^{\beta}, \text{outside the circle} \\ |z_2|=|-iae^{-\beta}|=ae^{-\beta}, \text{inside the circle} \end{array}\right.$$

since $ae^{-\beta} < ae^{\beta}$.

We know that at the **stagnation points (critical points),** there are two branches of the streamlines which are at right angles to each other. Thus the liquid inside the loop formed at the stagnation points will not be carried by the stream but will circulate round the cylinder

Pressure (Force) on the circular cylinder :- From (1), we have

$$\frac{dW}{dz} = U - \frac{Ua^2}{z^2} + \frac{ik}{z}$$

Therefore, by Blasius theorem,

$$X - iy = \frac{i\rho}{2}\int_C \left(\frac{dW}{dz}\right)^2 dz.$$

$$= -\pi\rho \text{ (sum of the residues of } \left(\frac{dW}{dz}\right)^2 \text{ within the circle } |z| = a)$$

$$\left|\ \text{By Cauchy's Residue theorem as } \left(\frac{dW}{dz}\right)^2 \text{ is a meromorphic function}\right.$$

where X, Y are components of the pressure of the liquid and ρ is the density of the liquid

Now, $$\left(\frac{dW}{dz}\right)^2 = U^2\left(1-\frac{a^2}{z^2}\right)^2 + \frac{2ikU}{z}\left(1-\frac{a^2}{z^2}\right) - \frac{k^2}{z^2}$$

The only pole inside the cylinder $|z| = a$ is $z = 0$ i.e. a simple pole. The residue at $z = 0$ is $2ikU$

Therefore, $$X - iy = -\pi\rho(2i\,kU)$$

$$\Rightarrow \quad X = 0,\ Y = 2\pi k\rho U$$

This represents an upward thrust on the cylinder due to circulation. The lifting tendency ($k \neq 0$) is called the **Magnus effect.** The moment M is obtained to be **zero**, since residue is zero in that case.

2.14. Exercise. Show that the complex potential

$W = U\left(z + \frac{a^2}{z}\right) + ik \log z$ represents a possible flow part a circular cylinder. Sketch the streamlines, find the stagnation points and calculate the force on the cylinder.

2.15. Example. Verify that $W = iK \log\left(\frac{z - ia}{z + ia}\right)$, K and a both real, is the complex potential of a steady flow of liquid about a circular cylinder, the plane y = 0 being a rigid boundary. Find the force exerted by the liquid on unit length of the cylinder.

Solution. Putting $W = \phi + i\psi$, we get

$$\phi + i\psi = iK \log \frac{z - ia}{z + ia}$$

$$= ik\left[\log\frac{|z - ia|}{|z + ia|} + i\tan^{-1}\frac{y - a}{x} - i\tan^{-1}\frac{y + a}{x}\right]$$

$$\Rightarrow \qquad \psi = K \log \frac{|z - ia|}{|z + ia|}$$

The streamlines ψ = constant are given by

$$\frac{|z - ia|}{|z + ia|} = \text{constant} = \lambda \text{ (say)}$$

For $\lambda \neq 1$, these are non-intersecting coaxial circles having $z = \pm\, ia$ as the limiting points i.e. circles of zero radius. In particular, for $\lambda = 1$, we get a streamline which is the perpendicular bisector of the line segment joining the points $\pm\, ia$ and it is the radical axis of the coaxial system. No fluid crosses a streamline and so a rigid boundary may be introduced along any circle λ = constant of the coaxial system, including the perpendicular bisector $\lambda = 1$

We note that for $\lambda = 1$, $|z - ia| = |z + ia|$

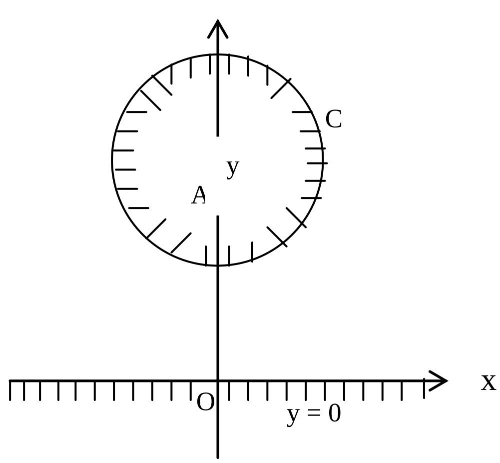
$$\Rightarrow \qquad x^2 + (y-a)^2 = x^2 + (y+a)^2 \quad \Rightarrow y = 0$$

Hence we can introduce rigid boundaries along

(i) a particular circle λ = constant ($\neq 1$)

(ii) along the plane $y = 0$ ($\lambda = 1$)

and this establishes the result of the first part of the question. The circular section C of the cylinder and the rigid plane $y = 0$ are shown in the fig. Circle C is any member of the above mentioned λ-system of coaxial circles and it encloses the point A(0, a) whereas the point B(0, –a) is external to it.



Since W = iK [log (z − ia) − log (z + ia)]

$$\Rightarrow \qquad W' = \frac{dW}{dz} = iK\left(\frac{1}{z-ia} - \frac{1}{z+ia}\right)$$

Therefore, by Blasis theorem,

$$X - iY = \frac{i\rho}{2}\int_C \left(\frac{dW}{dz}\right)^2 dz$$

i.e.

$$X - iY = \frac{i\rho}{2}\int_C \left[iK\left(\frac{1}{z-ia} - \frac{1}{z+ia}\right)\right]^2 dz$$

$$= \frac{-iK^2\rho}{2}\int_C \left[\frac{1}{(z-ia)^2} + \frac{1}{(z+ia)^2} - \frac{2}{(z-ia)(z+ia)}\right] dz$$

The integrand has double poles at z = $\pm$ ia. Out of these poles only z = ia lies within C. Thus, we find residue at z = ia. It is only the last term of the integrand which gives a non-zero contribution to the contour integral and the appropriate residue at z = ia is

$$\operatorname*{Lt}_{z\to ia}\left[(z-ia)\frac{-2}{(z-ia)(z+ia)}\right] = \frac{-2}{2ia} = \frac{-1}{ia} = \frac{i}{a}$$

Hence by Cauchy-Residue theorem, we get

$$X - iY = \frac{-K^2\rho}{2}\left[(2\pi i)\frac{i}{a}\right] = \frac{i\pi K^2\rho}{a}$$

$$\Rightarrow \qquad X = 0,\ Y = \frac{-\pi K^2\rho}{a}$$

which shows that the liquid exerts a downward force on the cylinder of amount $\frac{\pi K^2\rho}{a}$ per unit length. In case of moment M, the sum of residues is obtained to be zero and thus M = 0

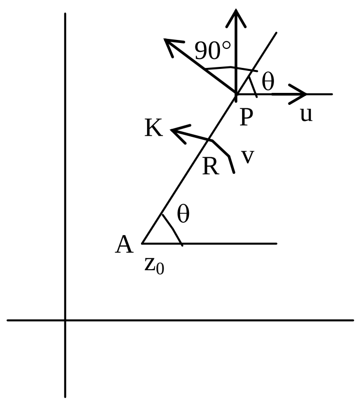
2.16. Motion of a Vortex Filament. We find the velocity of the point P(z) due to a vortex filament K at $z = z_0$. We know that, the complex potential is

$$W = iK \log (z - z_0)$$

$$\Rightarrow \qquad \bar{q} = -\frac{dW}{dz} = -\frac{iK}{z - z_0} = \frac{-iK}{Re^{i\theta}} = \frac{-iK}{R} e^{-i\theta}$$

where $z - z_0 = Re^{i\theta}$.

$$AP = R, \arg (z - z_0) = \theta$$



$$\therefore \qquad u - iv = \frac{-iK}{Re} e^{-i\theta}$$

$$= \frac{-iK}{R}(\cos\theta - i\sin\theta)$$

$$\Rightarrow \qquad U = \frac{-K}{R}\sin\theta, \quad v = \frac{K}{R}\cos\theta, \; q = \frac{K}{R}$$

Therefore, $\qquad \dfrac{v}{u} = -\cot\theta = \tan(90 + \theta)$

Thus, the direction of motion at P is perpendicular to AP with speed K/R in the sense given by the rotation of the vortex at A.

3. Motion of Rectilinear Vortex (Line Vortex)

The stream function ψ at a distance r < a (the radius of a cylindrical vortex) is determined by $\quad \xi = \nabla^2\psi$. Using polar co-ordinates, we get

$$\nabla^2\psi = \frac{d^2\psi}{dr^2}+\frac{1}{r}\frac{d\psi}{dr},$$ where ψ is a function of r only, due to symmetry $\left(\text{i.e.}\frac{d^2\psi}{d\theta^2}=0\right)$

Thus, we get

$$\nabla^2\psi = \frac{1}{r}\frac{d}{dr}\left(r\frac{d\psi}{dr}\right) = \xi,\ r < a \qquad (1)$$

Integrating (1) and noting that ξ is constant, we obtain

$$\frac{d\psi}{dr} = \frac{1}{2}r\xi + \frac{A}{r} \qquad (2)$$

But the radial and transverse components of velocity are

$$q_r = -\frac{1}{r}\frac{\partial\psi}{\partial\theta}, \qquad q_\theta = \frac{\partial\psi}{\partial r}$$

$$\therefore \qquad q_r = 0, \qquad q_\theta = \frac{1}{2}r\xi + \frac{A}{r}$$

The velocity cannot be infinite at the origin (r = 0) and so A = 0

Therefore, $\qquad q_\theta = \frac{1}{2}r\xi = 0$ at $r = 0$

Thus there is no motion at the centre of a circular vortex. Therefore, in case of a rectilinear vortex (line vortex), its motion is not due to itself but due to the presence of other vortices. Thus, if motion is due to n vortices of strengths K_s at the points z_s ($s = 1, 2, \ldots, n$), then the complex potential at a point P(z), not occupied by any vortex, is

$$W = \sum_{s=1}^{n} i\, K_s \log(z - z_s) \qquad (3)$$

and the complex velocity is given by

$$u - iv = -\frac{dW}{dz} = -\sum_{s=1}^{n} i\left(\frac{K_s}{z - z_s}\right) \qquad (4)$$

Further, the complex velocity of the vortex of strength K_r, which is produced only by the other vortices, is

$$u_r - i\,v_r = -\sum_{s=1}^{n} i\left(\frac{K_s}{z_r - z_s}\right), \text{ where } s \neq r. \qquad (5)$$

The result (5) is practically obtained as

$$W' = W - iK_r \log (z - z_r)$$

so that

$$u_r - iv_r = \left(-\frac{dW'}{dz}\right)_{z=z_r} = \left(-\frac{dW}{dz} + \frac{iK_r}{z - z_r}\right)_{z=z_r}$$

3.1.Centroid of Vortices. Let there be two vortices of strengths K_1 and K_2 at points $A(z = z_1)$ and $B(z = z_2)$ respectively, then

$$W = iK_1 \log (z - z_1) + iK_2 \log (z - z_2)$$

The velocity of A is due to the presence of other vortex at B and vice-versa. Thus

$$\dot{\bar{z}}_1 = \left(-\frac{dW}{dz}\right)_{z=z_1} = \frac{-iK_2}{z_1 - z_2}$$

and

$$\dot{\bar{z}}_2 = \left(-\frac{dW}{dz}\right)_{z=z_2} = \frac{-iK_1}{z_2 - z_1} = \frac{iK_1}{z_1 - z_2}$$

Therefore,

$$K_1 \dot{\bar{z}}_1 + K_2 \dot{\bar{z}}_2 = \frac{iK_1K_2}{z_1 - z_2} + \frac{iK_1K_2}{z_1 - z_2} = 0$$

or $$\frac{K_1\dot{\bar{z}}_1 + K_2\dot{\bar{z}}_2}{K_1 + K_2} = 0 \quad \text{i.e.} \quad \frac{d}{dt}\left(\frac{K_1 z_1 + K_2 z_2}{K_1 + K_2}\right) = 0$$

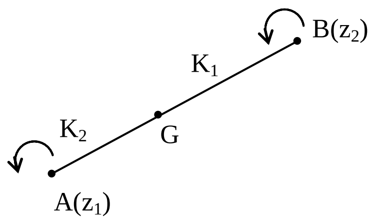
Integrating, we get

$$\frac{K_1 z_1 + K_2 z_2}{K_1 + K_2} = \text{constant}$$

The point $\dfrac{K_1z_1 + K_2z_2}{K_1 + K_2}$ divides AB in the ratio $K_2 : K_1$. This point remains fixed (not necessarily a stagnation point) and is called the centroid G of the vortices at A and B.

Further

$$\frac{AG}{K_2} = \frac{GB}{K_1} = \frac{AB}{K_1 + K_2}$$



Therefore, $\quad AG = \dfrac{K_2}{K_1 + K_2} AB$

$$v = r\ \omega = r\frac{d\theta}{dt}$$

The velocity of A is

$$|u_1 - iv_1| = \frac{K_2}{AB}$$

$$= \frac{K_2 AB}{K_1 + K_2} \cdot \frac{K_1 + K_2}{(AB)^2} = AG.\omega$$

where $\quad \omega = \dfrac{K_1 + K_2}{(AB)^2}$

Thus, A moves with a velocity AG.ω perpendicular to AG. Similarly B moves with a velocity GB.ω perpendicular to GB. So AB rotates with an angular velocity ω. Further, neither vortex has a component of velocity along AB, it follows that AB remains constant in length.

3.2. Vortex Pair. A pair of vortices of equal and opposite strengths is called a vortex pair.

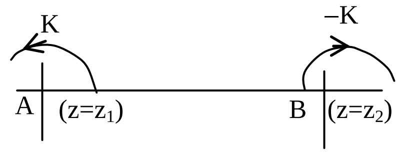
Let K and –K be the strengths of the two vortices at $A(z = z_1)$ and $B(z = z_2)$ respectively. Then the complex potential is

$$W = iK \log (z-z_1) - iK \log (z-z_2)$$

$$= W_1 + W_2 \text{ (say)}$$

The velocity at A is due to the presence of the vortex at B and vice-versa.

Therefore the velocity at A is given by



$$u_1 - iv_1 = \left(-\frac{dW_2}{dz}\right)_{z=z_1} = \frac{iK}{z_1 - z_2}$$

Similarly, the velocity at B is

$$u_2 - iv_2 = \left(-\frac{dW_1}{dz}\right)_{z=z_2}$$

$$= \frac{-iK}{z_2 - z_1} = \frac{iK}{z_1 - z_2}$$

$\Rightarrow$ $$q_1 = |u_1 - iv_1| = \frac{K}{AB},$$

$$q_2 = |u_2 - iv_2| = \frac{K}{AB} \qquad |\ |z_1 - z_2| = AB$$

Therefore, both the vortices have the same velocity.

Further, $W = iK \log \dfrac{z - z_1}{z - z_2}$

$\Rightarrow$ $$\phi + i\psi = iK\left[\log\left|\frac{z - z_1}{z - z_2}\right| + i(\theta_1 - \theta_2)\right] \qquad \left|\ \theta = \tan^{-1}\frac{y}{x}\right.$$

$\Rightarrow$ $$\psi = K \log \left|\frac{z - z_1}{z - z_2}\right| = K \log \frac{r_1}{r_2}$$

Therefore, the streamlines, ψ = constant, are $\dfrac{r_1}{r_2}$ = constant.

which are co-axial circles.

Thus the streamlines in case of a vortex pair are co-axial circles which have A and B as limiting points.

3.3. Example. A vortex of circulation $2\pi k$ is at rest at the point $z = na$ $(n > 1)$, in the presence of a plane circular impermeable boundary $|z| = a$, around which there is circulation $2\pi\lambda k$. Show that

$$\lambda = \frac{1}{n^2 - 1}$$

Show that there are two stagnation points on the circular boundary $z = ae^{i\theta}$ symmetrically placed about the real axis in the quadrants nearest to the vortex given by

$$\cos\theta = (3n^2 - 1)/2n^3.$$

and prove that θ is real.

Solution. The circulation of vortex is $2\pi k$ and thus the strength of vortex is k

Therefore, complex potential due to the vortex is

$$f(z) = ik \log (z - na)$$

$\Rightarrow$ $\bar{f}(z) = -ik \log (z - na)$ | k,n, a and the function form are real.

$$\Rightarrow \qquad \bar{f}(a^2/z) = -ik \log \left(\frac{a^2}{z} - na\right)$$

The complex potential, when the circular cylinder $|z| = a$ is introduced into the fluid, becomes $f(z) + \bar{f}(a^2/z)$, by circle theorem.

Now, there is a circulation $2\pi\lambda k$ around the cylinder. This is equivalent to the line vortex at $z = 0$ of strength λk.

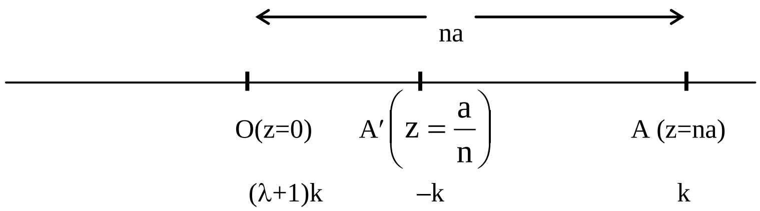
Thus the total complex potential is

$$W = ik \log (z - na) - ik \log \left(\frac{a^2}{z} - na\right) + i\lambda k \log z$$

$$= ik \log (z - na) - ik \log \left(z - \frac{a}{n}\right) + i\lambda k \log z + ik \log z + \text{constant.}$$

$$= ik \log (z - na) - ik \log \left(z - \frac{a}{n}\right) + ik\ (\lambda + 1) \log z + \text{constant} \qquad (1)$$

This is equivalent to the complex potential due to a vortex of strength k at $z = na$, $-k$ at $z = a/n$ and $(\lambda + 1)k$ at $z = 0$ as shown in the figure



The velocity at point A is due to the motion of other two vortices (i.e. excluding first term in (1))

Therefore,

$$\left(\frac{dW}{dz}\right)_{z=na} = \left(\frac{-ik}{na - \frac{a}{n}}\right) + \left(\frac{ik(\lambda+1)}{na}\right)$$

(Differentiating (1) and put z = an excluding Ist term of (1))

The vortex at A is at rest if

$$\left|\frac{dW}{dz}\right|_{z=na} = 0 \quad \Rightarrow \frac{k(\lambda+1)}{na} - \frac{k}{na - \frac{a}{n}} = 0$$

$$\Rightarrow \qquad \lambda = \frac{1}{n^2 - 1}$$

Hence the result

Now, from (1), we get

$$\frac{dW}{dz} = ik\left[\frac{1}{z - na} - \frac{1}{z - \frac{a}{n}} + \frac{\lambda+1}{z}\right]$$

Putting $z = a\,e^{i\theta}$ and simplifying, we get

$$\frac{dW}{dz} = -\frac{ik\,e^{i\theta}}{a}\frac{2n^3\cos\theta - 3n^2 + 1}{(n^2 - 2n\cos\theta + 1)(n^2 - 1)} \qquad \left|\ \text{usng}\ \lambda = \frac{1}{n^2 - 1}\right.$$

The stagnation points on the circle, if any, are given by

$$\frac{dW}{dz} = 0 \text{ for } z = ae^{i\theta}$$

Thus

$$\frac{dW}{dz} = 0 \quad \Rightarrow 2n^3 \cos\theta - 3n^2+1 = 0$$

$$\Rightarrow \qquad \cos\theta = \frac{3n^2-1}{2n^3} \qquad (2)$$

Now, we know that $-1 \le \cos\theta \le 1$ i.e. $|\cos\theta| \le 1$ therefore R.H.S. of (2) must lie within these limits for θ to be real

Let us write

$$f(n) = \frac{3n^2-1}{2n^3} = \frac{3}{2n} - \frac{1}{2n^3}$$

Then

$$f(1) = 1\ , \text{ and also, } f'(n) = -\frac{3}{2n^2} + \frac{3}{2n^4}$$

$$= \frac{3}{2n^4}(1-n^2) < 0 \text{ for } n > 1$$

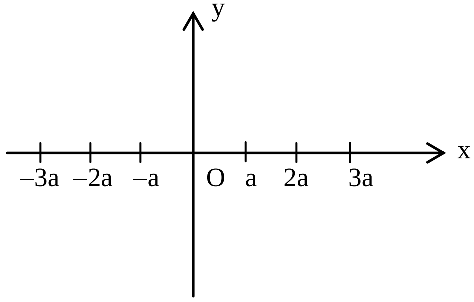
From here, we note that $f'(n) < 1$ for $n > 1$. Thus for $n > 1$, $f(n)$ decreases monotonically from 1 at $n = 1$ to 0 as $n \to \infty$. For all $n > 1$, real values of θ are obtained from (2). Two distinct values of θ are obtained for any given $n > 1$, one of the values is $\theta = \alpha$, where $0 < \alpha \le \pi/2$ and the other is $\theta = 2\pi - \alpha$. Hence the two stagnation points are symmetrically placed about the real axis in the quadrants nearest to the vortex.

4. Vortex Rows

When a body moves slowly through a liquid, rows of vortices are sometimes formed. There vortices can, when stable, be photographed. Here we consider infinite system of parallel line vortices and two dimensional flow will be presumed throughout.

4.1. Single Infinite Row of Vortices. We shall find the complex potential of an infinite row of parallel rectilinear vortices (line vortices) of same strength K and a distance ‘a’ apart.

First, let there be 2n+1 vortices with their centres on x-axis and the middle vortex having its centre at the origin. The vortices are placed at points $z = \pm\, na$, $n = 0, 1, 2, \ldots\ldots$, symmetrical about y-axis. The complex potential due to these vortices is



$$W = iK\log z + iK\log(z-a) + iK\log(z-2a) + \ldots + iK\log(z-na)$$

$$+\, iK\log(z+a) + iK\log(z+2a) + \ldots + iK\log(z+na)$$

$$= iK\log z\,(z^2-a^2)(z^2-2^2a^2)(z^2-3^2a^2)\ldots\ldots(z^2-n^2a^2)$$

$$= iK\log \frac{\pi z}{a}\left(1-\frac{z^2}{a^2}\right)\left(1-\frac{z^2}{2^2a^2}\right)\left(1-\frac{z^2}{3^2a^2}\right)\ldots\ldots\left(1-\frac{z^2}{n^2a^2}\right)$$

$$+\, iK\log \frac{a}{\pi}(-1)^n(a^2.\,2^2a^2.\,3^2a^2\ldots\ldots n^2a^2)$$

Ignoring the constant term and putting $\frac{\pi z}{a} = \theta,$ we get

$$W = iK\log\theta\left(1-\frac{\theta^2}{\pi^2}\right)\left(1-\frac{\theta^2}{n^2\pi^2}\right)\ldots\ldots\ldots\left(1-\frac{\theta^2}{n^2\pi^2}\right)$$

Making $n\to\infty$, we find

$$W = iK\log\sin\theta = iK\log\sin\frac{\pi z}{a} \qquad (1)$$

The velocity of the vortex at origin is given by

$$q_\theta = -\frac{d}{dz}\left[W - iK\log z\right]_{z=0} \qquad |\because \text{ The motion is due to other vortices}$$

$$= -\frac{d}{dz}\left[iK\log\sin\frac{\pi z}{a} - iK\log z\right]_{z=0}$$

$$= -iK\left[\frac{\pi}{a}\frac{\cos\frac{\pi z}{a}}{\sin\frac{\pi z}{a}} - \frac{1}{z}\right]_{z=0}$$

which is indeterminate form and $\to 0$ as $z \to 0$. Hence the velocity at z = 0 is zero. Similarly, all other vortices are at rest. Thus, the infinite row of vortices does not induce any velocity by itself.

Now, the velocity at any point of the fluid other than the vortices is given by

$$\bar{q} = u - iv = -\frac{dW}{dz} = \frac{-iK\pi}{a}\cot\frac{\pi z}{a}$$

$$= \frac{-iK\pi}{a}\cot\left[\frac{\pi}{a}(x+iy)\right] = \frac{-iK\pi}{a}\frac{\cos\frac{\pi}{a}(x+iy)}{\sin\frac{\pi}{a}(x+iy)}$$

$$= \frac{-iK\pi}{a}\frac{2\cos\frac{\pi}{a}(x+iy)\sin\frac{\pi}{a}(x-iy)}{2\sin\frac{\pi}{a}(x+iy)\sin\frac{\pi}{a}(x-iy)}$$

$$= \frac{-iK\pi}{a}\frac{\sin\frac{2\pi x}{a} - \sin\frac{2\pi yi}{a}}{\cos\frac{2\pi y}{a}i - \cos\frac{2\pi x}{a}}$$

$$= \frac{-iK\pi}{a}\left[\frac{\sin\frac{2\pi x}{a} - i\sinh\frac{2\pi y}{a}}{\cosh\frac{2\pi y}{a} - \cos\frac{2\pi x}{a}}\right]$$

$$\Rightarrow \qquad u = \frac{\frac{-K\pi}{a}\sinh\frac{2\pi y}{a}}{\cosh\frac{2\pi y}{a} - \cos\frac{2\pi x}{a}}, \quad v = \frac{\frac{K\pi}{a}\sin\frac{2\pi x}{a}}{\cosh\frac{2\pi y}{a} - \cos\frac{2\pi x}{a}}$$

Also, we have $W = \phi + i\psi = iK \log \sin\frac{\pi z}{a}$

and $\qquad \bar{w} = \phi - i\psi = -iK \log \sin\frac{\pi \bar{z}}{a}$

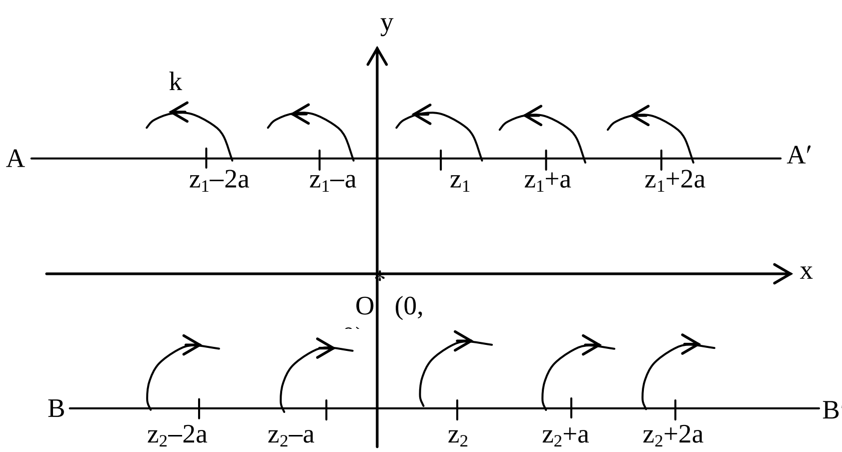
$$\therefore \qquad (\phi + i\psi) - (\phi - i\psi) = iK \log \sin \frac{\pi z}{a} - \left(- iK \log \sin \frac{\pi \bar{z}}{a} \right)$$

$$\Rightarrow \qquad 2\, i\psi = iK \log \sin \frac{\pi z}{a} \sin \frac{\pi \bar{z}}{a}$$

Streamlines, ψ = constant, are found to be

$$\cosh \frac{2\pi y}{a} - \cos \frac{2\pi x}{a} = \text{constant.}$$

4.2. Double Infinite Row of Vortices. Let us suppose that we have a system consisting of infinite number of vortices each of strength K evenly placed along a line AA′ parallel to x-axis and another system also consisting of infinite number of vortices each of strength –K placed similarly along a parallel line BB′. Let the line midway between these two lines of vortices be taken as the x-axis.



Let one vortex on infinite row AA′ be at $z = z_1$ and one vortex on infinite row BB′ be at $z = z_2$, so that the system consists of vortices K at $z = z_1 \pm na$ and vortices –K at $z = z_2 \pm na$, n = 1, 2, ….

The complex potential of the system is

$$W = iK \sum_{n=0}^{\infty} \log \left[\frac{(z - z_1 - na)(z - z_1 + na)}{(z - z_2 - na)(z - z_2 + na)} \right]$$

$$= iK \sum_{0}^{\infty} \log \left[\frac{(z - z_1)^2 - n^2 a^2}{(z - z_2)^2 - n^2 a^2} \right]$$

$$= iK \log \left(\frac{z - z_1}{z - z_2} \right) + iK \sum_{n=1}^{\infty} \log \frac{(z - z_1)^2 - n^2 a^2}{(z - z_2)^2 - n^2 a^2}$$

$$= iK \log \left(\frac{z - z_1}{z - z_2}\right) \prod_{n=1}^{\infty} \left[\frac{1 - (z - z_1)^2 / n^2 a^2}{1 - (z - z_2)^2 / n^2 a^2}\right] \qquad (1)$$

Now, since $\sin\theta = \theta \prod_{n=1}^{\infty} \left(1 - \frac{\theta^2}{n^2\pi^2}\right)$ $\forall$ θ real or complex,

we get, on setting $\theta = \frac{\pi(z - z_1)}{a} \equiv \frac{\pi(z - z_2)}{a}$

$$\sin \frac{\pi(z - z_1)}{a} = \frac{\pi(z - z_1)}{a} \prod_{n=1}^{\infty} \left(1 - \frac{(z - z_1)^2}{n^2 a^2}\right)$$

$$\sin \frac{\pi(z - z_2)}{a} = \frac{\pi(z - z_2)}{a} \prod_{n=1}^{\infty} \left(1 - \frac{(z - z_2)^2}{n^2 a^2}\right)$$

Therefore, equation (1) takes the form.

$$W = iK \log \left[\frac{\sin \pi \frac{(z - z_1)}{a}}{\sin \pi \frac{(z - z_2)}{a}}\right] \qquad (2)$$

The velocity at any point P(z), not occupied by a vortex filament, is

$$u - iv = -\frac{dW}{dz} = -iK\lambda \left[\cot \lambda (z - z_1) - \cot \lambda (z - z_2)\right], \text{ where } \lambda = \pi/a$$

$$= 2iK\lambda \sin \lambda(z_2 - z_1)/[\cos \lambda(z_2 - z_1) - \cos\lambda (2z - z_1 - z_2)] \quad (3)$$

To find the velocity (u_1, v_1) of the vortex K at $z = z_1$, we have

$$u_1 - iv_1 = -\left[\frac{d}{dz}\{W - iK \log(z - z_1)\}\right]_{z=z_1}$$

$$= iK \left[\lambda \cot\lambda(z - z_2) - \lambda \cot\lambda(z - z_1) + \frac{1}{z - z_1}\right]_{z=z_1}$$

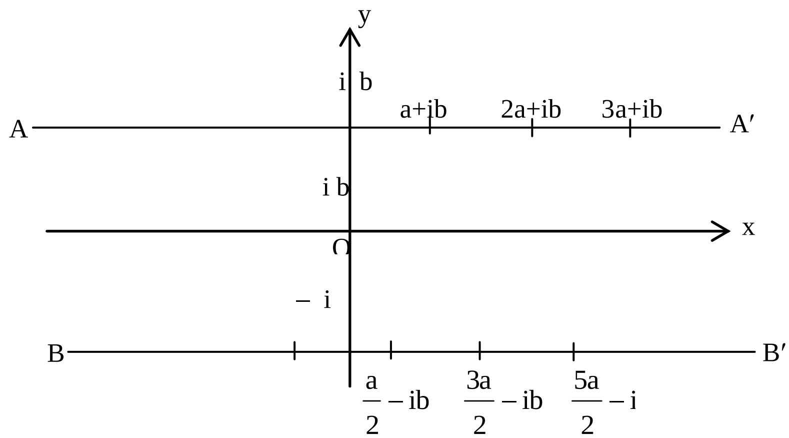
Since

$$\left[\cot\lambda(z - z_1) - \frac{1}{\lambda(z - z_1)}\right] \to 0 \text{ as } z \to z_1$$

Therefore,

$$u_1 - iv_1 = iK\lambda \cot\lambda\,(z_1 - z_2),\ \lambda = \pi/a \qquad (4)$$

4.3. Karman Vortex Street. This consists of two parallel infinite rows AA′ and BB′ of vortices of equal spacing ‘a’ so arranged that each vortex of strength K of AA′ is exactly above the mid-point of the join of two vortices of BB′ each of strength –K ; as shown in the figure



Therefore, the complex potential, in this, case is

$$W = iK \log \frac{\sin\frac{\pi}{a}(z - ib)}{\sin\frac{\pi}{a}\left(z - \frac{a}{2} + ib\right)}$$

(Similar to (2) of previous article on putting $z_1 = ib$, $z_2 = \frac{a}{2} - ib$).

The velocity of the vortex at $z = ib$ is

$$u_1 - iv_1 = \frac{\pi}{a} iK \cot\frac{\pi}{a}\left(ib - \frac{a}{2} + ib\right) \qquad \text{| Similar to (4) of previous article}$$

$$= \frac{\pi}{a} iK \cot\pi\left(\frac{2ib}{a} - \frac{1}{2}\right)$$

$$= \frac{\pi}{a} iK \cot\left(\frac{-\pi}{2} + \frac{2\pi ib}{a}\right)$$

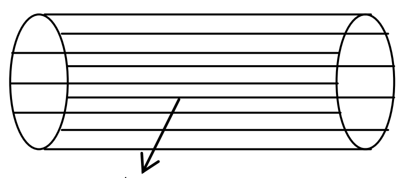
$$= -\frac{\pi iK}{a} \tan\left(\frac{2\pi ib}{a}\right)$$

$$= \frac{\pi K}{a} \tanh \frac{2\pi b}{a}$$

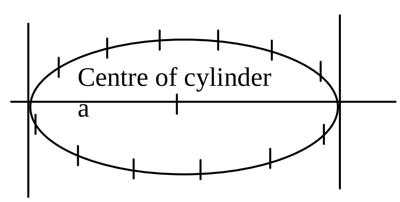
$$\Rightarrow \qquad u_1 = \frac{\pi K}{a} \tanh \frac{2\pi b}{a}, \quad v_1 = 0 \ .$$

It can be shown that each of vortices at the rows. AA′ and BB′ move with the same velocity. This means that the vortex configuration remains unaltered at all times, since both AA′ and BB′ have the same velocity $\frac{\pi K}{a} \tanh \frac{2\pi b}{a}$ in x-direction. Hence the street moves through the liquid with this velocity.

4.4. Example. If n rectilinear vortices of the same strength k are symmetrically arranged along generators of a circular cylinder of radius ‘a’ in an infinite liquid, prove that the vortices will move round the cylinder uniformly in time $\frac{4\pi a^2}{K(n-1)}$. Also find the velocity at any point of the fluid.



Solution. Since then rectilinear vortices of strength k are symmetrically distributed around the circular cylinder, the angular distance between any two consecutive vortices is $\frac{2\pi}{n}$. Let the line through the centre of the cylinder and one of the vortices be taken as x-axis Thus, the vortices are at points



$z = ae^0, ae^{2\pi i/n}, a\,e^{4\pi i/n}, \ldots.. ae^{2\pi(n-1)i/n}$

which are n distinct roots of the equation $z^n - a^n = 0$

$$| \because z^n - a^n = 0 \quad \Rightarrow z^n = a^n e^{2\pi ir} \quad \Rightarrow z = a^{2\pi ir/n}$$

Total complex potential of the system is

$$W = iK \log (z-a) + iK \log (z-ae^{2\pi i/n}) + \ldots\ldots\ldots\ldots$$

$$= iK \log (z-a)(z-a\,e^{2\pi i/n}) \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots.$$

$$= iK \log (z^n - a^n)$$

The velocity at any point outside the vortices is

$$u - iv = -\frac{dW}{dz} = \frac{-niK\,z^{n-1}}{z^n - a^n}$$

The velocity at the point z = a is

$$q_a = \left| -\frac{d}{dz}\left[W - iK\log(z-a)\right]_{z=a} \right|$$

$$= \left| -iK\frac{d}{dz}\left[\log\frac{z^n - a^n}{z-a}\right]_{z=a} \right|$$

$$= K\ \frac{d}{dz}\left[\log\left(z^{n-1} + a\,z^{n-2} + \ldots\ldots + a^{n-1}\right)\right]_{z=a}$$

$$= \frac{K[(n-1)z^{n-2} + a(n-2)z^{n-3} + \ldots\ldots + a^{n-2}]_{z=a}}{\left[z^{n-1} + a\,z^{n-2} + \ldots\ldots\ldots\ldots + a^{n-1}\right]_{z=a}}$$

$$= \frac{K}{a}\left[\frac{(n-1)+(n-2)+\ldots\ldots\ +2+1}{n}\right] \quad = \frac{K}{a}\frac{n-1}{2}$$

Therefore, time period is given by

$$T = \frac{2\pi a}{\frac{K(n-1)}{2a}} \qquad\qquad \left| \; T = \frac{\text{distance}}{\text{velocity}} \right.$$

i.e. $$T = \frac{4\pi a^2}{K(n-1)}$$

Hence the result.

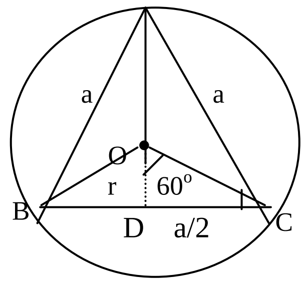
4.5. Remark. If we use $K = \frac{k}{2\pi}$ i.e .$k = 2\pi K$, then

$$T = \frac{4\pi a^2}{\frac{k}{2\pi}(n-1)} = \frac{8\pi^2 a^2}{(n-1)k}$$

4.6. Example. Three parallel rectilinear vortices of the same strength K and in the same sense meet any plane perpendicular to them in an equilateral triangle of side a. Show that the vortices move round the same cylinder with uniform speed in time $\frac{2\pi a^2}{3K}$.

Solution. Here, the vortices are situated at points $z = re^{2\pi pi/3}$ where p = 0, 1, 2,

A



From Δ OCD, $\frac{a}{2} = r\cos 30^{o} = r\frac{\sqrt{3}}{2}$

$\Rightarrow$ $r = \frac{a}{\sqrt{3}}$, where r is the radius of the cylinder.

The complex potential of the system is

$$W = iK \log (z - re^{0}) + iK \log (z - re^{2\pi i/3}) + iK \log (z - re^{4\pi i/3})$$

$$= iK \log (z^3 - r^3)$$

For the motion of vortex at A, we have

$$u_A - iv_A = \left| -\frac{d}{dz}\left[W - iK\log(z-r)\right]_{z=r} \right|$$

$$= \left| -iK\frac{d}{dz}\left[\log\frac{z^3-r^3}{z-r}\right]_{z=r} \right|$$

$$= K\frac{d}{dz}\left[\log(r^2+z^2+z^r)\right]_{z=r}$$

$$= K\left[\frac{2z+r}{r^2+z^2+z^r}\right]_{z=r} = K.\frac{1}{r}$$

Therefore, if T be the time period during which the vortex A moves round the cylinder, then

$$T = \frac{2\pi r}{K/r} = \frac{2\pi r^2}{K} = \frac{2\pi}{K}\left(\frac{a}{\sqrt{3}}\right)^2 = \frac{2\pi a^2}{3K}$$

Hence the result.

5. Wave Motion in a Gas

When studying impulsive motion of incompressible fluids, we have observed that a small disturbance applied at any point of such a fluid is transmitted instantaneously throughout the whole field of the fluid. In case of compressible fluids, such as air, a small disturbance applied at a point of the fluid is propagated throughout the fluid as a wave motion. Before studying wave propagation, in compressible fluids, we first discuss some elementary concepts of wave motion. We first treat wave motion in one-dimension and then generalize the results to propagation in two or three dimensions.

A wave is a disturbance in a medium such that there is no permanent displacement of the medium and the energy is propagated to the distant points.

5.1. One-dimensional Wave. The one-dimensional wave equation is

$$\nabla^2\phi = \frac{\partial^2\phi}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2\phi}{\partial t^2} \tag{1}$$

The function $\phi = \phi(x, t)$ is known as **wave function.** We find that

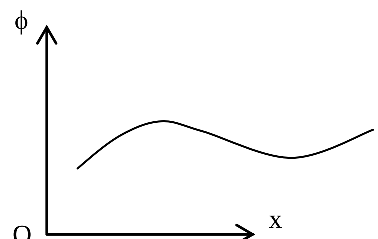
$$\phi(x, t) = f(x-ct) \tag{2}$$

is a solution of the wave equation (1). The shape of the disturbance ϕ is known as **wave profile.** For t = 0, we get

$$\phi = f(x)$$

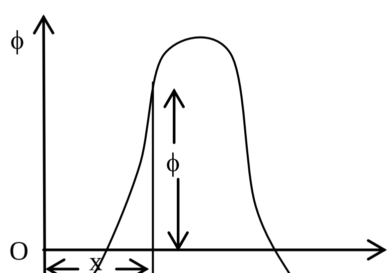
and the graph varies with t. We note that

$$\begin{aligned}\phi(x + cT, t + T) &= f[(x + cT)-c(t + T)] \\ &= f(x-ct) \\ &= \phi(x, t)\end{aligned}$$

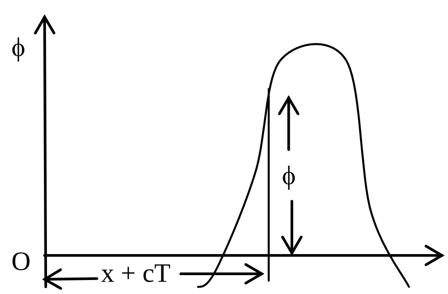


(3)

This shows that the value of ϕ at distance x and time t is equal to the value of ϕ at distance x + cT and time t + T, i.e., the wave profile at time t has moved through a distance cT along the x-axis at time T with constant speed c.



(Profile at t)



(Profile at t +T)

Thus the shape of the wave profile in (3) remains unchanged when it has moved a distance cT. For this reason the wave profile represented by equation (2) is called a wave without change of shape or **undistorted wave.** Equation (2) represents a wave which propagates with time. Such a wave is also termed as **progressive wave**. Similarly, the function defined by

$$\phi(x, t) = g(x + ct) \tag{4}$$

satisfies the wave equation (1) and it represents a disturbance moving without distortion in the negative x-direction with speed c.

5.2. Principle of Superposition. We note that a wave equation is a second order homogeneous linear partial differential equation. If ϕ_1 and ϕ_2 are two solutions of it, then $\phi_1 + \phi_2$ is also a solution. Hence $\phi = \phi_1 + \phi_2$ also represents a wave. This principle is called the principle of superposition for wave motion. Clearly, the combination

$$\phi(x, t) = f(x-ct) + g(x + ct) \tag{5}$$

represents the superposition of a forward and a backward travelling wave, each moving with speed c. From equation (5), we can show that

$$\frac{\partial^2\phi}{\partial t^2} = c^2 \frac{\partial^2\phi}{\partial x^2} \tag{6}$$

where equation (6) is known as the one-dimensional wave equation and the form ϕ in (5) is its general solution.

5.3. Wave Equations in Two and Three Dimensions. If a disturbance takes place in three dimensions in such a way that the disturbance is constant over any plane perpendicular to the direction of propagation, then the wave is called a **plane wave** and any such plane is called a **wave-front.** If such a wave is travelling with speed c in a direction specified by the unit vector $\hat{n} = [l, m, n]$, then the function $f(lx + my + nz - ct)$ satisfies these requirements since the wave fronts have equations $lx + my + nz =$ constant at any considered time. Similarly $g(lx + my + nz + ct)$ would represent a disturbance travelling in the direction $-\hat{n}$ with the same speed. Hence the function

$$\phi(x, y, z, t) = f(lx + my + nz - ct) + g(lx + my + nz + ct) \tag{7}$$

represents the superposition of plane waves travelling with speeds c in the directions $\pm\hat{n}$. Finding the second-order derivatives of ϕ w.r.t. x, y, z, t and using the fact that $l^2 + m^2 + n^2 = 1$, we get the wave equation in three dimensions as

$$\nabla^2\phi = \frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} + \frac{\partial^2\phi}{\partial z^2} = \frac{1}{c^2}\frac{\partial^2\phi}{\partial t^2} \tag{8}$$

Equation (7) gives the general solution of (8). The solution (7) can also be expressed as

$$\phi = f(\hat{n}\cdot\vec{r} - ct) + g(\hat{n}\cdot\vec{r} + ct) \tag{9}$$

In two dimensions (xy-plane), the wave equation is

$$\nabla^2\phi = \frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} = \frac{1}{c^2}\frac{\partial^2\phi}{\partial t^2} \tag{10}$$

having general solution

$$\phi(x, y, t) = f(lx + my - ct) + g(lx + my + ct) \tag{11}$$

where $l^2 + m^2 = 1$.

5.4. Spherical Waves. Let us consider the three dimensional wave equation

$$\nabla^2\phi = \frac{1}{c^2}\frac{\partial^2\phi}{\partial t^2} \tag{12}$$

in spherical polar co-ordinates (r, θ, ψ). It can be written as

$$\frac{\partial^2\phi}{\partial r^2} + \frac{2}{r}\frac{\partial\phi}{\partial r} + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\phi}{\partial\theta}\right)$$

$$+ \frac{1}{r^2\sin\theta}\frac{\partial^2\phi}{\partial\psi^2} = \frac{1}{c^2}\frac{\partial^2\phi}{\partial t^2} \tag{13}$$

If there is spherical symmetry so that $\phi = \phi(r, t)$, equation (13) reduces to

$$\frac{\partial^2\phi}{\partial r^2} + \frac{2}{r}\frac{\partial\phi}{\partial r} = \frac{1}{c^2}\frac{\partial^2\phi}{\partial t^2}$$

or

$$\frac{\partial^2}{\partial r^2}(r\,\phi) = \frac{1}{c^2}\frac{\partial^2}{\partial t^2}(r\,\phi) \tag{14}$$

The general solution of the one-dimensional wave equation gives the solution of (14) for $r\phi$ as

$$r\phi = f(r - ct) + g(r + ct)$$

or

$$\phi(r, t) = \frac{1}{r}\{f(r-ct) + g(r + ct)\} \tag{15}$$

The above solution represents concentric spherical wave fronts with centre O and having radii which increase or decrease with speed c. Here, the wave profiles change because of the factor $\frac{1}{r}$ in the solution.

5.5. Progressive and Stationary Waves. Upto now, we have considered various types of wave equations whose solutions represent the superposition of wave fronts travelling in opposite directions with speed c. In each case, the wave profile remains unaltered, except in the case of spherical waves where it is diminished by the factor $\frac{1}{r}$. Such waves, plane of spherical, are called progressive waves because of their movement in some direction.

Now, let us consider one-dimensional wave equations in the special forms

$$\left.\begin{aligned} f(x-ct) &= a\cos m\left(\frac{x}{c}-t\right), \\ g(x+ct) &= a\cos m\left(\frac{x}{c}+t\right) \end{aligned}\right] \qquad (16)$$

where a, c, m are constants. If the wave profile is either a sine or a cosine function, then the waves are **harmonic waves.** Thus, (16) represents harmonic waves. Superposition of the functions in (16), gives

$$\phi(x, t) = f(x-ct) + g(x + ct)$$

$$= 2a \cos\left(\frac{mx}{c}\right)\cos(mt) \qquad (17)$$

This type of disturbance is known as **stationary wave,** since its profile does not move. Thus at all times $\phi = 0$ at the fixed positions where $x = \left(p+\frac{1}{2}\right)\frac{c\pi}{m}$, where p is an integer. These determine the positions of zero displacement, called **nodes.** The points where $x = \frac{pc\pi}{m}$ determine the positions of maximum displacement, called **antinodes.** In both forms $f(x-ct)$, $g(x+ct)$, the **amplitude** is a. The **period** or periodic time in each case and also in ϕ, is $\frac{2\pi}{m}$, denoted by T. The **angular frequency** is m and the **frequency in cycles per unit time,** denoted by n, is $\frac{m}{2\pi}$ so that $n = \frac{1}{T}$ i.e. $nT = 1$. If, keeping t constant, we increase or decrease x by an amount $\frac{2\pi c}{m}$ or whole-number multiple of it, then all f, g, ϕ remain unaltered. The quantity $\frac{2\pi c}{m}$ is called the **wavelength** of the harmonic wave or of he combination of harmonic waves which ϕ represents. The wavelength is denoted by λ. Thus, $\lambda = \frac{2\pi c}{m}$. The number of waves in a unit distance is called the **wave number**. If k is the wave number, then $\lambda k = 1$ i.e. $k = \frac{1}{\lambda}$. Also, we observe that the amplitude of ϕ is $2a \cos(m t)$, which varies with time. Further, if in the forms for f and g, the cosines are replaced by sines, then similar results follow.

Another convenient way of representing a progressive harmonic wave is by considering either real or imaginary part of

$$a \exp\left\{im\left(t \mp \frac{x}{c}\right)\right\}, i = \sqrt{-1}.$$

The harmonic wave motions are of two types as follows.

(i) **Transverse :** If the vibrations occur in planes at right angles to the direction of propagation, then the waves are known as transverse waves. e.g. light waves

(ii) **Longitudinal :** When the vibrations occurs in the direction of propagation, then the waves are called longitudinal waves. e.g. sound waves.

As an another illustration, one-dimensional longitudinal waves propagate on a rod and transverse waves propagate on a string.

5.6. Some Elementary Concepts of Thermodynamics. The measurable quantities of a compressible substance are its pressure p, density ρ and temperature T. It is found that these quantities are connected through a functional relation of the form

$$f(p, \rho, T) = 0 \qquad (1)$$

where f is a single-valued function of the variable p, ρ, T. Such an equation (1) is known as the **equation of state** of the substance. The form (1) depends on the nature of the substance and for certain kinds of gas it is of very simple form.

For some gases, the molecules have negligible volume and there are virtually no mutual attractions between the individual molecules. Such a gas is said to be a **perfect gas** and its equation of state (1) takes the simple form

$$p = R\rho T \text{ or } pv = RT \qquad (2)$$

where $v = \frac{1}{\rho}$ is the volume of unit mass of the gas and R is a constant for the particular gas under consideration. Let δQ be the amount of heat added to unit mass of a substance so as to produce a temperature increment δT. Then the rate of increase of heat added with temperature rise is $\frac{\partial Q}{\partial T}$. This defines a quantity known as the **specific heat** of the substance, which is the heat addition per unit mass of the substance required to produce unit temperature rise. The quantity $\frac{\partial Q}{\partial T}$ may not be unique. For gases, it will depend on the manner in which the heat has been supplied. We can associate a specific heat at constant pressure, denoted by C_p, and a specific heat at constant volume, denoted by C_v, which are defined as

$$C_p = \left(\frac{\partial Q}{\partial T}\right)_p, \; C_v = \left(\frac{\partial Q}{\partial T}\right)_v \qquad (3)$$

These quantities in (3) are unequal.

For a perfect gas, the kinetic theory shows that C_p and C_v are constant and that

$$C_p/C_v = \gamma \tag{4}$$

where γ is a constant termed as the **adiabatic constant.**

From first law of thermodynamics, the relation for dQ, for a perfect gas, using pv = RT, can be written in the form

$$dS = \frac{C_p}{v}dv + \frac{C_v}{p}dp \tag{5}$$

where $dS = \frac{dQ}{T}$.

But $\frac{\partial}{\partial p}\left(\frac{C_p}{v}\right) = 0, \frac{\partial}{\partial v}\left(\frac{C_v}{p}\right) = 0$

so that $\frac{\partial}{\partial p}\left(\frac{C_p}{v}\right) = \frac{\partial}{\partial v}\left(\frac{C_p}{p}\right)$

This shows that dS is an exact differential. So we may integrate it to get

$$S - S_0 = C_p \log v + C_v \log p$$

or $$\log (pv^\gamma) = \frac{S - S_o}{C_v}$$

or $$pv^\gamma = \exp\left(\frac{S - S_0}{C_v}\right) \tag{6}$$

The quantity S is called the **entropy** per unit mass and dS is the entropy differential. Flows for which S is constant are called **isentropic** and from equation (6), we find that they are characterised by

$$pv^\gamma = \text{constant, i.e., } p = K\rho^\gamma \tag{7}$$

where $v = \frac{1}{\rho}$ and K is a constant.

The change in a substance from a fixed state A to a fixed state B may be effected in many ways. A change from A to B in which the temperature T is kept constant is said to be **isothermal.** In case of a perfect gas, since $p = R\rho T$ is the equation of state, an isothermal change would be governed by Boyle's law given by $p \propto k\rho$ i.e. pv = constant.

An adiabatic change is one in which there is not heat exchange between the working substance and its surroundings. If a change is made so that the entropy of every single particle of the working substance remains constant, then such a change is termed as **isentropic.** When the entropy of every single particle of a substance of fixed mass is the same and remains constant in any change, then the change is said to be **homentropic**. The constant under reference is the same for each considered small quantity of gas in isentropic flow but a different constant attaches to each such quantity. For homentropic flow, however, the constant is the same throughout the entire volume of gas.

6. The Speed of Sound in a Gas

We suppose that a small disturbance is created within a non-viscous gas such that

(i) The disturbance is propagated as a wave motion, known as a **sound wave,** by small to-and-fro motions of the medium without resulting in mass transport of the medium itself.

(ii) Before the disturbance, the fluid is at rest and thus the motion is irrotational so that a velocity potential ϕ exists at each point of the fluid. The fluid velocity at any point is $\bar{q} = -\nabla\phi$.

(iii) The squares and products of all disturbances from the equilibrium state specified by pressure p_0 and density ρ_0 can be neglected. Also $q = |\bar{q}|$ is so small that q^2 can be neglected.

(iv) The isentropic law $p = k\rho^\gamma$ holds as a consequence of assuming that changes take place so rapidly that heat exchanges and hence entropy changes are negligible, where γ is an adiabatic constant.

We write $\rho = \rho_0(1 + s)$, where s is the **condensation** of the medium. This is a dimensionless quantity expressing the fractional increment of local fluid density during the disturbance over the undisturbed density ρ_0 of the medium. It is a function of time t and space co-ordinates (x, y, z) if the motion is three-dimensional. The equation of continuity

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \bar{q}) = 0 \qquad (1)$$

becomes

$$\frac{\partial s}{\partial t} - \nabla \cdot \{(1+s)\nabla\phi\} = 0 \qquad (2)$$

If we assume that the velocity $-\nabla\phi$ is so small in magnitude that $s\nabla\phi$ is negligible, then (2) simplifies to

$$\frac{\partial s}{\partial t} = \nabla^2\phi \qquad (3)$$

In the absence of body forces, the equation of motion becomes

$$\int \frac{dp}{\rho} - \frac{\partial \phi}{\partial t} = \text{constant} \qquad (4)$$

where we have neglected q^2.

Assumption (iv) implies that $p = k\rho^\gamma$, where $k = \dfrac{p_0}{\rho_0^\gamma}$. Therefore,

$$\frac{dp}{d\rho} = k\,\gamma\,\rho^{\gamma-1} = \frac{\gamma p_0}{\rho_0^\gamma}\,\rho^{\gamma-1}$$

$$= a_0^2\left(\frac{\rho}{\rho_0}\right)^{\gamma-1} = a_0^2(1+s)^{\gamma-1} \simeq a_0^2,$$

where s is small and

$$a_0^2 = \frac{\gamma p_0}{\rho_0} \tag{5}$$

Hence

$$\int \frac{dp}{\rho} = \int\left(\frac{dp}{d\rho}.\frac{d\rho}{\rho}\right) \simeq a_0^2 \int \frac{d\rho}{\rho}$$

$$= a_0^2 \log \rho + \text{constant} \tag{6}$$

From (4) and (6), we obtain

$$\frac{\partial\phi}{\partial t} = a_0^2 \log \rho + \text{constant}$$

$$= a_0^2 \{\log \rho_0 + \log (1 + s)\} + \text{constant}$$

$$\simeq a_0^2\, s + \text{constant, to the first order.}$$

Absorbing the constant into s, we get

$$\frac{\partial\phi}{\partial t} = a_0^2 s \tag{7}$$

Eliminating s from (3) and (7), we have

$$\frac{\partial^2\phi}{\partial t^2} = a_0^2\,\nabla^2\phi \tag{8}$$

Equation (8) is a wave type equation and shows that small disturbances are propagated in the gas with speed

$$a_0 = \left(\frac{dp}{d\rho}\right)^{1/2} = \left(\frac{\gamma p_0}{\rho_0}\right)^{1/2} \qquad (9)$$

This speed is called the **speed of sound** in the gas. A vibrating tuning fork would produce disturbances propagating with such a speed. Equation (9) is obtained under isentropic conditions. When we wish to emphasize this we write

$$a^2 = \left(\frac{\partial p}{\partial \rho}\right)_s .$$

7. Equation of Motion of a Gas

We know that the equation of continuity for a compressible fluid is

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \overline{q}) = 0 \qquad (1)$$

and Euler's equation of motion is

$$\frac{\partial \overline{q}}{\partial t} + (\overline{q} \cdot \nabla)\overline{q} = \overline{F} - \frac{1}{\rho}\nabla p \qquad (2)$$

In the case of steady motion under no body forces, (1) and (2) become

$$\nabla \cdot (\rho \overline{q}) = 0 \qquad (3)$$

$$(\overline{q} \cdot \nabla)\overline{q} = -\frac{1}{\rho}\nabla p \qquad (4)$$

For such flow, Bernoull's equation becomes

$$\frac{1}{2}\overline{q}^2 + \int \frac{dp}{\rho} = \text{constant} \qquad (5)$$

In the special case of isentropic flow for which the entropy of each particle remains constant along any streamline and for each such particle $p = k\rho^{\gamma}$ so that (5) reduces to

$$\frac{1}{2}\overline{q}^2 + \int kr\, \rho^{\gamma-2}\, d\rho = \text{constant}$$

or $$\frac{1}{2}\overline{q}^2 + \frac{k\gamma\rho^{\gamma-1}}{\gamma-1} = \text{constant}$$

or $$\frac{1}{2}\overline{q}^2 + \frac{a^2}{\gamma-1} = \text{constant} \qquad (6)$$

where $a^2 = \left(\dfrac{\partial p}{\partial \rho}\right)_s$

In (6), the constant is same along any streamline, but unless the flow is homentropic, it will vary from one streamline to another. The condition for constant entropy of a fluid particle in its steady motion along a streamline is

$$(\bar{q} \cdot \nabla)\, S = 0 \tag{7}$$

since the total rate of change of the particle's entropy S per unit mass is $\dfrac{\partial S}{\partial t} + (\bar{q} \cdot \nabla) S$ in the general time-varying case. In addition, the equation of state may be taken in either of the equivalent forms

$$f(p, \rho, T) = 0 \tag{8}$$

or

$$p = F(\rho, S) \tag{9}$$

where the forms f, F are known. Equation (9) is more convenient for discussing the cases of isentropic and homentropic flows. Equations (3), (4), (7) and (8) or (9) are distinct equations for determining p, ρ, $\bar{q}$ and S. Bernaulli's equation is really derived from the equations of motion, but the forms (5) and (6) are very useful. Thus the problem of determining the nature of gas flow is solvable.

Now, since

$$(\bar{q} \cdot \nabla)\bar{q} = \nabla\left(\frac{1}{2}\bar{q}^2\right) - \bar{q} \times \bar{\xi},$$

scalar multiplication of (4) by $\bar{q}$ gives

$$\bar{q} \cdot \nabla\left(\frac{1}{2}\bar{q}^2\right) = -\frac{1}{\rho}\bar{q} \cdot \nabla p \tag{10}$$

Using the equation of state in the form (9), we get

$$dp = \left(\frac{\partial p}{\partial \rho}\right)_S d\rho + \left(\frac{\partial p}{\partial S}\right)_\rho dS$$

so that

$$\nabla p = \left(\frac{\partial p}{\partial \rho}\right)_S \nabla\rho + \left(\frac{\partial p}{\partial S}\right)_\rho \nabla S$$

For homentropic flow,

$$\nabla S = 0 \text{ and } \nabla p = a^2\, \nabla\rho,$$

then (10) becomes

$$\bar{q} \cdot \nabla\left(\frac{1}{2}\bar{q}^2\right) = -\frac{a^2}{\rho}\bar{q} \cdot \nabla\rho$$

From (3), we get

$$\rho\, \nabla \cdot \bar{q} + (\nabla\rho) \cdot \bar{q} = 0$$

and so

$$\bar{q} \cdot \nabla\left(\frac{1}{2}\bar{q}^2\right) = a^2\, \nabla \cdot \bar{q} \qquad (11)$$

Equation (11) is anther important result.

8. Subsonic, Sonic and Supersonic Flows

Let q be the speed of a gas at a certain location and let a be the local speed of sound, where

$$a = \left(\frac{dp}{d\rho}\right)^{1/2} = \left(\frac{p\gamma}{\rho}\right)^{1/2}.$$

Then the local **Mach number** M is defined to be the dimensionless parameter $M = q/a$.

Now, we consider the following three cases

Case (i) : When $q = a$, $M = 1$ then the flow is said to be **sonic** since the speed of gas flow and the local speed of sound are the same.

Case (ii) : When $M < 1$, $q < a$ then the flow is **subsonic** i.e. the speed of the gas flow is less than the local speed of sound.

Case (iii) : When $M > 1$, $q > a$ then the flow is termed as **supersonic** i.e. the speed of gas flow exceeds the local speed of sound.

Subsonic and supersonic flows have many different physical features. To know what type of flow pattern is realized, we should know the Mach number. We examine these physical features by discussing the nature of spherical sound waves in a moving stream of gas.

8.1. Theorem : Show that for subsonic flow, the spherical disturbances speed throughout the entire field, whereas for supersonic flow, the disturbances are confined to the interior of the cone, the region outside the cone being unaffected by the disturbances.

Proof : Let us consider a source O emitting spherical sound waves in a gas at rest. Spherical wave fronts centred at O travel outwards from O and at time t after starting from O, the disturbance is spread uniformly over the surface S of the sphere with centre O and radius a.t, as shown in the figure 1.

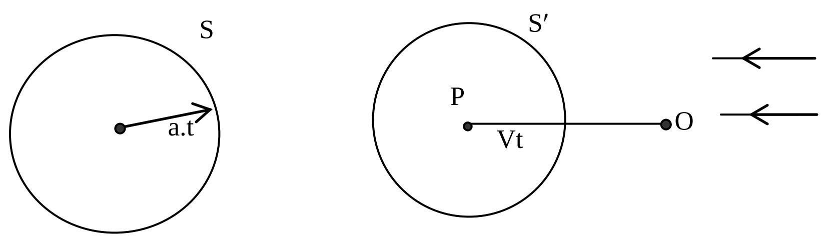


Figure 1.

Now, suppose that the gas flows with uniform velocity $\overline{v}$ past the source. Then at time t, every particle of S is displaced through a distance v t relative to S′ and the disturbance which was initially at O is now on the surface S′ of a sphere with centre P and radius a.t, where $\overline{OP} = \overline{v}t$. Here, M = v/a. When M < 1, v < a and O lies within S′. When M > 1, v > a and O lies outside S′. We discuss these two cases in turn.

Case (i) : When M < 1, let P_1, P_2, P_3,… denote the centres of the spherical disturbances at times t, 2t, 3t,…, the radii of the corresponding spheres being at, 2at, 3at,….

Also, $\overline{OP_1} \equiv \overline{v}t, \overline{OP_2} \equiv 2\overline{v}t, \overline{OP_3} \equiv 3\overline{v}t$ …, as shown in the figure 2.

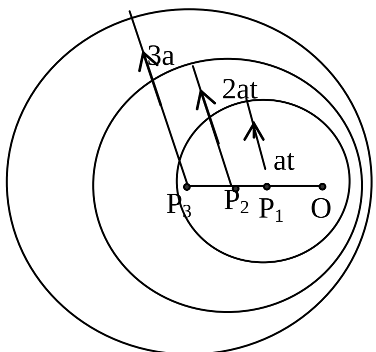


Figure 2

It is seen that the disturbances at times t, 2t, 3t,…, are on the boundaries of non-intersecting spheres.

Case (ii) : When M > 1, then O lies outside the spheres as shown in the figure 3. It is seen that the

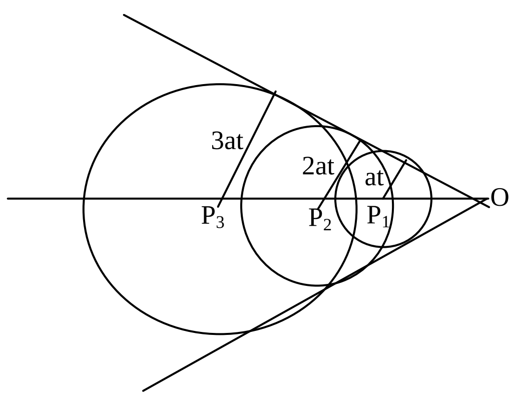


Figure 3

spheres intersect. They have an envelope which is a right circular cone having vertex O and axis $OP_1P_2P_3$…..

From the above two cases, we conclude that for subsonic flow the spherical disturbances spread throughout the entire field while for supersonic flow the disturbances are confined to the interior of the cone, the region outside the cone being silent, i.e., supersonic flow is characterised by a domain of dependence, which in the above case is the conical interior.

Corollary (1). In figure 3, let μ be the semi-vertical angle of the right-circular cone, then

$$\sin\mu = \frac{at}{vt} = \frac{2at}{2vt} = \frac{3at}{3vt}$$

i.e. $$\sin\mu = \frac{a}{v} = \frac{1}{M}.$$

The angle μ is termed as the **Mach angle** and it is real only when $M \geq 1$. It does not exist for subsonic flow. The cone is called a **Mach cone.**

Corollary (2). In two dimensions, the spheres in the above models become circles and the cone becomes pair of **Mach lines** or **Mach waves.**

8.3. Remark. If an aircraft is flying overhead at subsonic speed, any observer on the ground will hear the disturbance once the sound waves have spread out to meet him. However, if the aircraft is travelling at supersonic speed, disturbances will be confined to a domain of dependence relative to the aircraft and the observer will hear noise only when he comes within this domain. Thus, one may see a supersonic aircraft or missile travelling overhead but only hear the sound some little while afterwards when the domain of dependence encloses him.

9. Isentropic Gas Flow

We have obtained that Bernaulli's equation for isentropic gas flow along a streamline is

$$\frac{1}{2}\bar{q}^2 + \frac{a^2}{\gamma-1} = \text{constant} \qquad (1)$$

The L.H.S. of (1) shows that the maximum value of q, denoted by $q_{max.}$, occurs whenever $a = 0$. Such case corresponds to gases expanding to zero pressure and is entirely theoretical i.e. cannot be obtained in practice. Also we introduce the **critical speed of sound** a_* which is defined to be the value of a when $q = a$, and the **stagnation speed of sound** a_0 corresponding to $q = 0$ i.e. when the fluid is locally at rest. Then (1) can be written as

$$\frac{1}{2}\bar{q}^2 + \frac{a^2}{\gamma-1} = \frac{1}{2}q_{max}^2$$

$$= \frac{\gamma+1}{2(\gamma-1)}a_*^2 = \frac{a_0^2}{\gamma-1} \qquad (2)$$

Equation (2) gives three different forms of the constant on R.H.S. of (1).

Since $a^2 = \dfrac{\gamma p}{\rho}$, other forms of (2) are

$$\frac{1}{2}\bar{q}^2 + \frac{\gamma p}{\rho(\gamma-1)} = \frac{\gamma(\gamma+1)p_*}{2(\gamma-1)\rho_*} = \frac{\gamma p_0}{(\gamma-1)\rho_0} \qquad (3)$$

where p_* is critical pressure, ρ_* is critical density for local sonic flow with q = a, whereas, p_0 is stagnation pressure, ρ_0 is stagnation density for local condition of rest q = 0.

Now, for a perfect gas

$$\frac{\gamma p}{\rho(\gamma-1)} = \frac{\gamma RT}{\gamma-1} = C_p T \qquad (4)$$

where C_p $R\gamma/(\gamma-1)$ is the specific heat at constant pressure. Thus Bernoulli's equation along a streamline can be written as

$$\frac{1}{2}\bar{q}^2 + C_p T = C_p T_0 \qquad (5)$$

Here, T is the local temperature in Kelvins and T_0 is the stagnation temperature at a point on the same streamline where q = 0. Dividing both sides of (5) by $C_p T$, we get

$$\frac{T_0}{T} = 1 + \frac{1}{2}\bar{q}^2 \frac{\rho(\gamma-1)}{\gamma p}$$

$$= 1 + \frac{1}{2}(\gamma-1) M^2 \qquad (6)$$

where $M^2 = \dfrac{q^2}{a^2} = \rho q^2/\gamma p$, M being the Mach number. For isentropic flow, we have the relations

$$\frac{a_0^2}{a^2} = \frac{\gamma p_0}{\rho_0} \Big/ \frac{\gamma p}{\rho} = \frac{T_0}{T} = \left(\frac{p_0}{p}\right)^{(\gamma-1)/\gamma} = \left(\frac{\rho_0}{\rho}\right)^{\gamma-1} \qquad (7)$$

so that

$$\frac{p_0}{p} = \left[1 + \frac{1}{2}(\gamma-1)M^2\right]^{\gamma/(\gamma-1)} \qquad (8)$$

$$\frac{\rho_0}{\rho} = \left[1 + \frac{1}{2}(\gamma-1)M^2\right]^{1/(\gamma-1)} \qquad (9)$$

Other isentropic relations, which are easily found from (5), are

$$\frac{T}{T_0} = 1 - \frac{\gamma - 1}{2}\left(\frac{q}{a_0}\right)^2 = 1 - \frac{\gamma - 1}{\gamma + 1}\left(\frac{q}{a_*}\right)^2 \tag{10}$$

which result in the following relations for p/p_0 and ρ/ρ_0

$$\frac{p}{p_0} = \left[1 - \frac{\gamma - 1}{2}\left(\frac{q}{a_0}\right)^2\right]^{\gamma/(\gamma-1)} = \left[1 - \frac{\gamma - 1}{\gamma + 1}\left(\frac{q}{a_*}\right)^2\right]^{\gamma/(\gamma-1)} \tag{11}$$

$$\frac{\rho}{\rho_0} = \left[1 - \frac{\gamma - 1}{2}\left(\frac{q}{a_0}\right)^2\right]^{1/(\gamma-1)} = \left[1 - \frac{\gamma - 1}{\gamma + 1}\left(\frac{q}{a_*}\right)^2\right]^{1/(\gamma-1)} \tag{12}$$

At sonic or critical speeds ($M = 1$), $p = p_*$, $\rho = \rho_*$, $T = T_*$ and so

$$\frac{p_*}{p_0} = \left(\frac{2}{\gamma + 1}\right)^{\gamma/(\gamma-1)} \tag{13}$$

$$\frac{\rho_*}{\rho_0} = \left(\frac{2}{\gamma + 1}\right)^{1/(\gamma-1)} \tag{14}$$

$$\frac{T_*}{T_0} = \frac{2}{\gamma + 1} \tag{15}$$

For air, $\gamma \simeq 1.400$, equation (13), (14), (15) result in

$$\frac{p_*}{p_0} \simeq 0.528, \quad \frac{\rho_*}{\rho} \simeq 0.630, \quad \frac{T_*}{T_0} \simeq 0.833.$$

10. Reservoir Discharge through a Channel of Varying Section (Flow Through a Nozzle)

Let us consider a reservoir containing stationary gas at high pressure p_0, density ρ_0, temperature T_0. An open-ended axially symmetric channel is fitted to the reservoir and we assume that the gas discharges steadily and isentropically into the air at the open section where the pressure is less than p_0. Let the section of the channel vary so slowly that to a first order of approximation, the velocity is constant across any section. However, the velocity varies from section to section. Here, the flow can be considered as one-dimensional.

At a location of the channel where the cross-sectional area is A, let p be the pressure, ρ be the density and u be the gas speed. For steady flow, the equation of continuity across the section is

$$\rho u A = \text{constant}$$

Differentiating, we get

$$\rho u dA + \rho A\, du + uA\, d\rho = 0$$

$$\Rightarrow \qquad \frac{dA}{A} + \frac{d\rho}{\rho} + \frac{du}{u} = 0 \qquad (1)$$

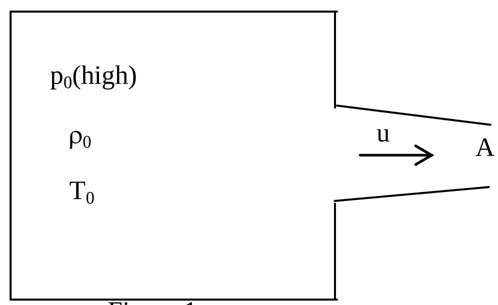


Figure 1.

Bernoulli's equation is

$$\frac{1}{2}u^2 + \int \frac{dp}{\rho} = \text{constant} \qquad (2)$$

Differentiating (2), we get

$$u\, du + \frac{dp}{\rho} = 0 \qquad (3)$$

Putting $dp = a^2 d\rho$ in (3) and eliminating $\frac{d\rho}{\rho}$ from (1) and (3), we obtain

$$(1-M^2)\, \frac{du}{u} = -\frac{dA}{A} \qquad (4)$$

where $M = u/a$ i.e. the local Mach number. We discuss the following two cases.

Case I : If $M < 1$, equation (4) shows that a decrease in A produces an increase in u and conversely. Thus, to accelerate subsonic flow through a channel it is necessary to decrease the channel section A downstream of the flow.

Case II : If $M > 1$, equation (4) show that A and u increase or decrease together. Thus, to accelerate supersonic flow it is necessary to widen the channel downstream of the flow.

Also, on putting $dp = a^2 d\rho$ in (3), we get

$$\frac{d\rho}{\rho} = -M^2 \frac{du}{u}. \qquad (5)$$

which indicates how the fluid density varies with changing Mach number. In fact equation (5) shows that for a given speed increment there is a density drop whose magnitude increases with increasing Mach number. Further, for $M \geq 1$, the drop in density is so large that the channel must expand to satisfy continuity requirements.

From equation (4), if $\frac{dA}{A} = 0$, then either $M = 1$ or $du = 0$. The case $du = 0$ is realized in incompressible flow where the speed of the flow reaches a maximum at the stage when the channel section attains a minimum area of cross-section. For compressible fluids, M may be unity when the section A is a minimum.

To summarise the above results we may say that if, starting with subsonic flow in a channel, we decrease the section downstream, then the flow is accelerated until the section has attained a certain minimum at which the Mach number is unity. If beyond this minimum section we now widen the channel, then the flow can be accelerated downstream of the section to produce supersonic flow. This illustrates the principle of **flow through a nozzle.** The minimum section is termed as **throat,** as shown in figure 2.

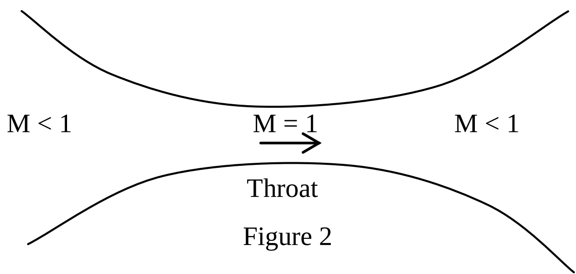


Figure 2

10.1. Maximum Mass Flow Through a Nozzle. We consider a channel which is tapering steadily to a minimum section at the outlet. Let A be the section at the outlet of the channel where the velocity is u, pressure p and density ρ. Then, applying Bernoulli's equation along a stream-line from the reservoir to the section A, we get

$$\frac{1}{2}u^2 + \left(\frac{\gamma}{\gamma-1}\right)\frac{p}{\rho} = \left(\frac{\gamma}{\gamma-1}\right)\frac{p_0}{\rho_0} \qquad (1)$$

so that

$$u = \left[\frac{2\gamma p_0}{(\gamma-1)\rho_0}\left(1 - \frac{p\rho_0}{p_0\rho}\right)\right]^{1/2} \qquad (2)$$

Using $\quad \frac{\rho_0}{\rho} = \left(\frac{p_0}{p}\right)^{1/\gamma}$, (2) becomes

$$u = \left[\frac{2\gamma p_0}{(\gamma-1)\rho_0}\left\{1 - \left(\frac{p}{p_0}\right)^{(\gamma-1)/\gamma}\right\}\right]^{1/2} \qquad (3)$$

The mass flux per unit time across the section A is

$$m = \rho\, u\, A = \rho_0\, u\, A(p/p_0)^{1/\gamma} \qquad (4)$$

Using u from (3) in (4), we get

$$m = A\left[\frac{2\gamma p_0\rho_0}{\gamma-1}\left(\frac{p}{p_0}\right)^{2/\gamma}\left\{1-\left(\frac{p}{p_0}\right)^{(\gamma-1)/\gamma}\right\}\right]^{1/2} \qquad (5)$$

To find the stationary values of m for fixed A and variable p/p_0, we write $P = p/p_0$ so that

$$m^2 = k\,(P^{2/\gamma} - P^{(\gamma+1)/\gamma}) \qquad (6)$$

where k is constant

Differentiating (6) w.r.t. P, we get

$$2m\,\frac{dm}{dP} = \frac{k(\gamma+1)}{\gamma}P^{(2-\gamma)/\gamma}\left\{\frac{2}{\gamma+1} - P^{(\gamma-1)/\gamma}\right\} \qquad (7)$$

$$\frac{dp}{dP} = 0 \text{ when } P = \{2/(\gamma+1)\}^{\gamma/(\gamma-1)} = P_* \qquad (8)$$

Also, $\frac{dm}{dP} < 0 \text{ when } P > P_*, \frac{dm}{dP} > 0 \text{ when } P < P_*$ These inequalities depend on the assumption that for any gas $\gamma > 1$. From these inequalities, we conclude that m is maximum when $P = P_*$

i.e.

$$\frac{p}{p_0} = \left(\frac{2}{\gamma+1}\right)^{\gamma/(\gamma-1)} \qquad (9)$$

and

$$m_{max.} = A\left\{\left(\frac{2\gamma}{\gamma+1}\right)p_0\rho_0\left(\frac{2}{\gamma+1}\right)^{2/(\gamma-1)}\right\}^{1/2} \qquad (10)$$

Thus, we find that $P_* = \frac{p_*}{p_0}$, where p_* is the pressure at a point where M = 1. It therefore follows that for maximum isentropic mass flow, conditions at the exit plane are sonic.

From (5) and (10), we obtain

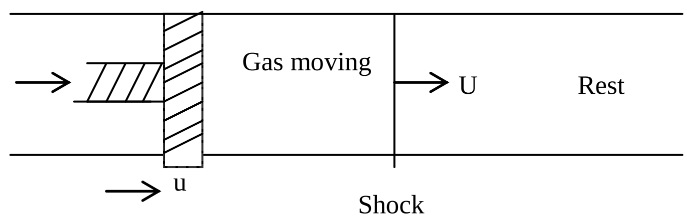
$$\frac{m}{m_{max.}} = \left(\frac{\gamma+1}{\gamma-1}\right)^{1/2}\left[\frac{1}{2}(\gamma+1)^{1/(\gamma-1)}\left(\frac{p}{p_0}\right)^{1/\gamma}\right]\left[1-\left(\frac{p}{p_0}\right)^{(\gamma-1)/\gamma}\right]^{1/2}$$

(11)

which gives the variation of $m/m_{max.}$ with p/p_0.

11. Shock Waves

Shock waves are not waves in real sense. These are plane discontinuities, pulse-like in nature and are sometimes more appropriately called **shock fronts.** For the formation of shock waves, we consider a piston which is being driven with uniform velocity u into a long open tube, known as **shock tube.**



The gas particles within the vicinity of the piston acquire the uniform velocity u but those some way ahead of the piston are at rest. A plane normal discontinuity or shock front travels forwards with velocity U(> u) into the virgin gas as the piston advances into the tube. The shock is the mechanism by which the gas between it and the piston acquires the velocity u. The existence of the shock can be detected experimentally by certain delicate kinds of photographic methods such as shadowgraph. The velocity U > a, the local speed of sound in the fluid. A simple physical explanation of the shock formation in this case is as follows :

Suppose we approximate the continuous motion of the piston by a set of forward-moving pulses, each of short duration. When the piston makes the first short movement forward, a small disturbance is propagated forward into the gas at the speed of sound. This small amplitude wave (or sound wave) heats the gas slightly and since $a \propto \sqrt{T}$, where T is the temperature, the second pulse will be propagated as another sound wave at a speed slightly in excess of the first one. Similarly, the third pulse will be propagated at a speed slightly in excess of the second and so on. Thus the discreet pulses cause a train of sound waves of ever increasing velocity to be propagated through the gas. The model discussed here is a simplified model explaining shock formation in the tube when the piston is activated with constant velocity.

Let us consider another more analytical model in which first of all we consider a sound wave of velocity a moving into a gas at rest (figure. 1). The pressure ahead of the wave is p, then density ρ and the particle velocity zero. The pressure immediately behind the wave is $p + \delta p$, the density $\rho + \delta\rho$ and the particle velocity δu. Here, the disturbance is assumed to be weak so δp, $\delta\rho$, δu are small. Figure 2 shows the equivalent model obtained when the sound wave is brought to rest by imposing a backward velocity a on the entire system.

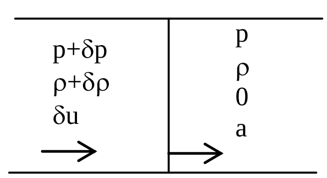


Figure 1.

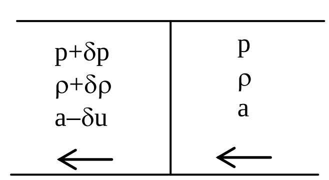


Figure 2.

Let us consider figure 2, where we apply the equation of continuity across the stationary wave to obtain

$$\rho a = (\rho + \delta\rho)(a - \delta u) \qquad (1)$$

Here, we have considered the mass flux per unit time across unit area of the wave.

Thus $d\rho/\rho = du/a$

But $$a^2 = \frac{\gamma p}{\rho} = \gamma\left(\frac{p_0}{\rho_0}\right)\left(\frac{\rho}{\rho_0}\right)^{\gamma-1} \tag{2}$$

So, we have

$$du = a\frac{d\rho}{\rho} = \gamma^{1/2}\, p_0^{1/2}\, \rho_0^{-\gamma/2}\; \rho^{(\gamma-3)/2}\, d\rho$$

which, on integration, gives

$$u = \int_{\rho_0}^{\rho}\frac{du}{d\rho}d\rho = \frac{2}{\gamma-1}\frac{\gamma^{1/2}p_0^{1/2}}{\rho_0^{\gamma/2}}\left(\rho^{(\gamma-1)/2} - \rho_0{}^{(\gamma-1)/2}\right)$$

$$= \left(\frac{2a_0}{\gamma-1}\right)\left[\left(\frac{\rho}{\rho_0}\right)^{(\gamma-1)/2} - 1\right] \tag{3}$$

Hence

$$(\rho/\rho_0)^{(\gamma-1)/2} = 1 + \frac{1}{2}(\gamma-1)\,(u/a_0) \tag{4}$$

where a_0 is the speed of sound in the undisturbed gas. Now

$$\frac{a^2}{a_0^2} = \left(\frac{\gamma p}{\rho}\right)\left(\frac{\rho_0}{\gamma p_0}\right) = \left(\frac{\rho_0}{\rho}\right)\left(\frac{p}{p_0}\right) = \left(\frac{\rho}{\rho_0}\right)^{\gamma-1} \tag{5}$$

Hence

$$a = a_0\,(\rho/\rho_0)^{(\gamma-1)/2} = a_0 + \frac{1}{2}(\gamma-1)u \tag{6}$$

Now, each small disturbance propagates itself at a velocity equal to local speed of sound relative to the fluid. Thus if the fluid moves with velocity u, then velocity of propagation of disturbance

$$= u + a = a_0 + \frac{1}{2}(\gamma+1)u \tag{7}$$

Hence in a short time interval τ, the distance moved by the disturbance is

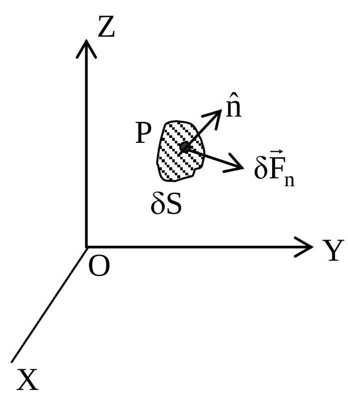
$$(u + a)\tau = \left[a_0 + \frac{1}{2}(\gamma+1)u\right]\tau \tag{8}$$

Equation (8) shows that in a given interval τ, the points of high velocity move farther to the right than those of low velocity. The type of shock wave just considered is a **normal shock,** since it is perpendicular to the incident gas stream. Another type of shocks are **oblique shocks** which are inclined at oblique angles to the direction of flow.

UNIT – IV

1. Stress Components in a Real Fluid

Let δS be a small rigid plane area inserted at a point P in a viscous fluid. Cartesian co-ordinates (x, y, z) are referred to a set of fixed axes OX, OY, OZ. Suppose that $\delta \overline{F}_n$ is the force exerted by the moving fluid on one side of δS, the unit vector $\hat{n}$ being taken to specify the normal at P to δS on this side. We know that in the case of an inviscit fluid, $\delta \overline{F}_n$ is aligned with $\hat{n}$. For a viscous fluid, however, frictional forces are called into play between the fluid and the surface so that $\delta \overline{F}_n$ will also have a component tangential to δS. We suppose the Cartesian components of $\delta \overline{F}_n$ to be $(\delta F_{nx}, \delta F_{ny}, \delta F_{nz})$ so that



$$\delta \overline{F}_n = \delta F_{nx}\, \hat{i} + \delta F_{ny}\, \hat{j} + \delta F_{nz}\, \hat{k}\,.$$

Then the components of stress parallel to the axes are defined to be σ_{nx}, σ_{ny}, σ_{nz}, where

$$\sigma_{nx} = \underset{\delta S \to 0}{\mathrm{lt}} \frac{\delta F_{nx}}{\delta S} = \frac{dF_{nx}}{dS},$$

$$\sigma_{ny} = \underset{\delta S \to 0}{\mathrm{lt}} \frac{\delta F_{ny}}{\delta S} = \frac{dF_{ny}}{dS},$$

$$\sigma_{nz} = \underset{\delta S \to 0}{\mathrm{lt}} \frac{\delta F_{nz}}{\delta S} = \frac{dF_{nz}}{dS}.$$

In the components σ_{nx}, σ_{ny}, σ_{nz}, the first suffix n denotes the direction of the normal to the elemental plane δS whereas the second suffix x or y or z denotes the direction in which the component is measured.

If we identify $\hat{n}$ in turn with the unit vectors $\hat{i}, \hat{j}, \hat{k}$ in $(\overline{OX}), (\overline{OY}), (\overline{OZ})$, which is achieved by suitably re-orientating δS, we obtain the following three sets of stress components

$$\sigma_{xx}, \quad \sigma_{xy}, \quad \sigma_{xz};$$

$$\sigma_{yx}, \quad \sigma_{yy}, \quad \sigma_{yz};$$

$$\sigma_{zx}, \quad \sigma_{zy}, \quad \sigma_{zz}.$$

The diagonal elements σ_{xx}, σ_{yy}, σ_{zz} of this array are called normal or direct stresses. The remaining six elements are called shearing stresses. For an inviscid fluid, we have

$$\sigma_{xx} = \sigma_{yy} = \sigma_{zz} = -p$$

$$\sigma_{xy} = \sigma_{xz} = \sigma_{yx} = \sigma_{yz} = \sigma_{zx} = \sigma_{zy} = 0$$

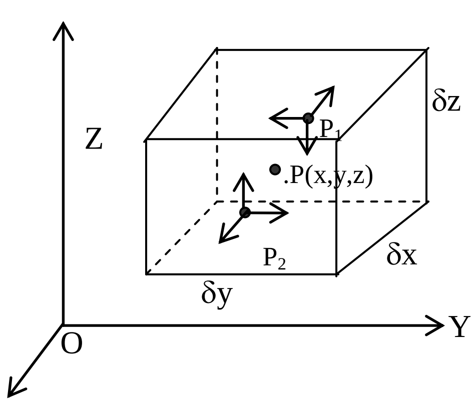
Here, we consider the normal stresses as positive when they are tensile and negative when they are compressive, so that p is the hydrostatic pressure. The matrix

$$\begin{bmatrix} \sigma_{xx} & \sigma_{xy} & \sigma_{xz} \\ \sigma_{yx} & \sigma_{yy} & \sigma_{yz} \\ \sigma_{zx} & \sigma_{zy} & \sigma_{zz} \end{bmatrix} \qquad (1)$$

is called the stress matrix. If its components are known, we can calculate the total forces on any area at any chosen point. The quantities $\sigma_{ij}(i, j = x, y, z)$ are called the components of the stress tensor whose matrix is of the form (1). Further we observe that σ_{ij} is a tensor of order two.

2. Relation Between Rectangular (Cartesian) Components of Stress

Let us consider the motion of a small rectangular parallelopiped of viscous fluid, its centre being P(x, y, z) and its edges of lengths δx, δy, δz, parallel to fixed Cartesian axes, as shown in the figure.



X

Let ρ be the density of the fluid. The mass $\rho\delta x\ \delta y\ \delta z$ of the fluid element remains constant and the element is presumed to move alongwith the fluid. In the figure, the points P_1 and P_2 have been taken on the centre of the faces so that they have co-ordinates $\left(x - \frac{\delta x}{2}, y, z\right)$ and $\left(x + \frac{\delta x}{2}, y, z\right)$ respectively.

At P(x, y, z), the force components parallel to $\overline{OX}, \overline{OY}, \overline{OZ}$ on the surface area $\delta y.\ \delta z$ through P and having $\hat{i}$ as unit normal, are

$$(\sigma_{xx}\delta y\ \delta z,\quad \sigma_{xy}\ \delta y\delta z,\quad \sigma_{xz}\ \delta y\delta z)$$

At $P_2\left(x + \frac{\delta x}{2}, y, z\right)$, since $\hat{i}$ is the unit normal measured outwards from the fluid, the corresponding force components across the parallel plane of area $\delta y\delta z$, are

$$\left[\left\{\sigma_{xx} + \frac{\delta x}{2}\left(\frac{\partial\sigma_{xx}}{\partial x}\right)\right\}\delta y\delta z, \left\{\sigma_{xy} + \frac{\delta x}{2}\left(\frac{\partial\sigma_{xy}}{\partial x}\right)\right\}\delta y\delta z, \left\{\sigma_{xz} + \frac{\delta x}{2}\left(\frac{\partial\sigma_{xz}}{\partial x}\right)\right\}\delta y\delta z\right].$$

For the parallel plane through $P_1\left(x - \frac{\delta x}{2}, y, z\right)$, since $-\hat{i}$ is the unit normal drawn outwards from the fluid element, the corresponding components are

$$\left[-\left\{\sigma_{xx} - \frac{\delta x}{2}\left(\frac{\partial\sigma_{xx}}{\partial x}\right)\right\}\delta y\delta z, -\left\{\sigma_{xy} - \frac{\delta x}{2}\left(\frac{\partial\sigma_{xy}}{\partial x}\right)\right\}\delta y\delta z, -\left\{\sigma_{xz} - \frac{\delta x}{2}\left(\frac{\partial\sigma_{xz}}{\partial x}\right)\right\}\delta y\delta z\right]$$

The forces on the parallel planes through P_1 and P_2 are equivalent to a single force at P with components

$$\left[\frac{\partial\sigma_{xx}}{\partial x}, \frac{\partial\sigma_{xy}}{\partial x}, \frac{\partial\sigma_{xz}}{\partial x}\right]\delta x\ \delta y\ \delta z$$

together with couples whose moments (upto third order terms) are

$$\begin{cases} -\sigma_{xz}\ \delta x\ \delta y\ \delta z \text{ about Oy,} \\ \sigma_{xy}\ \delta x\delta y\delta z \text{ about Oz.} \end{cases}$$

Similarly, the pair of faces perpendicular to the y axis give a force at P having components

$$\left[\frac{\partial \sigma_{yx}}{\partial y}, \frac{\partial \sigma_{yy}}{\partial y}, \frac{\partial \sigma_{yz}}{\partial y}\right] \delta x\ \delta y\ \delta z$$

together with couples of moments

$$\begin{cases} -\sigma_{yx}\ \delta x\ \delta y\ \delta z \text{ about } Oz, \\ \sigma_{yz}\ \delta x \delta y \delta z \text{ about } Ox. \end{cases}$$

The pair of faces perpendicular to the z-axis give a force at P having components

$$\left[\frac{\partial \sigma_{zx}}{\partial z}, \frac{\partial \sigma_{zy}}{\partial z}, \frac{\partial \sigma_{zz}}{\partial z}\right] \delta x\ \delta y\ \delta z$$

together with couples of moments

$$\begin{cases} -\sigma_{zy}\ \delta x\ \delta y\ \delta z \text{ about } Ox, \\ \sigma_{zx}\ \delta x \delta y \delta z \text{ about } Oy. \end{cases}$$

Combining the surface forces of all six faces of the parallelopiped, we observe that they reduce to a single force at P having components

$$\left[\left(\frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{yx}}{\partial y} + \frac{\partial \sigma_{zx}}{\partial z}\right), \left(\frac{\partial \sigma_{xy}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial y} + \frac{\partial \sigma_{zy}}{\partial z}\right), \left(\frac{\partial \sigma_{xz}}{\partial x} + \frac{\partial \sigma_{yz}}{\partial y} + \frac{\partial \sigma_{zz}}{\partial z}\right)\right] \delta x\ \delta y\ \delta z,$$

together with a vector couple having Cartesian components

$$[(\sigma_{yz} - \sigma_{zy}), (\sigma_{zx} - \sigma_{xz}), (\sigma_{xy} - \sigma_{yx})]\ \delta x\ \delta y\ \delta z.$$

Now, suppose the external body forces acting at P are [X, Y, Z] per unit mass, so that the total body force on the element has components [X, Y, Z] $\rho\ \delta x\ \delta y\ \delta z$. Let us take moments about $\hat{i}$ –direction through P. Then, we have

Total moment of forces = Moment of inertia about axis × Angular acceleration

i.e. $(\sigma_{yz}-\sigma_{zy})\ \delta x\ \delta y\ \delta z$ + terms of 4th order in $\delta x, \delta y\ \delta z$ = terms of 5th order in $\delta x, \delta y, \delta z$.

Thus, to the third order of smallness in δx, δy, δz, we obtain

$$(\sigma_{yz} - \sigma_{zy})\, \delta x\, \delta y\, \delta z = 0$$

Hence, as the considered fluid element becomes vanishingly small, we obtain

$$\sigma_{yz} = \sigma_{zy}.$$

Similarly, we get

$$\sigma_{zx} = \sigma_{xz}, \qquad \sigma_{xy} = \sigma_{yx}$$

Thus, the stress matrix is diagonally symmetric and contains only six unknowns. In other words, we have proved that

$$\sigma_{ij} = \sigma_{ji},\ (i, j = x, y, z)$$

i.e. σ_{ij} is symmetric.

In fact, σ_{ij} is a symmetric second order Cartesian tensor.

2.1. Transnational Motion of Fluid Element. Considering the surface forces and body forces, we note (from the previous article) that the total force component in the $\hat{i}$ –direction, acting on the fluid element at point P(x, y, z), is

$$\left(\frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{yx}}{\partial y} + \frac{\partial \sigma_{zx}}{\partial z}\right)\delta x\, \delta y\, \delta z + X\, \rho \delta x\, \delta y\, \delta z \qquad (1)$$

where (X, Y, Z) is the body force per unit mass and ρ being the density of the viscous fluid. As the mass $\rho\, \delta x\, \delta y\, \delta z$ is considered constant, if $\bar{q} = (u, v, w)$ be the velocity of point P at time t, then the equation of motion in the $\hat{i}$ –direction is

$$\left(\frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{yx}}{\partial y} + \frac{\partial \sigma_{zx}}{\partial z}\right)\delta x\, \delta y\, \delta z + \rho\, X\, \delta x\, \delta y\, \delta z = (\rho \delta x\, \delta y\, \delta z)\frac{du}{dt}$$

or
$$\frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{yx}}{\partial y} + \frac{\partial \sigma_{zx}}{\partial z} + \rho X = \rho \frac{du}{dt} \qquad (2)$$

If u = u(x, y, z, t), then

$$\frac{du}{dt} = \frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} \text{ where } \frac{d}{dt} \equiv \frac{\partial}{\partial t} + \bar{q}\cdot\nabla$$

Thus, (2) becomes

$$\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}+w\frac{\partial u}{\partial z}=X+\frac{1}{\rho}\left(\frac{\partial\sigma_{xx}}{\partial x}+\frac{\partial\sigma_{yx}}{\partial y}+\frac{\partial\sigma_{zx}}{\partial z}\right) \quad (3)$$

Similarly the equations of motion in $\hat{j}$ and $\hat{k}$ directions are

$$\frac{\partial v}{\partial t}+u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}+w\frac{\partial v}{\partial z}=Y+\frac{1}{\rho}\left(\frac{\partial\sigma_{xy}}{\partial x}+\frac{\partial\sigma_{yy}}{\partial y}+\frac{\partial\sigma_{zy}}{\partial z}\right) \quad (4)$$

$$\frac{\partial w}{\partial t}+u\frac{\partial w}{\partial x}+v\frac{\partial w}{\partial y}+w\frac{\partial w}{\partial z}=Z+\frac{1}{\rho}\left(\frac{\partial\sigma_{xz}}{\partial x}+\frac{\partial\sigma_{yz}}{\partial y}+\frac{\partial\sigma_{zz}}{\partial z}\right) \quad (5)$$

Equations (3), (4), (5) provide the equations of motion of the fluid element at P(x, y, z).

In tensor form, if the co-ordinates are x_i, the velocity components u_i, the body force components X_i, where i = 1, 2, 3, the equations of motion can be expressed as

$$\frac{\partial u_i}{\partial t}+u_j\,u_{i,j}=X_i+\frac{1}{\rho}\,\sigma_{ji,j} \quad (i, j = 1, 2, 3).$$

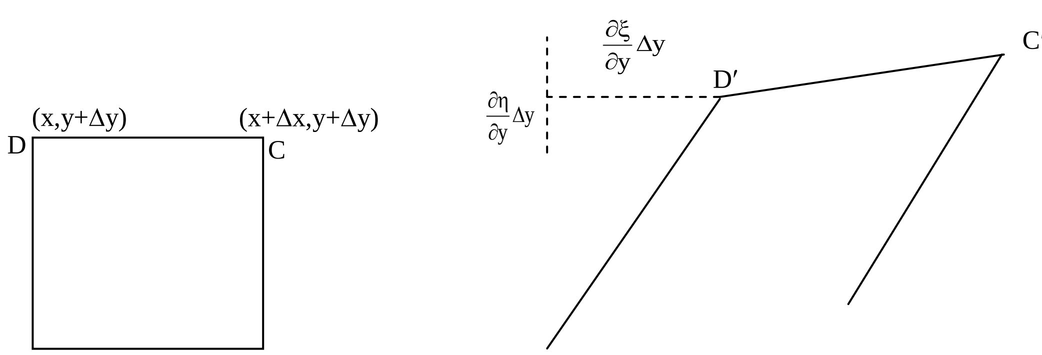
3. Nature of Strains (Rates of Strain)

The change in the relative position of the parts of the body under some force, is termed as **deformation.** By Hooke’s law, the stress is proportional to strain in case of elastic bodies, while in case of non-elastic bodies the stress is proportional to the **rate of strain.**

Strain is of two kinds, the normal and the shearing. The ratio of change in length to the original length of a line element is called **normal (or direct) strain.** The shearing strain measures the change in angle between two line elements from the natural state to some standard state. We shall consider two dimensional case and then extend it to three dimensions. Let us consider a rectangular element ABCD of an elastic solid with co-ordinates of A as (x, y) and length of sides as Δx and Δy in the natural state.

Let the point A. be defined to a point A′(x +ξ, y +η) then

$$B(x+\Delta x, y) \text{ goes to } B'\left(x+\xi+\Delta x+\frac{\partial\xi}{\partial x}\Delta x,\ y+\eta+\frac{\partial\eta}{\partial x}\Delta x\right)$$

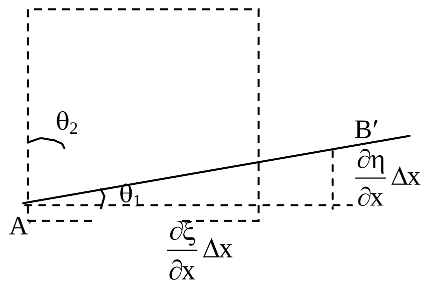


Δy

Δx

A(x,y) B(x+Δx,y)

(Before deformation)



(After deformation)

The point D(x, y +Δy) goes to the point

$$D'(x + \xi + \frac{\partial \xi}{\partial y}\Delta y,\ \ y + \eta + \Delta y + \frac{\partial \eta}{\partial y}\Delta y)\,.$$

Therefore, projected lengths of A′B′ along x and y axes are $\Delta x + \dfrac{\partial \xi}{\partial x}\Delta x$ and $\dfrac{\partial \eta}{\partial x}\Delta x$

Thus,

$$(A'\,B')^2 = \left(\Delta x + \frac{\partial \xi}{\partial x}\Delta x\right)^2 + \left(\frac{\partial \eta}{\partial x}\Delta x\right)^2 \qquad (1)$$

The normal strain along x-axis is defined by

$$\in_{xx} = \frac{A'B' - AB}{AB}$$

$$\Rightarrow \qquad A'B' = (1+\in_{xx})\,AB = (1 + \in_{xx})\,\Delta x \qquad |\,AB = \Delta x \qquad (2)$$

From (1) & (2), we have

$$(1+\in_{xx})^2\,(\Delta x)^2 = (\Delta x)^2\left[\left(1+\frac{\partial \xi}{\partial x}\right)^2 + \left(\frac{\partial \eta}{\partial x}\right)^2\right]$$

$$\Rightarrow \qquad (1+\in_{xx})^2 = \left(1+\frac{\partial \xi}{\partial x}\right)^2 + \left(\frac{\partial \eta}{\partial x}\right)^2$$

From here, to the first order terms only, we get

$$\in_{xx} = \frac{\partial \xi}{\partial x}\,.$$

Similarly, the normal strain along the y-axis is

$$\in_{yy} = \frac{\partial \eta}{\partial y}$$

The shearing strain γ_{xy} at the point A is the change in the angle between the sides AB and AD. The right angle $\underline{|DAB}$ between AB and AD is diminished by $\gamma_{xy} = \theta_1 + \theta_2 = \tan\theta_1 + \tan\theta_2$, θ_1 & θ_2 being small.

i.e. $$\gamma_{xy} = \frac{\frac{\partial \eta}{\partial x}\Delta x}{\left(1+\frac{\partial \xi}{\partial x}\right)\Delta x} + \frac{\frac{\partial \xi}{\partial y}\Delta y}{\left(1+\frac{\partial \eta}{\partial y}\right)\Delta y}$$

$$= \frac{\partial \eta}{\partial x}\left(1+\frac{\partial \xi}{\partial x}\right)^{-1} + \frac{\partial \xi}{\partial y}\left(1+\frac{\partial \eta}{\partial y}\right)^{-1}$$

$$\in_{xy} = \frac{1}{2}(\gamma_{xy}) = \frac{1}{2}\left(\frac{\partial \eta}{\partial x} + \frac{\partial \xi}{\partial y}\right), \qquad \text{upto first order.}$$

We observe that the strains have the nature of change in displacement in a given unit length in a given direction. Hence strain is a tensor of order two.

In the case of fluids, there is no resistance to deformation but only to the time rate of deformation. Hence in fluid dynamics the rate of change of strain with time i.e. rate of strain is to be used in place of strain in elasticity. Thus, for viscous fluids, replacing strains by rates of strain, the corresponding results are obtained to be

$$\in_{xx} = \frac{\partial}{\partial t}\left(\frac{\partial \xi}{\partial x}\right) = \frac{\partial}{\partial x}\left(\frac{\partial \xi}{\partial t}\right) = \frac{\partial}{\partial x}(u) = \frac{\partial u}{\partial x}$$

$$\in_{yy} = \frac{\partial v}{\partial y}, v = \frac{\partial \eta}{\partial t}$$

$$\in_{xy} = \frac{1}{2}\frac{\partial}{\partial t}\left(\frac{\partial \eta}{\partial x} + \frac{\partial \xi}{\partial y}\right) = \frac{1}{2}\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right)$$

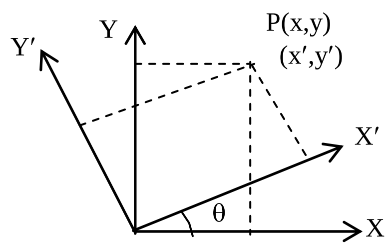
In case of three dimensions, these become

$$\left.\begin{aligned}
\in_{xx} &= \frac{\partial u}{\partial x}, \in_{yy} = \frac{\partial v}{\partial y}, \in_{zz} = \frac{\partial w}{\partial z} \\
\in_{xy} &= \frac{1}{2}(\gamma_{xy}) = \frac{1}{2}\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right) \\
\in_{yz} &= \frac{1}{2}(\gamma_{yz}) = \frac{1}{2}\left(\frac{\partial w}{\partial y} + \frac{\partial v}{\partial z}\right) \\
\in_{zx} &= \frac{1}{2}(\gamma_{zx}) = \frac{1}{2}\left(\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}\right)
\end{aligned}\right\} \qquad \text{(A)}$$

where u, v, w are the velocity components of the viscous fluid along x, y, z axis respectively.

The six quantities $\in_{xx}$, $\in_{yy}$, $\in_{zz}$, γ_{xy}, γ_{yz}, γ_{zx} in (A) are called components of the rates of strain or **gradients of velocity**

3.1. Transformation of Rates of Strain. We shall obtain the rates of strain in term of the new co-ordinates x′, y′, changing from x, y to x′, y′. Let us obtain the new axes by rotating the original axes through angle θ and let $l = \cos\theta$, $m = \sin\theta$



Then $x' = lx + my$, $y' = -mx + ly$

$\Rightarrow$ $x = lx' - my'$, $y = mx' + ly'$

Further, $$\frac{\partial}{\partial t}(x') = \frac{\partial}{\partial t}(lx + my)$$

$\Rightarrow$ $u' = lu + mv$

and $v' = -mu + lv$

Also, $(OP)^2 = x^2 + y^2 = x'^2 + y'^2$ | $\because$ they are still perpendicular

Now, $$\in'_{xx} = \frac{\partial u'}{\partial x'} = \left(\frac{\partial u'}{\partial x}\right)\frac{\partial x}{\partial x'} + \left(\frac{\partial u'}{\partial y}\right)\frac{\partial y}{\partial x'}$$

or $$\in'_{xx} = \left(l\frac{\partial u}{\partial x} + m\frac{\partial v}{\partial x}\right)l + \left(l\frac{\partial u}{\partial y} + m\frac{\partial v}{\partial y}\right)m$$

$$= l^2\frac{\partial u}{\partial x} + m^2\frac{\partial v}{\partial y} + lm\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right)$$

$$= l^2 \in_{xx} + m^2 \in_{yy} + lm.\gamma_{xy}$$

Similarly $\quad \in'_{yy} = \dfrac{\partial v'}{\partial y'} = m^2 \in_{xx} + l^2 \in_{yy} - lm\, \gamma_{xy}$

$$\gamma'_{xy} = \frac{\partial u'}{\partial y'} + \frac{\partial v'}{\partial x'} = 2lm\,(\in_{yy} - \in_{xx}) + (l^2 - m^2)\,\gamma_{xy}.$$

which are the rates of strain of the new system in terms of rates of strain in the original system. If we put back l = cosθ, m = sinθ, then

$$\left.\begin{aligned}
\in'_{xx} &= \frac{\in_{xx} + \in_{yy}}{2} + \frac{\in_{xx} - \in_{yy}}{2}\cos 2\theta + \frac{\gamma_{xy}}{2}\sin 2\theta \\
\in'_{yy} &= \frac{\in_{xx} + \in_{yy}}{2} - \frac{\in_{xx} - \in_{yy}}{2}\cos 2\theta - \frac{\gamma_{xy}}{2}\sin 2\theta \\
\in'_{xy} &= \frac{1}{2}(\gamma'_{xy}) = -\frac{\in_{xx} - \in_{yy}}{2}\sin 2\theta + \frac{\gamma_{xy}}{2}\cos 2\theta
\end{aligned}\right\} \qquad \text{(B)}$$

These equations give the transformation formulae for the rates of strain.

We observe that the rate of strain is also a tensor of order two, there must exist at least two invariants of the rate of strain to the choice of co-ordinate systems. These can be obtained as follows.

$$\in'_{xx} + \in'_{yy} = (l^2 + m^2)\,(\in_{xx} + \in_{yy})$$

$$= \in_{xx} + \in_{yy} = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = \text{div}\ \bar{q}, \qquad \bar{q} = (u, v) \qquad (1)$$

$$\in'_{xx} \in'_{yy} - \frac{(\gamma'_{xy})^2}{4} = (l^2 \in_{xx} + m^2 \in_{yy} + lm\gamma_{xy})\,(m^2 \in_{xx} + l^2 \in_{yy} - lm\,\gamma_{xy})$$

$$- \frac{1}{4}\,[2lm\,(\in_{yy} - \in_{xx}) + (l^2 - m^2)\,\gamma_{xy}]^2$$

$$= (l^4 + 2\,l^2 m^2 + m^4) \in_{xx} \in_{yy} - \frac{\gamma_{xy}^2}{4}\,(l^4 + 2l^2 m^2 + m^4)$$

$$= \in_{xx} \in_{yy} - \frac{\gamma_{xy}^2}{4} \qquad (2)$$

Equation (1) shows that the divergence of the velocity vector at a given point is independent of the orientation of the co-ordinate axes. Equation (2 is related to the dissipation function. i.e. loss of energy due to viscosity.

Let us now consider the general case of the rates of strain in three dimensions. The direction cosines between x, y, z and x′, y′, z′ are related as follows.

	x	y	Z
x′	l_1	m_1	n_1
y′	l_2	m_2	n_2
z′	l_3	m_3	n_3

The relations between co-ordinates in the two systems are

$$x' = l_1x + m_1y \ \ n_1z$$

$$y' = l_2x + m_2y + n_2z$$

$$z' = l_3x + m_3y + n_3z$$

and

$$x = l_1x' + l_2\, y' + l_3\, z'$$

$$y = m_1x' + m_2y' + m_3z'$$

$$z = n_1x' + n_2y' + n_3z'$$

From here, we get

$$u' = l_1u + m_1v + n_1w$$

$$v' = l_2u + m_2v + n_2w$$

$$w' = l_3u + m_3v + n_3w$$

We shall use these relations to find out the rates of strain w. r. t. the new co-ordinates x′, y′, z′.

Let us work out

$$\in'_{xx} = \frac{\partial u'}{\partial x'} = \frac{\partial u'}{\partial x}\frac{\partial x}{\partial x'} + \frac{\partial u'}{\partial y}\frac{\partial y}{\partial x'} + \frac{\partial u'}{\partial z}\frac{\partial z}{\partial x'}$$

$$= \left(l_1 \frac{\partial u}{\partial x} + m_1 \frac{\partial v}{\partial x} + n_1 \frac{\partial w}{\partial x}\right) l_1$$

$$+ \left(l_1 \frac{\partial u}{\partial y} + m_1 \frac{\partial v}{\partial y} + n_1 \frac{\partial w}{\partial y}\right) m_1$$

$$+ \left(l_1 \frac{\partial u}{\partial z} + m_1 \frac{\partial v}{\partial z} + n_1 \frac{\partial w}{\partial z}\right) n_1$$

$$= l_1^2 \in_{xx} + m_1^2 \in_{yy} + n_1^2 \in_{zz} + l_1 m_1 \left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right)$$

$$+ m_1 n_1 \left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right) + n_1 l_1 \left(\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}\right)$$

$$= l_1^2 \in_{xx} + m_1^2 \in_{yy} + n_1^2 \in_{zz} + l_1 m_1 \gamma_{xy} + m_1 n_1 \gamma_{yz} + n_1 l_1 \gamma_{zx}$$

Similarly, we have

$$\in'_{yy} = \frac{\partial v'}{\partial y'} = l_2^2 \in_{xx} + m_2^2 \in_{yy} + n_2^2 \in_{zz} + l_2 m_2 \gamma_{xy} + m_2 n_2 \gamma_{yz} + n_2 l_2 \gamma_{zx}$$

$$\in'_{zz} = \frac{\partial w'}{\partial z'} = l_3^2 \in_{xx} + m_3^2 \in_{yy} + n_3^2 \in_{zz} + l_3 m_3 \gamma_{xy} + m_3 n_3 \gamma_{yz} + n_3 l_3 \gamma_{zx}$$

$$\gamma'_{xy} = \frac{\partial v'}{\partial x'} + \frac{\partial u'}{\partial y'} = \frac{\partial v'}{\partial x}\frac{\partial x}{\partial x'} + \frac{\partial v'}{\partial y}\frac{\partial y}{\partial x'} + \frac{\partial v'}{\partial z}\frac{\partial z}{\partial x'} + \frac{\partial u'}{\partial x}\frac{\partial x}{\partial y'} + \frac{\partial u'}{\partial y}\frac{\partial y}{\partial y'} + \frac{\partial u'}{\partial z}\frac{\partial z}{\partial y'}$$

$$= 2 l_1 l_2 \in_{xx} + 2 m_1 m_2 \in_{yy} + 2 n_1 n_2 \in_{zz}$$

$$+ (l_1 m_2 + m_1 l_2) \gamma_{xy} + (m_1 n_2 + n_1 m_2) \gamma_{yz} + (n_1 l_2 + l_1 n_2) \gamma_{zx}$$

$$\gamma'_{yz} = \frac{\partial w'}{\partial y'} + \frac{\partial v'}{\partial z'} = 2 l_2 l_3 \in_{xx} + 2 m_2 m_3 \in_{yy} + 2 n_2 n_3 \in_{zz}$$

$$+ (l_2 m_3 + m_2 l_3) \gamma_{xy} + (m_2 n_3 + n_2 m_3) \gamma_{yz} + (n_2 l_3 + l_2 n_3) \gamma_{zx}$$

$$\gamma'_{zx} = \frac{\partial u'}{\partial z'} + \frac{\partial w'}{\partial x'} = 2l_3 l_1 \in_{xx} + 2m_3\, m_1 \in_{yy} + 2n_3 n_1 \in_{zz}$$

$$+ (l_3 m_1 + m_3 l_1)\, \gamma_{xy} + (m_3 n_1 + n_3 m_1)\, \gamma_{yz} + (n_3 l_1 + l_3 n_1)\, \gamma_{zx}$$

From here, we find

$$\begin{aligned}
\in'_{xx} + \in'_{yy} + \in'_{zz} &= (l_1^2 + l_2^2 + l_3^2) \in_{xx} + (m_1^2 + m_2^2 + m_3^2) \in_{yy} \\
&\quad + (n_1^2 + n_2^2 + n_3^2) \in_{zz} + (l_1 m_1 + l_2 m_2 + l_3 m_3)\, \gamma_{xy} \\
&\quad + (m_1 n_1 + m_2 n_2 + m_3 n_3)\, \gamma_{yz} + (n_1 l_1 + n_2 l_2 + n_3 l_3)\, \gamma_{zx} \\
&= \in_{xx} + \in_{yy} + \in_{zz}
\end{aligned}$$

where we have used the orthogonality relations

$$l_1^2 + l_2^2 + l_3^2 = 1 \text{ etc}$$

and $\qquad l_1 m_1 + l_2 m_2 + l_3 m_3 = 0$ etc.

Thus we conclude that

$$\in'_{xx} + \in'_{yy} + \in'_{zz} = \in_{xx} + \in_{yy} + \in_{zz}$$

$$= \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = \text{div } \bar{q}$$

is invariant.

Similarly,

$$\in'_{xx} \in'_{yy} + \in'_{yy} \in'_{zz} + \in'_{zz} \in'_{xx} - \frac{1}{4} [(\gamma'_{xy})^2 + (\gamma'_{yz})^2 + (\gamma'_{zx})^2]$$

$$= \in_{xx} \in_{yy} + \in_{yy} \in_{zz} + \in_{zz} \in_{xx} - \frac{1}{4} [(\gamma_{xy})^2 + (\gamma_{yz})^2 + (\gamma_{zx})^2]$$

is also invariant.

3.2. Remark. The stress tensor σ_{ij} and the rates of strain $\in_{ij}$ follow the same rules of transformation. Thus, the three equations in (B) can also be written for stress components so that we get the relations between the original and the new stress components as

$$\left.\begin{aligned}
\sigma'_{xx} &= \frac{\sigma_{xx}+\sigma_{yy}}{2}+\frac{\sigma_{xx}-\sigma_{yy}}{2}\cos 2\theta+\sigma_{xy}\sin 2\theta \\
\sigma'_{yy} &= \frac{\sigma_{xx}+\sigma_{yy}}{2}-\frac{\sigma_{xx}-\sigma_{yy}}{2}\cos 2\theta-\sigma_{xy}\sin 2\theta \\
\sigma'_{xy} &= -\frac{\sigma_{xx}-\sigma_{yy}}{2}\sin 2\theta+\sigma_{xy}\cos 2\theta
\end{aligned}\right\} \quad (C)$$

4. Relations Between the Stress and Gradients of Velocity

(Equivalence of Hooke's Law in Case of Viscous Fluids)

In elasticity, generalized Hooke's law gives a relation between the stress and the strain components.

For viscous fluid, the following assumptions are to be made to find the relations between the stress and the rate of strain.

(i) The stress components may be expressed as linear functions of rates of strain components.

(ii) The relations between stress and rates of strain are invariant w.r.t rotation and reflection of co-ordinate axes (symmetry).

(iii) The stress components reduce to the hydrostatic pressure when all the gradients of velocity are zero.

i.e. $\sigma_{xx} = -p = \sigma_{yy} = \sigma_{zz},\ \in_{xx} = \dfrac{\partial u}{\partial x} = 0 = \in_{yy} = \in_{zz}.$

First we consider two dimensional case and then we extend it to three dimensions.

Under the assumption (i), we can write

$$\begin{aligned}
\sigma_{xx} &= A_1 \in_{xx} + B_1 \in_{yy} + C_1 \gamma_{xy} + D_1 \\
\sigma_{yy} &= A_2 \in_{xx} + B_2 \in_{yy} + C_2 \gamma_{xy} + D_2 \qquad (1) \\
\sigma_{xy} &= A_3 \in_{xx} + B_3 \in_{yy} + C_3 \gamma_{xy} + D_3
\end{aligned}$$

where A′s, B′s, C′s and D′s are constants to be determined.

From the assumption (ii), we have

$$\begin{aligned}
\sigma'_{xx} &= A_1 \in'_{xx} + B_1 \in'_{yy} + C_1 \gamma'_{xy} + D_1 \\
\sigma'_{yy} &= A_2 \in'_{xx} + B_2 \in'_{yy} + C_2 \gamma'_{xy} + D_2 \qquad (2)
\end{aligned}$$

$$\sigma'_{xy} = A_3 \in'_{xx} + B_3 \in'_{yy} + C_3 \gamma'_{xy} + D_3$$

But the relations between the original and the new stress components are (from equation (C))

$$\left.\begin{aligned}
\sigma'_{xx} &= \frac{\sigma_{xx}+\sigma_{yy}}{2} + \frac{\sigma_{xx}-\sigma_{yy}}{2}\cos 2\theta + \sigma_{xy}\sin 2\theta \\
\sigma'_{yy} &= \frac{\sigma_{xx}+\sigma_{yy}}{2} - \frac{\sigma_{xx}-\sigma_{yy}}{2}\cos 2\theta - \sigma_{xy}\sin 2\theta \\
\sigma'_{xy} &= -\frac{\sigma_{xx}-\sigma_{yy}}{2}\sin 2\theta + \sigma_{xy}\cos 2\theta
\end{aligned}\right\} \quad (3)$$

Using the equation (1) in 1st of (3), we get

$$\begin{aligned}
\sigma'_{xx} &= \frac{1}{2}(A_1 + A_2) \in_{xx} + \frac{1}{2}(B_1+B_2) \in_{yy} + \frac{1}{2}(C_1+C_2)\gamma_{xy} \\
&+ \frac{1}{2}(D_1+D_2) + \frac{1}{2}(A_1-A_2) \in_{xx} \cos 2\theta \\
&+ \frac{1}{2}(B_1-B_2) \in_{yy} \cos 2\theta + \frac{1}{2}(C_1-C_2)\gamma_{xy}\cos 2\theta \\
&+ \frac{1}{2}(D_1-D_2)\cos 2\theta + (A_3 \in_{xx} + B_3 \in_{yy} + C_3\gamma_{xy} + D_3)\sin 2\theta \qquad (4)
\end{aligned}$$

Also, the relations between the original and the new rates of strain are

$$\left.\begin{aligned}
\in'_{xx} &= \frac{\in_{xx}+\in_{yy}}{2} + \frac{\in_{xx}-\in_{yy}}{2}\cos 2\theta + \frac{\gamma_{xy}}{2}\sin 2\theta \\
\in'_{yy} &= \frac{\in_{xx}+\in_{yy}}{2} - \frac{\in_{xx}-\in_{yy}}{2}\cos 2\theta - \frac{\gamma_{xy}}{2}\sin 2\theta \\
\gamma'_{xy} &= -\frac{\in_{xx}-\in_{yy}}{2}\sin 2\theta + \frac{\gamma_{xy}}{2}\cos 2\theta
\end{aligned}\right\} \quad (5)$$

Using equation (5) in 1st of equations (2), we get

$$\begin{aligned}
\sigma'_{xx} &= \frac{A_1}{2}(\in_{xx} + \in_{yy}) + \frac{A_1}{2}(\in_{xx} - \in_{yy})\cos 2\theta + \frac{A_1}{2}\gamma_{xy}\sin 2\theta \\
&+ \frac{B_1}{2}(\in_{xx} + \in_{yy}) - \frac{B_1}{2}(\in_{xx} - \in_{yy})\cos 2\theta - \frac{B_1}{2}\gamma_{xy}\sin 2\theta
\end{aligned}$$

$$- C_1(\in_{xx} - \in_{yy}) \sin 2\theta + C_1 \gamma_{xy} \cos 2\theta + D_1 \qquad (6)$$

Comparing co-efficients in (4) & (6), we get

$$\frac{A_1}{2}(1+\cos 2\theta) + \frac{A_2}{2}(1-\cos 2\theta) + A_3 \sin 2\theta$$

$$= \frac{A_1}{2}(1 + \cos 2\theta) + \frac{B_1}{2}(1-\cos 2\theta) - C_1 \sin 2\theta \qquad | \in_{xx}$$

$$\frac{B_1}{2}(1+\cos 2\theta) + \frac{B_2}{2}(1-\cos 2\theta) + B_3 \sin 2\theta$$

$$= \frac{A_1}{2}(1 - \cos 2\theta) + \frac{B_1}{2}(1+\cos 2\theta) + C_1 \sin 2\theta \qquad | \in_{yy}$$

$$\frac{C_1}{2}(1+\cos 2\theta) + \frac{C_2}{2}(1-\cos 2\theta) + C_3 \sin 2\theta$$

$$= \frac{A_1}{2}\sin 2\theta - \frac{B_1}{2}\sin 2\theta + C_1 \cos 2\theta \qquad | \gamma_{xy}$$

$$\frac{D_1}{2}(1+\cos 2\theta) + \frac{D_2}{2}(1-\cos 2\theta) + D_3 \sin 2\theta = D_1$$

From these equations, we get

$$A_2 = B_1 = B(\text{say}),\ B_2 = A_1 = A(\text{say})$$

$$C_2 = A_3 = - C_1 = -B_3 = -C(\text{say})$$

$$C_3 = \frac{A_1 - B_1}{2} = \frac{A - B}{2},\ D_1 = D_2 = D\ (\text{say}),\ D_3 = 0$$

The stress components in terms of the rates of strain are now obtained to be

$$\left.\begin{aligned} \sigma_{xx} &= A\in_{xx} + B\in_{yy} + C\gamma_{xy} + D \\ \sigma_{yy} &= B\in_{xx} + A\in_{yy} - C\gamma_{xy} + D \\ \sigma_{xy} &= -C(\in_{xx} - \in_{yy}) + \frac{A-B}{2}\gamma_{xy} \end{aligned}\right] \qquad (7)$$

To find A, B, C and D, we make use of the assumption that there is symmetry of the fluid about the co-ordinate axes.

Let us take the symmetry w.r.t. the y-axis. If (x_1, y_1) are the new co-ordinates of the point with co-ordinates (x, y), then

$$x_1 = -x,\ y_1 = y$$

i.e. $\qquad u_1 = -u,\ v_1 = v$

The rates of strain w.r.t. (x_1, y_1) co-ordinates are

$$\in_{x_1x_1} = \frac{\partial u_1}{\partial x_1} = \frac{-\partial u}{\partial x_1} = -\frac{\partial u}{\partial x}\frac{\partial u}{\partial x_1} - \frac{\partial u}{\partial y}\frac{\partial y}{\partial x_1}$$

$$= \frac{\partial u}{\partial x} = \in_{xx} \qquad \Big|\ \because \frac{\partial x}{\partial x_1} = -1,\ \frac{\partial y}{\partial x_1} = 0$$

Similarly,

$$\in_{y_1y_1} = \in_{yy},\quad \gamma_{x_1y_1} = -\gamma_{xy}$$

and

$$\sigma_{x_1x_1} = \sigma_{xx},\ \sigma_{y_1y_1} = \sigma_{yy},\ \sigma_{x_1y_1} = -\sigma_{xy}$$

Using these in (7), we get

$$\left.\begin{aligned}
\sigma_{x_1x_1} &= A\in_{x_1x_1} + B\in_{y_1y_1} - C\gamma_{x_1y_1} + D \\
\sigma_{y_1y_1} &= B\in_{x_1x_1} + A\in_{y_1y_1} + C\gamma_{x_1y_1} + D \\
\sigma_{x_1y_1} &= C(\in_{x_1x_1} - \in_{y_1y_1}) + \frac{A-B}{2}\gamma_{x_1y_1}
\end{aligned}\right] \qquad (8)$$

The relations (7) are invariant where there is a symmetry w.r.t. any co-ordinate transformation and so

$$\left.\begin{aligned}
\sigma_{x_1x_1} &= A\in_{x_1x_1} + B\in_{y_1y_1} + C\gamma_{x_1y_1} + D \\
\sigma_{y_1y_1} &= B\in_{x_1x_1} + A\in_{y_1y_1} - C\gamma_{x_1y_1} + D \\
\sigma_{x_1y_1} &= -C(\in_{x_1x_1} - \in_{y_1y_1}) + \frac{A-B}{2}\gamma_{x_1y_1}
\end{aligned}\right\} \qquad (9)$$

Comparing (8) & (9), we find C = 0. According to the assumption (iii), we have

$$\sigma_{xx} = \sigma_{yy} = -p,\ \in_{xx} = \in_{yy} = 0$$

Thus from (7), we find $D = -p$, since $C = 0$.

The last equation in (7) becomes

$\sigma_{xy} = \dfrac{A-B}{2}\ \gamma_{xy} = \mu\ \gamma_{xy}$, where $\mu = \dfrac{A-B}{2}$ is called the **co efficient of viscosity.**

The relations in (7) are now,

$$\sigma_{xx} = A \in_{xx} + B \in_{yy} - p = (A-B) \in_{xx} + B (\in_{xx} + \in_{yy}) - p$$

$$= 2\mu \in_{xx} + B\, \nabla\cdot \bar{q} - p$$

$$\left|\begin{array}{l} \bar{q} = (u, v) \\ \in_{xx} + \in_{yy} = \dfrac{\partial u}{\partial x} + \dfrac{\partial v}{\partial y} = \nabla \cdot \bar{q} \end{array}\right.$$

$$\sigma_{yy} = 2\mu \in_{yy} + B\nabla\cdot\bar{q} - p.$$

$$\sigma_{xy} = \mu\, \gamma_{xy} = 2\mu \in_{xy}$$

These are the required relations between the stress components and the rates of strain in two dimensions.

For three dimensional case, we can write.

$$\left.\begin{aligned} \sigma_{xx} &= 2\mu\in_{xx} + B\nabla\cdot\bar{q} - p = 2\mu\frac{\partial u}{\partial x} + \lambda\nabla\cdot\bar{q} - p \\ \sigma_{yy} &= 2\mu\in_{yy} + B\nabla\cdot\bar{q} - p = 2\mu\frac{\partial v}{\partial y} + \lambda\nabla\cdot\bar{q} - p \\ \sigma_{zz} &= 2\mu\in_{zz} + B\nabla\cdot\bar{q} - p = 2\mu\frac{\partial w}{\partial z} + \lambda\nabla\cdot\bar{q} - p \end{aligned}\right\} \quad (10)$$

$$\left.\begin{aligned} \sigma_{xy} &= \mu\, \gamma_{xy} = \mu\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right), \\ \sigma_{yz} &= \mu\, \gamma_{yz} = \mu\left(\frac{\partial w}{\partial y} + \frac{\partial v}{\partial z}\right) \\ \sigma_{zx} &= \mu\, \gamma_{zx} = \mu\left(\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}\right) \end{aligned}\right\} \quad (11)$$

where $B \equiv \lambda$.

Also, $\sigma_{xx} + \sigma_{yy} + \sigma_{zz} = 2\mu(\in_{xx} + \in_{yy} + \in_{zz}) + 3\lambda\, \nabla\cdot\overline{q} - 3p$

$$= 2\mu\, \nabla\cdot\overline{q} + 3\lambda\, \nabla\cdot\overline{q} - 3p$$

$$= (2\mu + 3\lambda)\, \nabla\cdot\overline{q} - 3p$$

For incompressible fluid $\nabla\cdot\overline{q} = 0$.

$\Rightarrow \qquad \sigma_{xx} + \sigma_{yy} + \sigma_{zz} = -3p$

i.e. $\qquad \dfrac{\sigma_{xx} + \sigma_{yy} + \sigma_{zz}}{3} = -p$

This shows that the mean normal stress is equal to the hydrostatic pressure (i.e. constant)

4.1. Remarks : (i) For compressible fluids, $B \equiv \lambda = -\dfrac{2\mu}{3}$

(ii) Equations (10) and (11) may be combined in tensor form. Thus, if x_i denote the Cartesian co-ordinates, u_i the velocity components (i = 1, 2, 3), then (10) & (11) may be collectively written as

$$\sigma_{ij} = (\lambda\theta - p)\, S_{ij} + \mu(u_{i,j} + u_{j,i}),\ (i, j = 1, 2, 3)$$

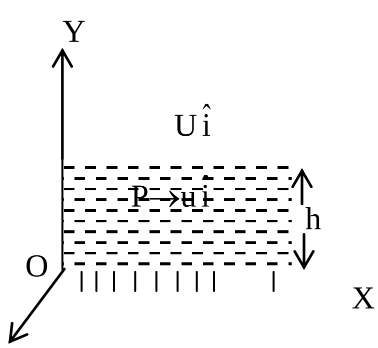
where $\theta = \text{div}\ \overline{q} = u_{j,i}$,

$$p = -\frac{1}{3}\sigma_{i,i},\ \theta = 0 \text{ for incompressible flow,}$$

$$\lambda = -\frac{2}{3}\mu \text{ for compressible flow.}$$

(iv) For viscous fluids, stress is linearly proportional to rate of strain. This law is known as Newton's law of viscosity and such fluids are known as **Newtonian fluids**.

4.2. The Co-efficient of Viscosity and Laminar Flow :



Z

The figure shows two parallel planes y = 0, y = h, a small distance h apart, the space between them being occupied by a thin film of viscous fluid. The plane y = 0 is held fixed and the upper plane is given a constant velocity $U\hat{i}$. If U is not very large, the layers of liquid in contact with y = 0 are at rest and those in contact with y = h are moving with velocity $U\hat{i}$ i.e. there is no slip between fluid and either surface. A velocity gradient is set up in the fluid between the planes. At some point P(x, y, z) in between the planes, the fluid velocity will be $U\hat{i}$, where 0 < u < U and u is independent of x and z. Thus, when y is fixed, u is fixed i.e. fluid moves in layers parallel to two planes. Such flow is termed as **Laminar flow.** Due to viscosity of the fluid there is friction between these layers. Experimental work shows that the shearing stress on the moving plane is proportional to U/h when h is sufficiently small. Thus, we write this stress in the form

$$\sigma_{yx} = \mu \lim_{h\to 0} \frac{U}{h} = \mu \frac{du}{dy}$$

where μ is the co-efficient of viscosity. In aerodynamics, a more important quality is the Kinematic co-efficient of viscosity ν defined by

$$\nu = \mu/\rho.$$

For most fluids μ depends on the pressure and temperature. For gases, according to the Kinetic theory, μ is independent of the pressure but decreases with the temperature.

5. Navier-Stoke's Equations of Motion (Conservation of Linear Momentum)

Let us consider a mass of volume τ enclosed by the surface S in motion at time t. Let $d\tau$ be an element of volume, then the mass of this element is $\rho d\tau$, ρ being the density of the viscous fluid.

Let the element moves with the velocity $\bar{q}$. The inertial force on the element is

$$\rho d\tau \left(\frac{d\bar{q}}{dt}\right) \qquad\qquad |\bar{F} = m\bar{a}$$

The resultant of inertial forces (or the rate of change of linear momentum) is

$$\overline{F}_I = \iiint \rho \frac{d\overline{q}}{dt} d\tau \qquad (1)$$

Let $\overline{X}$ be the body force per unit mass, then the resultant of body force is

$$\overline{F}_B = \iiint \rho \overline{X} d\tau \qquad (2)$$

The surface force on an element $d\overline{A}$ of the surface is given by the vector

$$\bar{f} = fx\hat{i}_x + fy\hat{i}_y + fz\hat{i}_z$$

$$= (\overline{P}_x . d\overline{A})\hat{i}_x + (\overline{P}_y . d\overline{A})\hat{i}_y + (\overline{P}_z . d\overline{A})\hat{i}_z \qquad (3)$$

where $\hat{i}_x, \hat{i}_y, \hat{i}_z$ are unit vectors, $d\overline{A}$ is the vectorial area of the element and $\overline{P}x, \overline{P}y, \overline{P}z$ are components of stress vector, given by

$$\left.\begin{aligned} \overline{P}_x &= \sigma_{xx}\,\hat{i}_x + \sigma_{xy}\,\hat{i}_y + \sigma_{xz}\,\hat{i}_z \\ \overline{P}_y &= \sigma_{yx}\,\hat{i}_x + \sigma_{yy}\,\hat{i}_y + \sigma_{yz}\,\hat{i}_z \\ \overline{P}_z &= \sigma_{zx}\,\hat{i}_x + \sigma_{zy}\,\hat{i}_y + \sigma_{zz}\,\hat{i}_z \end{aligned}\right] \qquad | \; T_i^x = \tau_{ij}\, x_j \qquad (4)$$

The resultant of the surface forces is given by

$$\overline{F}_S = \hat{i}_x \iint \overline{P}_x . d\overline{A} + \hat{i}_y \iint \overline{P}_y . d\overline{A} + \hat{i}_z \iint \overline{P}_z . d\overline{A} \qquad (5)$$

Using Gauss divergence theorem this can be written as

$$\overline{F}_S = \hat{i}_x \iiint \nabla \cdot \overline{P}_x d\tau + \hat{i}_y \iiint \nabla \cdot \overline{P}_y d\tau + \hat{i}_z \iiint \nabla \cdot \overline{P}_z d\tau \quad (6) \quad \Big| \because d\overline{A} = \hat{n}\, dS$$

Let us use the law of conservation of momentum. By this law, the time rate of change of linear momentum is equal to the total force on the fluid mass. Equating the resultant of body and surface forces with that of inertial forces, we obtain.

$$\iiint \rho \frac{d\overline{q}}{dt} d\tau = \iiint \rho \overline{X} d\tau + \hat{i}_x \iiint \nabla \cdot \overline{P}_x d\tau + \hat{i}_y \iiint \nabla \cdot \overline{P}_y d\tau + \hat{i}_z \iiint \nabla \cdot \overline{P}_z d\tau \qquad (7)$$

Since $d\tau$ is an arbitrary volume element, so we have

$$\rho \frac{d\overline{q}}{dt} = \rho \overline{X} + \nabla \cdot \overline{P}_x\, \hat{i}_x + \nabla \cdot \overline{P}_y\, \hat{i}_y + \nabla \cdot \overline{P}_z\, \hat{i}_z \qquad (8)$$

This is the required equation of motion in vector form using the values of $\overline{P}_x, \overline{P}_y, \overline{P}_z$, we get

$$\nabla \cdot \overline{P}_x = \frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{xy}}{\partial y} + \frac{\partial \sigma_{xz}}{\partial z}$$

$$\nabla \cdot \overline{P}_y = \frac{\partial \sigma_{yx}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial y} + \frac{\partial \sigma_{yz}}{\partial z}$$

$$\nabla \cdot \overline{P}_x = \frac{\partial \sigma_{zx}}{\partial x} + \frac{\partial \sigma_{zy}}{\partial y} + \frac{\partial \sigma_{zz}}{\partial z}$$

and let $\overline{q} = (u, v, w)$, $\overline{X} = (X_x, X_y, X_z)$ then the equations of motion can be put as

$$\left.\begin{aligned} \rho \frac{du}{dt} &= \rho X_x + \frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{xy}}{\partial y} + \frac{\partial \sigma_{xz}}{\partial z} \\ \rho \frac{dv}{dt} &= \rho X_y + \frac{\partial \sigma_{yx}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial y} + \frac{\partial \sigma_{yz}}{\partial z} \\ \rho \frac{dw}{dt} &= \rho X_z + \frac{\partial \sigma_{zx}}{\partial x} + \frac{\partial \sigma_{zy}}{\partial y} + \frac{\partial \sigma_{zz}}{\partial z} \end{aligned}\right\} \qquad (9)$$

These are the equations of motion in terms of the stress components. (We have also drawn these equations previously)

Also, we know that

$$\frac{d}{dt} \equiv \frac{\partial}{\partial t} + \overline{q} \cdot \nabla \equiv \frac{\partial}{\partial t} + u \frac{\partial}{\partial x} + v \frac{\partial}{\partial y} + w \frac{\partial}{\partial z}$$

and the relations between stress and rates of strain are

$$\sigma_{xx} = 2\mu \frac{\partial u}{\partial x} + \lambda \nabla \cdot \overline{q} - p$$

$$\sigma_{yy} = 2\mu \frac{\partial v}{\partial y} + \lambda \nabla \cdot \overline{q} - p$$

$$\sigma_{zz} = 2\mu \frac{\partial w}{\partial z} + \lambda \nabla \cdot \overline{q} - p$$

$$\sigma_{xy} = \mu \gamma_{xy} = \mu \left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \right)$$

$$\sigma_{yz} = \mu\left(\frac{\partial w}{\partial y} + \frac{\partial v}{\partial z}\right),$$

$$\sigma_{zx} = \mu\left(\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}\right)$$

Using these in (9), we get

$$\left.\begin{aligned}
\rho\frac{du}{dt} &= \rho X_x - \frac{\partial p}{\partial x} + \frac{\partial}{\partial x}\left[\mu\left(2\frac{\partial u}{\partial x} - \frac{2}{3}\nabla\cdot\overline{q}\right)\right] + \frac{\partial}{\partial y}\left[\mu\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)\right] + \frac{\partial}{\partial z}\left[\mu\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right)\right] \\
\rho\frac{dv}{dt} &= \rho X_y - \frac{\partial p}{\partial y} + \frac{\partial}{\partial y}\left[\mu\left(2\frac{\partial v}{\partial y} - \frac{2}{3}\nabla\cdot\overline{q}\right)\right] + \frac{\partial}{\partial z}\left[\mu\left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right)\right] + \frac{\partial}{\partial x}\left[\mu\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right)\right] \\
\rho\frac{dw}{dt} &= \rho X_z - \frac{\partial p}{\partial z} + \frac{\partial}{\partial z}\left[\mu\left(2\frac{\partial w}{\partial z} - \frac{2}{3}\nabla\cdot\overline{q}\right)\right] + \frac{\partial}{\partial x}\left[\mu\left(\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}\right)\right] + \frac{\partial}{\partial y}\left[\mu\left(\frac{\partial w}{\partial y} + \frac{\partial v}{\partial z}\right)\right]
\end{aligned}\right\}$$

(10)

where $\lambda = -\frac{2\mu}{3}$ compressible fluids.

The equation in (10) are called Navier-Stoke's equations for a viscous **compressible** fluid.

5.1. Deductions (i) If μ = co-efficient of viscosity = constant, then Navier-Stoke's equations (10) become

$$\rho\frac{du}{dt} = \rho X_x - \frac{\partial p}{\partial x} + \frac{1}{3}\mu\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right) + \mu\nabla^2 u$$

$$\rho\frac{du}{dt} = \rho X_x - \frac{\partial p}{\partial x} + \frac{1}{3}\mu\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right) + \mu\nabla^2 u$$

$$\rho\frac{dw}{dt} = \rho X_z - \frac{\partial p}{\partial z} + \frac{1}{3}\mu\frac{\partial}{\partial z}\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right) + \mu\nabla^2 w$$

which can be expression in vector form as

$$\rho\frac{d\overline{q}}{dt} = \rho\left[\frac{\partial\overline{q}}{\partial t} + (\overline{q}\cdot\nabla)\overline{q}\right] = \rho\overline{X} - \nabla p + \mu\nabla^2\overline{q} + \frac{\mu}{3}\nabla(\nabla\cdot\overline{q})$$

(ii) For incompressible fluid, ρ = constant,

$$\mu = \text{constant}, \qquad \nabla \cdot \overline{q} = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0.$$

Thus the equations become

$$\frac{d\overline{q}}{dt} = \frac{\partial \overline{q}}{\partial t} + (\overline{q} \cdot \nabla)\overline{q} = \overline{X} - \frac{1}{\rho}\nabla p + \frac{\mu}{\rho}\nabla^2 \overline{q}$$

i.e.
$$\frac{d\overline{q}}{dt} = \overline{X} - \frac{\nabla p}{\rho} + \nu \nabla^2 \overline{q}$$

where $\nu = \mu/\rho$ is called the Kinematic co-efficient of viscosity.

For steady motion with no body forces, we have

$$(\overline{q} \cdot \nabla)\overline{q} = \frac{-\nabla p}{\rho} + \frac{\mu}{\rho}\nabla^2 \overline{q} \qquad \left| \frac{\partial \overline{q}}{\partial t} = 0, \ \overline{X} = 0 \right.$$

(iii) If there is no shear at all i.e $\mu = 0$, then

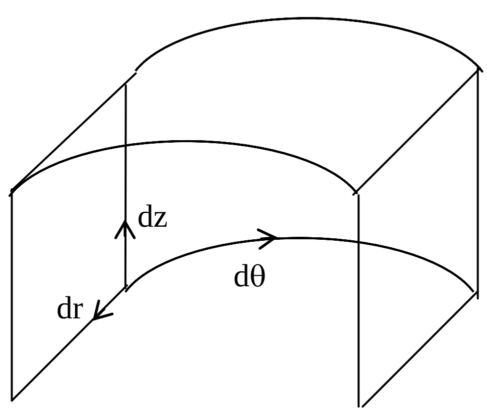
$$\frac{d\overline{q}}{dt} = \frac{\partial \overline{q}}{\partial t} + (\overline{q} \cdot \nabla)\overline{q} = \overline{X} - \frac{\nabla p}{\rho}$$

These are Euler's dynamical equations for an incompressible non-viscous fluid.

5.2. Equations of Motion in Cylindrical Co-ordinates (r, θ, z). In cylindrical co-ordinates (r, θ z), we have $\overline{q} = (q_r, q_\theta, q_z)$ and the acceleration is given by.

$$\frac{d\overline{q}}{dt} = \hat{i}_r\left(\frac{dq_r}{dt} - \frac{q_\theta^2}{r}\right) + \hat{i}_\theta\left(\frac{dq_\theta}{dt} + \frac{q_r q_\theta}{r}\right) + \hat{i}_z \frac{dq_z}{dt} \qquad (1)$$

where $\hat{i}_r, \hat{i}_\theta, \hat{i}_z$ are the unit vectors in the directions of r, θ, z increasing.



(The surface forces are obtained on cylindrical volume)

Thus, in cylindrical co-ordinates; the resultant inertial force is

$$\overline{F}_I = \iiint \rho \frac{d\overline{q}}{dt} d\tau$$

$$= \iiint \left[\hat{i}_r \left(\frac{dq_r}{dt} - \frac{q_\theta^2}{r} \right) + \hat{i}_\theta \left(\frac{dq_\theta}{dt} + \frac{q_r q_\theta}{r} \right) + \hat{i}_z \frac{dq_z}{dt} \right] \rho d\tau \qquad (2)$$

The components of stress vector, $\overline{P}_r, \overline{P}_\theta, \overline{P}_z$ in cylindrical co-ordinates are given by

$$\overline{P}_r = \hat{i}_r \sigma_{rr} + \hat{i}_\theta \sigma_{r\theta} + \hat{i}_z \sigma_{rz}$$
$$\overline{P}_\theta = \hat{i}_r \sigma_{\theta r} + \hat{i}_\theta \sigma_{\theta\theta} + \hat{i}_z \sigma_{\theta z}$$
$$\overline{P}_z = \hat{i}_r \sigma_{zr} + \hat{i}_\theta \sigma_{z\theta} + \hat{i}_z \sigma_{zz}$$

In cylindrical co-ordinates we have

$$\nabla \cdot \overline{P}_r = \frac{1}{r} \left[\frac{\partial}{\partial r}(r\sigma_{rr}) + \frac{\partial}{\partial \theta}(\sigma_{r\theta}) + \frac{\partial}{\partial z}(r\sigma_{rz}) \right] - \frac{\sigma_{\theta\theta}}{r}$$

$$\nabla \cdot \overline{P}_\theta = \frac{1}{r} \left[\frac{\partial}{\partial r}(r\sigma_{\theta r}) + \frac{\partial}{\partial \theta}(\sigma_{\theta\theta}) + \frac{\partial}{\partial z}(r\sigma_{\theta z}) \right] + \frac{\sigma_{r\theta}}{r}$$

$$\nabla \cdot \overline{P}_z = \frac{1}{r} \left[\frac{\partial}{\partial r}(r\sigma_{zr}) + \frac{\partial}{\partial \theta}(\sigma_{z\theta}) + \frac{\partial}{\partial z}(r\sigma_{zz}) \right]$$

Therefore, the equations of motion in vector form

$$\rho \frac{d\overline{q}}{dt} = \rho\overline{X} + \hat{i}_r (\nabla \cdot \overline{P}_r) + \hat{i}_\theta (\nabla \cdot \overline{P}_\theta) + \hat{i}_z (\nabla \cdot \overline{P}_z)$$

reduces to

$$\rho \left(\frac{dq_r}{dt} - \frac{q_\theta^2}{r} \right) = \rho X_r + \frac{1}{r} \left[\frac{\partial}{\partial r}(r\sigma_{rr}) + \frac{\partial}{\partial \theta}(r\sigma_\theta) + \frac{\partial}{\partial z}(r\sigma_{rz}) \right] - \frac{\sigma_{\theta\theta}}{r}$$

$$= \rho X_r + \frac{\partial \sigma_{rr}}{\partial r} + \frac{1}{r}\frac{\partial \sigma_{r\theta}}{\partial \theta} + \frac{\partial \sigma_{rz}}{\partial z} + \frac{\sigma_{rr} - \sigma_{\theta\theta}}{r}$$

$$\rho\left(\frac{dq_\theta}{dt} + \frac{q_r q_\theta}{r}\right) = \rho X_\theta + \frac{\partial}{\partial r}(\sigma_{r\theta}) + \frac{1}{r}\frac{\partial}{\partial \theta}(\sigma_{\theta\theta}) + \frac{\partial}{\partial z}(\sigma_{\theta z}) + 2\frac{\sigma_{r\theta}}{r}$$

$$\rho\frac{dq_z}{dt} = \rho X_z + \frac{\partial \sigma_{rz}}{\partial r} + \frac{1}{r}\frac{\partial \sigma_{\theta z}}{\partial \theta} + \frac{\partial \sigma_{zz}}{\partial z} + \frac{\sigma_{rz}}{r}$$

where $\overline{X} = (X_r, X_\theta, X_z), \frac{d}{dt} \equiv \frac{\partial}{\partial t} + q_r\frac{\partial}{\partial r} + \frac{q_\theta}{r}\frac{\partial}{\partial \theta} + q_z\frac{\partial}{\partial z}$

The relations between the stress components and the rates of strain, in cylindrical co-ordinates are

$$\sigma_{rr} = 2\mu \in_{rr} - \frac{2\mu}{3}\nabla\cdot\overline{q} - p$$

$$\sigma_{\theta\theta} = 2\mu \in_{\theta\theta} - \frac{2\mu}{3}\nabla\cdot\overline{q} - p \qquad \Bigg|\ \lambda = \frac{-2\mu}{3}$$

$$\sigma_{zz} = 2\mu \in_{zz} - \frac{2\mu}{3}\nabla\cdot\overline{q} - p$$

$\sigma_{r\theta} = \mu\, \gamma_{r\theta}$, $\quad \sigma_{rz} = \mu\, \gamma_{rz}$, $\quad \sigma_{\theta z} = \mu\gamma_{\theta z}$, where

$$\in_{rr} = \frac{\partial q_r}{\partial r}, \qquad \in_{\theta\theta} = \frac{1}{r}\frac{\partial q_\theta}{\partial \theta} + \frac{q_r}{r},$$

$$\in_{zz} = \frac{\partial q_z}{\partial z}, \quad \gamma_{r\theta} = \frac{\partial q_\theta}{\partial r} - \frac{q_\theta}{r} + \frac{1}{r}\frac{\partial q_r}{\partial \theta}$$

$$\gamma_{\theta z} = \frac{1}{r}\frac{\partial q_z}{\partial \theta} + \frac{\partial q_\theta}{\partial z}, \qquad \gamma_{rz} = \frac{\partial q_r}{\partial z} + \frac{\partial q_z}{\partial r}$$

Using the above relations, the equations of motion (Navier Stoke's equation) in cylindrical co-ordinates become

$$\rho\left(\frac{dq_r}{dt} - \frac{q_\theta^2}{r}\right) = \rho X_r - \frac{\partial p}{\partial r} + \frac{\partial}{\partial r}\left[\mu\left(2\frac{\partial q_r}{\partial r} - \frac{2}{3}\nabla\cdot\overline{q}\right)\right]$$

$$+ \frac{1}{r}\frac{\partial}{\partial\theta}\left[\mu\left(\frac{1}{r}\frac{\partial q_r}{\partial\theta} + \frac{\partial q_\theta}{\partial r} - \frac{q_\theta}{r}\right)\right]$$

$$+ \frac{\partial}{\partial z}\left[\mu\left(\frac{\partial q_r}{\partial z} + \frac{\partial q_z}{\partial r}\right)\right] + \frac{2\mu}{r}\left(\frac{\partial q_r}{\partial r} - \frac{1}{r}\frac{\partial q_\theta}{\partial\theta} - \frac{q_r}{r}\right)$$

$$\rho\left(\frac{dq_\theta}{dt} + \frac{q_r q_\theta}{r}\right) = \rho X_\theta - \frac{1}{r}\frac{\partial p}{\partial\theta} + \frac{1}{r}\frac{\partial}{\partial\theta}\left[\mu\left(\frac{2}{r}\frac{\partial q_\theta}{\partial\theta} + \frac{2q_r}{r} - \frac{2}{3}\nabla\cdot\bar{q}\right)\right]$$

$$+ \frac{\partial}{\partial r}\left[\mu\left(\frac{1}{r}\frac{\partial q_r}{\partial\theta} + \frac{\partial q_\theta}{\partial r} - \frac{q_\theta}{r}\right)\right]$$

$$+ \frac{\partial}{\partial z}\left[\mu\left(\frac{1}{r}\frac{\partial q_z}{\partial\theta} + \frac{\partial q_\theta}{\partial z}\right)\right] + \frac{2\mu}{r}\left(\frac{1}{r}\frac{\partial q_r}{\partial\theta} + \frac{\partial q_\theta}{\partial r} - \frac{q_\theta}{r}\right)$$

$$\rho\frac{dq_z}{dt} = \rho X_z - \frac{\partial p}{\partial z} + \frac{\partial}{\partial z}\left[\mu\left(2\frac{\partial q_z}{\partial z} - \frac{2}{3}\nabla\cdot\bar{q}\right)\right]$$

$$+ \frac{1}{r}\frac{\partial}{\partial\theta}\left[\mu\left(\frac{1}{r}\frac{\partial q_z}{\partial\theta} + \frac{\partial q_\theta}{\partial z}\right)\right] + \frac{\partial}{\partial r}\left[\mu\left(\frac{\partial q_r}{\partial z} + \frac{\partial q_z}{\partial r}\right)\right]$$

$$+ \frac{\mu}{r}\left(\frac{\partial q_r}{\partial z} + \frac{\partial q_z}{\partial r}\right)$$

where $\nabla\cdot\bar{q} = \dfrac{\partial q_r}{\partial r} + \dfrac{1}{r}\dfrac{\partial q_\theta}{\partial\theta} + \dfrac{\partial q_z}{\partial z} + \dfrac{qr}{r}$.

5.3. Special cases. (i) If ρ = constant and μ = constant, then $\nabla\cdot\bar{q} = 0$ and the equations of motion are

$$\rho\left(\frac{dq_r}{dt} - \frac{q_\theta^2}{r}\right) = \rho X_r - \frac{\partial p}{\partial r} + \mu\left(\nabla^2 q_r - \frac{q_r}{r^2} - \frac{2}{r^2}\frac{\partial q_\theta}{\partial\theta}\right)$$

$$\rho\left(\frac{dq_\theta}{dt} - \frac{q_r q_\theta}{r}\right) = \rho X_\theta - \frac{1}{r}\frac{\partial p}{\partial\theta} + \mu\left(\sigma^2 q_\theta + \frac{2}{r^2}\frac{\partial q_r}{\partial\theta} - \frac{q_\theta}{r^2}\right)$$

$$\rho \frac{dq_z}{dt} = \rho X_z - \frac{\partial p}{\partial z} + \mu \nabla^2 q_z.$$

where $\nabla^2 \equiv \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2} + \frac{\partial^2}{\partial z^2}$

(ii) If the fluid is non-viscous then $\mu = 0$ and if it is incompressible, then $\nabla \cdot \bar{q} = 0$, ρ = constant and the equations of motion become

$$\rho\left(\frac{dq_r}{dt} - \frac{q_\theta^2}{r}\right) = \rho X_r - \frac{\partial p}{\partial r}$$

$$\rho\left(\frac{dq_\theta}{dt} + \frac{q_r q_\theta}{r}\right) = \rho X_\theta - \frac{1}{r}\frac{\partial p}{\partial \theta}$$

$$\rho\left(\frac{dq_z}{dt}\right) = \rho X_z - \frac{\partial p}{\partial z}$$

These are Euler's dynamical equations in cylindrical co-ordinates.

5.4. Equations of Motion in Spherical Co-ordinates (r, θ, ψ). We know that the velocity and acceleration. components in spherical co-ordinates (r, θ, ψ) are

$$q_r = r \cos\theta \sin\theta \frac{d\theta}{dt} + \sin^2\theta \frac{dr}{dt},\ q_\theta = r\frac{d\theta}{dt},$$

$$q_\psi = r \sin\theta \frac{d\psi}{dt}$$

and

$$a_r = \frac{dq_r}{dt} - \frac{q_\theta^2 + q_\psi^2}{r}$$

$$a_\theta = \frac{dq_\theta}{dt} + \frac{q_r q_\theta}{r} - \frac{q_\psi^2 \cot\theta}{r}$$

$$a_\psi = \frac{dq_\psi}{dt} + \frac{q_r q_\psi}{r} + \frac{q_\theta q_\psi \cot\theta}{r}$$

The equations of motion for a viscous incompressible fluid of constant viscosity μ are :

$$\rho \frac{d\bar{q}}{dt} = \rho \bar{X} - \nabla p + \mu \nabla^2 \bar{q}$$

In spherical co-ordinates,

$$\nabla p = \left(\frac{\partial p}{\partial r}, \frac{1}{r}\frac{\partial p}{\partial \theta}, \frac{1}{r\sin\theta}\frac{\partial p}{\partial \psi}\right),$$

$$\overline{X} = (X_r, X_\theta, X_\psi)$$

Let us simplify

$$\nabla^2 \overline{q} = \nabla(\nabla\cdot\overline{q}) - \nabla\times(\nabla\times\overline{q})$$

But $\nabla\cdot\overline{q} = \frac{1}{r}\frac{\partial}{\partial r}(r^2 q_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta q_\theta) + \frac{1}{r\sin\theta}\frac{\partial q\psi}{\partial\psi}$

Further $\quad \nabla\times\overline{q} = \frac{1}{h_1h_2h_3}\begin{vmatrix} h_1\,\hat{i}_r & h_2\,\hat{i}_\theta & h_3\,\hat{i}_\psi \\ \partial/\partial r & \partial/\partial\theta & \partial/\partial\psi \\ h_1 q_r & h_2 q_\theta & h_3 q_\psi \end{vmatrix}$

$$= \frac{1}{r^2\sin\theta}\begin{vmatrix} \hat{i}_r & r\,\hat{i}_\theta & r\sin\theta\,\hat{i}_\psi \\ \partial/\partial r & \partial/\partial\theta & \partial/\partial\psi \\ q_r & r q_\theta & r\sin\theta\, q_\psi \end{vmatrix}$$

$$= \frac{1}{r^2\sin\theta}\left[\hat{i}_r\left\{\frac{\partial}{\partial\theta}(r\sin\theta q_\psi) - \frac{\partial}{\partial\psi}(rq_\theta)\right\} + \hat{i}_\theta r\left\{\frac{\partial q_r}{\partial\psi} - \frac{\partial}{\partial r}(r\sin\theta q_\psi)\right\}\right.$$

$$\left. + \ \hat{i}_\psi r\sin\theta\left\{\frac{\partial}{\partial r}(r\,q_\theta) - \frac{\partial q_r}{\partial\theta}\right\}\right]$$

Then

$$\nabla^2\overline{q} = \hat{i}_r\left[\nabla^2 q_r - \frac{2}{r^2}q_r - \frac{2}{r^2\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta q_\theta) - \frac{2}{r^2\sin\theta}\frac{\partial q_\psi}{\partial\psi}\right]$$

$$+ \ \hat{i}_\theta\left[\nabla^2 q_\theta - \frac{q_\theta}{r^2\sin^2\theta} + \frac{2}{r^2}\frac{\partial q_r}{\partial\theta} - \frac{2\cos\theta}{r^2\sin^2\theta}\frac{\partial q_\psi}{\partial\psi}\right]$$

$$+ \ \hat{i}_\psi\left[\nabla^2 q_\psi - \frac{q_\psi}{r^2\sin^2\theta} + \frac{2}{r^2\sin\theta}\frac{\partial q_r}{\partial\psi} + \frac{2\cos\theta}{r^2\sin^2\theta}\frac{\partial q_\theta}{\partial\psi}\right]$$

Thus, the equations of motion for a viscous incompressible fluid in spherical co-ordinates are

$$\rho\left(\frac{dq_r}{dt}-\frac{q_\theta^2+q_\psi^2}{r}\right)=\rho X_r-\frac{\partial p}{\partial r}$$

$$+\mu\left[\nabla^2 q_r-\frac{2}{r^2}q_r-\frac{2}{r^2\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta q_\theta)-\frac{2}{r^2\sin\theta}\frac{\partial q_\psi}{\partial\psi}\right]$$

$$\rho\left(\frac{dq_\theta}{dt}+\frac{q_r q_\theta}{r}-\frac{q_\psi^2\cot\theta}{r}\right)=\rho X_\theta-\frac{1}{r}\frac{\partial p}{\partial\theta}$$

$$+\mu\left[\nabla^2 q_\theta-\frac{q_\theta}{r^2\sin^2\theta}+\frac{2}{r^2}\frac{\partial q_r}{\partial\theta}-\frac{2\cos\theta}{r^2\sin^2\theta}\frac{\partial q_\psi}{\partial\psi}\right]$$

$$\rho\left(\frac{dq_\psi}{dt}+\frac{q_r q_\psi}{r}+\frac{q_\theta q_\psi\cot\theta}{r}\right)=\rho X_\psi-\frac{1}{r\sin\theta}\frac{\partial p}{\partial\psi}$$

$$+\mu\left[\nabla^2 q_\psi-\frac{q_\psi}{r^2\sin^2\theta}+\frac{2}{r^2\sin\theta}\frac{\partial q_r}{\partial\psi}+\frac{2\cos\theta}{r^2\sin^2\theta}\frac{\partial q_\theta}{\partial\psi}\right]$$

where $\nabla^2\equiv\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right)+\frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\frac{\partial}{\partial\theta}\right)+\frac{1}{r^2\sin^2\theta}\frac{\partial^2}{\partial\psi^2}$

If we put $\mu = 0$ in the above equations, we get the equations of motion for ideal fluid.

6. Steady Flow Between Parallel Planes

For a viscous incompressible fluid in steady flow, the Navier Stoke’s equation with negligible body forces, are

$$\frac{d\bar{q}}{dt}=\frac{-\nabla p}{\rho}+\frac{\mu}{\rho}\nabla^2\bar{q}=\frac{-\nabla p}{\rho}+\nu\nabla^2\bar{q},\ \nu=\frac{\mu}{\rho}$$

In Cartesina co-ordinates; these are

$$u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}+w\frac{\partial u}{\partial z}=-\frac{1}{\rho}\frac{\partial p}{\partial x}+\nu\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}\right)$$

$$\Big|\because \text{For steady case, } \frac{du}{dt} = \frac{\partial u}{\partial t} + (\bar{q}\cdot\nabla)u = (\bar{q}\cdot\nabla)u,\ \frac{\partial}{\partial t} = 0$$

$$= \left(u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y} + w\frac{\partial}{\partial z}\right)u$$

$$= u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z}$$

$$u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z} = -\frac{1}{\rho}\frac{\partial p}{\partial y} + \nu\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial z^2}\right)$$

$$u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z} = -\frac{1}{\rho}\frac{\partial p}{\partial z} + \nu\left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} + \frac{\partial^2 w}{\partial z^2}\right) \qquad (1)$$

The equation of continuity for incompressible flow is

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0 \qquad (2) \qquad \Big|\because \nabla\cdot\bar{q} = 0$$

The equations (1) are non-linear 2nd order partial differential equations and there is no known general method for solving them. However, we shall find some exact solutions of the Navier-Stoke's equations in some special cases. This is one of those cases.

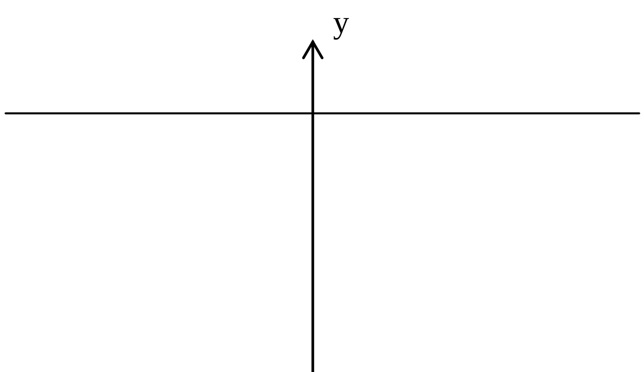
Let us consider a two dimensional steady laminar flow of a viscous in compressible fluid between two parallel straight plates. Let x-axis be the direction of flow, y-axis be perpendicular to it and z-axis be parallel to the width of the plates and let h be the distance between the plates.

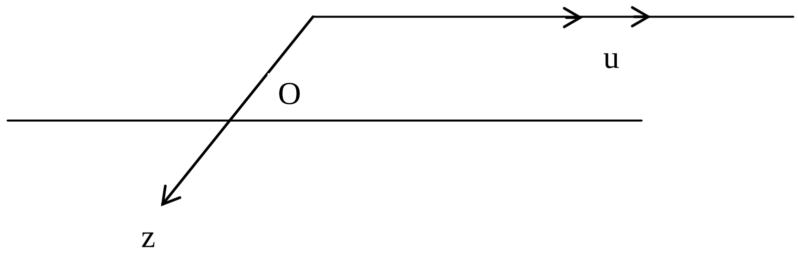
We have the conditions

$$v = 0,\ w = 0 \text{ and } \frac{\partial}{\partial z} \equiv 0 \qquad (3)$$

From the continuity equation (2), we have

$$\frac{\partial u}{\partial x} = 0 \Rightarrow u = u(y) \qquad (4)$$





The second equation of equations (1) gives

$$-\frac{\partial p}{\partial y} = 0 \Rightarrow p = p(x) \qquad (5)$$

The 3rd equation of equations (1) is identically satisfied and the 1st equation gives

$$0 = -\frac{1}{\rho}\frac{dp}{dx} + \nu\frac{d^2u}{dy^2} \Rightarrow \frac{dp}{dx} = \mu\frac{d^2u}{dy^2} \qquad \Big| \because \frac{\mu}{\rho} = \nu \qquad (6)$$

Since u is a function of y only, so $\frac{dp}{dx}$ is either a function of y or a constant. But from (5), p is a function of x alone.

Hence $\frac{dp}{dx}$ is constant. i.e. pressure gradient is constant.

Integrating equation (6) w.r.t y twice, we get the general solution to be

$$u = \frac{1}{\mu}\frac{dp}{dx}\frac{y^2}{2} + Ay + B \qquad (7)$$

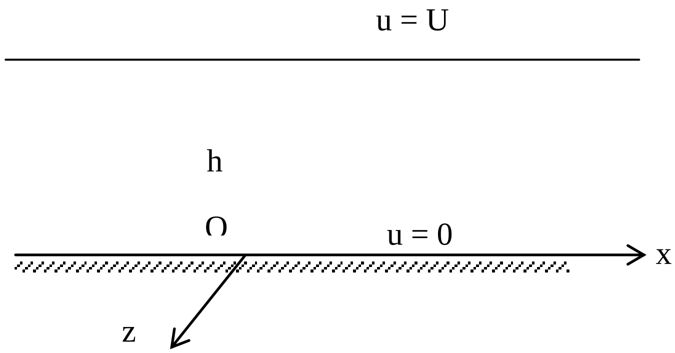
where A and B are constants to be determined from the boundary conditions.

Now we take the following particular cases

6.1. Couette's Flow : It is the flow between two parallel planes (flat plates) one of which is at rest and other moving with velocity U parallel to the fixed plate. Here, the constants A and B in (7) are determined from the conditions

$$\left.\begin{array}{l} u = 0,\ y = 0 \\ u = U,\ y = h \end{array}\right\} \qquad (8)$$

and



Using these conditions, we get

$$B = 0, \;\; U = \frac{1}{\mu}\left(\frac{dp}{dx}\right)\frac{h^2}{2} + Ah$$

$$\Rightarrow \qquad A = \frac{U}{h} - \frac{h}{2\mu}\left(\frac{dp}{dx}\right), \; B = 0 \qquad (9)$$

Therefore, the solution (7) becomes

$$u = \frac{1}{\mu}\left(\frac{dp}{dx}\right)\frac{y^2}{2} + y\left[\frac{U}{h} - \frac{h}{2\mu}\left(\frac{dp}{dx}\right)\right] \qquad (10)$$

$$= \frac{y^2 - hy}{2\mu}\left(\frac{dp}{dx}\right) + \frac{Uy}{h} \qquad (*)$$

$$= \frac{U}{h}y - \frac{h^2}{2\mu}\frac{dp}{dx}\frac{y}{h}\left(1 - \frac{y}{h}\right) \qquad (11)$$

We note that equation (10) represents a parabolic curve.

This equation is known as the equation of Couette's flow. Thus the velocity profile for Couetle's flow is parabolic. The flow Q per unit breadth is given by

$$Q = \int_0^h u\,dy = \int_0^h \left[\frac{1}{m}\frac{dp}{dx}\frac{y^2}{2} + y\left(\frac{U}{h} - \frac{h}{2\mu}\frac{dp}{dx}\right)\right]dy$$

$$= \frac{hU}{2} - \frac{h^3}{12\mu}\frac{dp}{dx} \qquad (**)$$

$$= \frac{hU}{2} + \frac{h^3}{12\mu}P, \quad P = -\frac{dp}{dx} \qquad (12)$$

In non-dimensional form (11) can be written as

$$\frac{u}{U} = \frac{y}{h} + \alpha \frac{y}{h}\left(1 - \frac{y}{h}\right) \qquad (13)$$

where $$\alpha = \frac{h^2}{2\mu U}\left(-\frac{dp}{dx}\right) \qquad (14)$$

α is the non-dimensional pressure gradient. If $\alpha > 0$, the pressure is decreasing in the direction of flow and the velocity is positive between the plates. If $\alpha < 0$, the equation (13) can be put as

$$\frac{u}{U} = \frac{y}{h}(1+\alpha) - \frac{\alpha y^2}{h^2} \qquad (15)$$

The pressure is increasing in the direction of flow and the reverse flow begins when $\alpha < -1$

| $\because$ y is small. i.e. y^2 is neglected

If $\alpha = 0$ $\left(\text{i.e.} \frac{dp}{dx} = 0\right)$, then the particular case is known as simple Couette's flow and the velocity is given by

$$\frac{u}{U} = \frac{y}{h}$$

which gives u = 0 where y = 0 i.e. on the stationary plane.

(i) Average and Extreme Values of Velocity : The average velocity of a Couette's flow between two parallel straight plates is given by

$$u_0 = \frac{1}{h}\int_0^h u\, d_y \qquad (16)$$ | $\because$ u = u(y)

Using the value of u from (13), we get

$$u_0 = \frac{1}{h}\int_0^h \left[\frac{Uy}{h} + U\alpha \frac{y}{h}\left(1 - \frac{y}{h}\right)\right] dy$$

$$= \frac{Uh^2}{2h^2} + U\alpha\left(\frac{h^2}{2h^2} - \frac{h^3}{3h^3}\right)$$

$$= \frac{U}{2} + \frac{U\alpha}{6} = \left(\frac{1}{2} + \frac{\alpha}{6}\right)U \qquad (17)$$

$$= \frac{U}{2} - \frac{\mu^2}{12\mu}\frac{dp}{dx} = \frac{U}{2} + \frac{h^2}{12\mu}P,\ P = -\frac{dp}{dx} \qquad (18)$$

In the case of a simple Couette's flow, the velocity increases from zero on the stationary plate to U on the moving plate such that the average velocity is $\frac{U}{2}$. When the non-dimensional pressure gradient is $\alpha = -3$, then from (17), we get $u_0 = 0$. This means that there is no flow because the pressure gradient is balanced by the viscous force.

For maximum & minimum values of u, we have

$$\frac{du}{dy} = 0 \Rightarrow \frac{U}{h} + U\alpha\left(\frac{1}{h} - \frac{2y}{h^2}\right) = 0$$

$$\Rightarrow y = \left(\frac{1+\alpha}{2\alpha}\right)h \qquad (19)$$

From here, $\frac{y}{h} = 1$, when $\alpha = 1$

and $\frac{y}{h} = 0$, when $\alpha = -1$

So, from (13), we get

$$u = \left[\frac{1+\alpha}{2\alpha} + \alpha\left(\frac{1+\alpha}{2\alpha}\right)\left(1 - \frac{1+\alpha}{2\alpha}\right)\right]U$$

$$= \frac{(1+\alpha)^2}{4\alpha}U$$

and thus u is maximum for $\alpha \geq 1$ and minimum for $\alpha \leq -1$.

(ii) Shearing Stress : The shearing stress (drag per unit area) in a Couette's flow is given by

$$\sigma_{yx} = \mu\frac{du}{dy} = \mu\frac{U}{h} + \frac{\mu\alpha U}{h}\left(1 - \frac{2y}{h}\right) \qquad (20)$$

$$= \frac{\mu U}{h}, \text{ for a simple Couette's flow } (\alpha = 0).$$

When $y = \frac{h}{2}$, then the second term in (20) vanishes. Thus the shearing stress is independent of α on the line midway between the flow. The shearing stress at the stationary plane is positive for $\alpha > -1$ and negative for $\alpha < -1$.
| y = 0 at stationary plate

The velocity gradient at the stationary plate is zero for $\alpha = -1$ and the shearing stress is zero for $\alpha = -1$.

Thus $\sigma_{yx} \gtreqless 0$ when $\alpha \gtreqless -1$.

Further, drag per unit area on the lower and the upper plates are obtained from (20) by putting y = 0 and y = h, as

$$\frac{\mu U}{h} + \frac{\mu\alpha U}{h} \text{ and } \frac{\mu U}{h} - \frac{\mu\alpha U}{h}$$

combining the two results, drag per unit area on the two plates is

$$\frac{\mu U}{h} \pm \frac{\mu\alpha U}{h} \text{ i.e. } \frac{\mu U}{h} \mp \frac{h}{2}\frac{dp}{dx} \qquad (***)$$

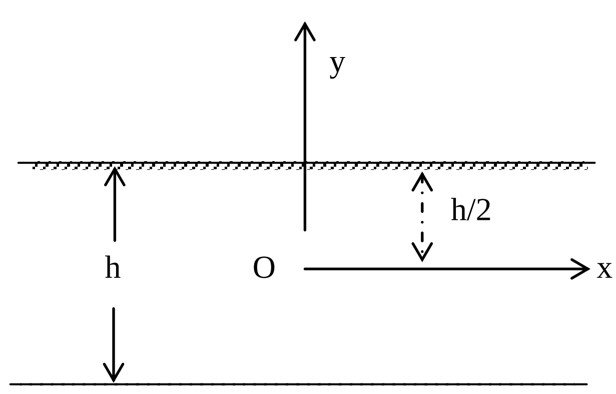
i.e. $$\frac{\mu U}{h} \pm \frac{Ph}{2},\ P = -\frac{dp}{dx}$$

6.2. Plane Poiseuille Flow : A flow between two parallel stationary plates is said to be a plane Poiseuille Flow.

The origin is taken on the line midway between the plates which are placed at a distance h and x-axis is along this line.

The conditions to be used in this problem are

$$u = 0, \text{ when } y = \pm \frac{h}{2} \qquad (21)$$



Using these conditions in (7), we get

$$A = 0,\ B = \frac{1}{\mu}\left(-\frac{dp}{dx}\right)\frac{h^2}{8}$$

and thus the solution (7) is

$$u = \frac{1}{\mu}\left(\frac{dp}{dx}\right)\left(\frac{y^2}{2} - \frac{h^2}{8}\right) \qquad (22)$$

This represents a parabola and thus the laminar flow in a Plane Poiseuille Flow is parabolic.

(i) Average and Maximum Velocity : For extreme values of u, we have $\frac{du}{dy} = 0$ and thus from (22), we get

$$\frac{1}{\mu}\left(\frac{dp}{dx}\right) y = 0 \quad \Rightarrow y = 0$$

Therefore , $$U_{max} = \frac{h^2}{8\mu}\left(-\frac{dp}{dx}\right) \qquad (23)$$

The average velocity in the plane Poiseuille flow is defined by

$$u_0 = \frac{1}{h}\int_{-h/2}^{h/2} u\, dy$$

Using the value of u from (22), we get

$$u_0 = \frac{1}{h}\int_{-h/2}^{h/2} \frac{-h^2}{8\mu}\frac{dp}{dx}\left(1 - \frac{4y^2}{h^2}\right)dy$$

$$= \frac{2}{3}\left(\frac{-h^2}{8\mu}\frac{dp}{d_x}\right) = \frac{2}{3}U_{max} \qquad (24)$$

From (23) & (24), decrease in the pressure is given by

$$\frac{dp}{dx} = -\frac{8\mu}{h^2}u_{max.} = \frac{-8\mu}{h^2}\frac{3}{2}u_0 = \frac{-12\mu}{h^2}u_0 \qquad (25)$$

This further shows that $\frac{dp}{dx}$ is a negative constant.

(ii) Shearing Stress : The shearing stress at a plate (lower plate) for a plane Poiseuille Flow is

$$(\sigma_{yx})_{y=\frac{-h}{2}} = \left(\mu\frac{du}{dy}\right)_{y=\frac{h}{2}} = -\mu\frac{1}{\mu}.\frac{dp}{dx}.\frac{h}{2}$$

$$= -\frac{h}{2}\frac{dp}{dx}$$

$$= \frac{4\mu}{h}u_{max.} \qquad (26)$$

The local frictional (skin) co-efficient C_f is defined by

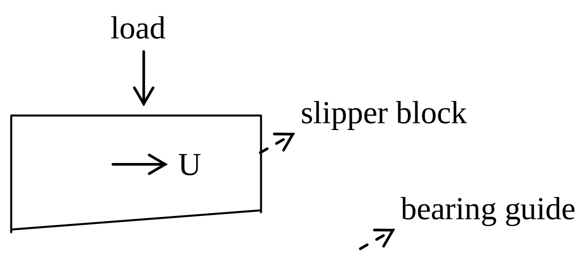
$$C_f = \frac{(\sigma_{yx})_{-h/2}}{\rho u_{0/2}^2} = \frac{4\mu}{h}u_{max.}\Big/\frac{\rho u_0^2}{2}$$

$$= \frac{4\mu}{\rho h}\left(\frac{3}{2}\frac{u_0}{u_0^2/2}\right) = \frac{12v}{h u_0} = \frac{12}{R_e}$$

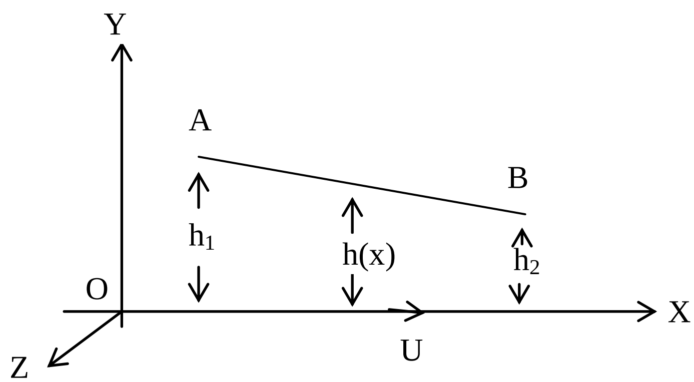
Where $R_e = \frac{u_0 h}{v}$ is the Reynolds number of the flow based on the average velocity and the channel height.

7. Theory of Lubrication

The hydrodynamic theory of Couette flow can be applied in the study of lubrication by considering an example of the slipper bearing which consists of a sliding block moving over a stationary guide and inclined at a small angle with respect to the stationary pad. The gap between the sliding block and the pad is always much smaller than the length of the block and is filled with a lubricant, usually oil. For such a case viscous forces are predominant. The theory of lubrication was first developed by Osborne Reynolds in 1886, and the discussion is due to Lord Rayleigh (1918).



In order to make the motion steady, a system of co-ordinates is chosen in which the slipper block is stationary and the pad moves with a uniform velocity U in the x-direction. Since the slipper block is inclined relative to the guide, a pressure difference is set up in the gap between the slipper and the guide. At high velocities, extreme pressure difference can be created to support heavy loads in the direction normal to the guide. Let the block be so wide in the z-direction that the problem may be treated as two-dimensional.



Let (a, h_1), (b, h_2) and (x, h) be the co-ordinates of A, B and any point on AB. Since the addition of a constant pressure throughout the fluid will make no difference to the solution, so we may for convenience assume that p = 0 beyond the ends of the block. Since the inclination of the plane faces is small, (i.e. the faces are nearly parallel) the velocity u at any point is given by

$$u = \frac{y^2 - hy}{2\mu}\frac{dp}{dx} + \frac{Uy}{h}$$ (from (∗) of previous article) and the flow Q in x-direction is

$$Q = \frac{hU}{2} - \frac{h^3}{12\mu}\frac{dp}{dx}$$ (from (∗∗) of previous article)

The condition of continuity requires that Q must be independent of x i.e. Q = Q(y). Hence

$$\frac{1}{2}hU - \frac{h^3}{12\mu}\frac{dp}{d_x} = \text{constant} = \frac{1}{2}h_0U$$

$$\Rightarrow \qquad \frac{dp}{d_x} = 6\mu U\left(\frac{h-h^0}{h^3}\right) \qquad (1)$$

where h_0 is the value of hat the points of maximum pressure $\left(\text{s.t.} \frac{dp}{d_x} = 0\right)$.

Now, the equation of AB is

$$h - h_1 = \frac{h_2 - h_1}{b-a}(x-a)$$

$$\Rightarrow \qquad \frac{dh}{dx} = \frac{h_2 - h_1}{b-a} = \frac{h_2 - h_1}{l} \qquad (2)$$

where l is the length of the block and $\frac{dh}{dx}$ is the slope of the line AB.

From (1) and (2), we get

$$\frac{dp}{dh} = \frac{dp}{dx}\cdot\frac{dx}{dh} = \frac{6\mu Ul}{h_2 - h_1}\left(\frac{1}{h^2} - \frac{h_0}{h^3}\right) \qquad (3)$$

Integrating, we find

$$p = \frac{6\mu Ul}{h_2 - h_1}\left(-\frac{1}{h} + \frac{h_0}{2h^2}\right) + C$$

$$= \frac{3\mu Ul}{h_2 - h_1}\left(\frac{h_0 - 2h}{h^2}\right) + C \qquad (4)$$

We now determine h_0 and C so that

$$p = 0 \text{ when } h = h_1 \text{ and when } h = h_2$$

This gives

$$h_0 = \frac{2h_1h_2}{h_1 + h_2}, \qquad C = \frac{6\mu Ul}{(h_2 - h_1)(h_2 + h_1)}$$

and thus $$p = \frac{6\mu U l(h-h_1)(h-h_2)}{h^2(h_2^2-h_1^2)}$$

or $$p = \frac{6\mu U l}{h_1^2-h_2^2}\frac{(h_1-h)(h-h_2)}{h^2} \qquad (5)$$

This suggests that $p > 0$ if $h_1 > h_2$ i.e. the stream contracts in the direction of motion. $P > 0$ yields thrust rather than a suction. So we conclude that a necessary condition for lubrication is that the relative motion should tend to drag the fluid from the wider to narrower part of the intervening space i.e. the stream should be convergent.

The total pressure (thrust)P is given by

$$P = \int_a^b p\,dx = \int_{h_1}^{h_2} p\left(\frac{dx}{dh}\right)dh$$

$$= \frac{l}{h_2-h_1}\int_{h_1}^{h_2} p\,dh \qquad \text{| using (2)}$$

$$= \frac{6\mu U l^2}{(h_1^2-h_2^2)(h_2-h_1)}\int_{h_1}^{h_2} \frac{(h_1-h)(h-h_2)}{h^2}\,dh$$

$$= \frac{6\mu U l^2}{(h_1-h_2)^2}\int_{h_1}^{h_2} \frac{(h-h_1)(h-h_2)}{h^2(h_1+h_2)}\,dh \qquad (6)$$

To find the integral in (6), we observe that

$$\int_{h_1}^{h_2} \frac{(h-h_1)(h-h_2)}{h^2}\,dh = \int_{h_1}^{h_2}\left(1+\frac{h_1h_2}{h^2}-\frac{h_1}{h}-\frac{h_2}{h}\right)dh$$

$$= \left[h - \frac{h_1h_2}{h} - h_1\log h - h_2\log h\right]_{h_1}^{h_2}$$

$$= -2(h_1-h_2) + (h_1+h_2)\log\left(\frac{h_1}{h_2}\right)$$

$$\Rightarrow \qquad \frac{1}{h_1+h_2}\int_{h_1}^{h_2} \frac{(h-h_1)(h-h_2)}{h^2}\,dh = \log(h_1/h_2) - 2\frac{h_1-h_2}{h_1+h_2}$$

$$= \log k - 2\left(\frac{k-1}{k+1}\right), \quad k = h_1/h_2.$$

Thus (6) becomes

$$P = \frac{6\mu U l^2}{h_2^2 (k-1)^2}\left[\log k - 2\left(\frac{k-1}{k+1}\right)\right] \tag{7}$$

Now, the tangential stress (drag) at the section h is

$$(\sigma_{yx})_{y=h} = \frac{\mu U}{h} + \frac{h}{2}\frac{dp}{dx} \qquad \text{| From (***) of previous article.}$$

and thus the total frictional force experienced by the moving fluid is

$$F = \int_a^b (\sigma_{yx})_{y=h}\, dx$$

$$= \int_a^b \left[\frac{\mu U}{h} + \frac{h}{2} 6\mu U\left(\frac{h-h_0}{h^3}\right)\right] dx \qquad \text{| using (1)}$$

$$= \mu U \int_{h_1}^{h_2} \left(\frac{u}{h} - \frac{3h_0}{h^2}\right)\frac{l}{h_2 - h_1}\, dh \qquad \text{| using (2)}$$

$$= \frac{\mu U l}{h_2 - h_1}\int_{h_1}^{h_2} \left(\frac{u}{h} - \frac{1}{h^2}.\frac{6h_1 h_2}{h_1 + h_2}\right) dh$$

$$= \frac{2\mu U l}{h_2 (k-1)}\left[2\log k - 3\left(\frac{k-1}{k+1}\right)\right] \tag{8}$$

Comparing (7) and (8) we see that the ratio F/P of the total friction to the total load is independent of both μ and U, but proportional to h if the scale of h is altered.

It has been found by Reynolds and Rayleigh that the value of k which makes P a maximum is 2.2 (approx.) and that this makes $P = 0.16\,\frac{\mu U l^2}{h_2^2}$, $F = 0.75\,\frac{\mu U l}{h_2}$.

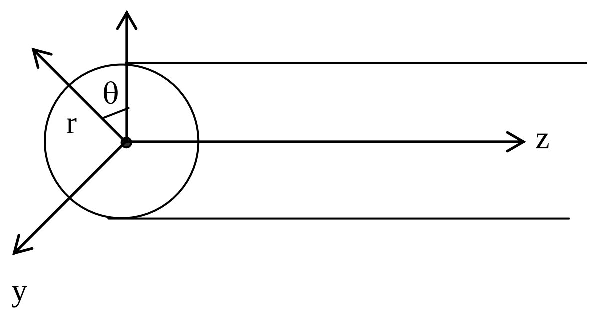
For this case, $F/P = 4.7\,\frac{h_2}{l}$.

By making h_2 small enough compared to l, we can ensure a small frictional drag i.e. good lubrication.

8. Steady Flow Through Tube of Uniform Circular Cross-section (Poiseuille's Flow or Hagen-Poiseuill's Flow)

We consider a laminar flow, in the absence of body forces, through a long tube of uniform circular cross-section with axial symmetry.

Let z-axis be taken along the axis of the tube and the flow be in the direction of z-axis. Since the flow is along z-axis, the radial and transverse components of velocity are absent.



Thus $q_r = q_\theta = 0$ $\qquad\qquad \bar{q} = (q_r, q_\theta, q_z)$

The continuity equation for a viscous incompressible fluid gives.

$$\frac{\partial q_z}{\partial z} = 0 \;\Rightarrow q_z = q_z(r) \qquad (1) \qquad | \because \text{ axial symmetry i.e. independent of } \theta$$

The equations of motion in cylindrical co-ords are

$$\rho\left(\frac{dq_r}{dt} - \frac{{q_\theta}^2}{r}\right) = \rho \cdot X_r - \frac{\partial p}{\partial r} + \mu\left(\nabla^2 q_r - \frac{q_r}{r^2} - \frac{2}{r^2}\frac{\partial q_\theta}{\partial \theta}\right)$$

$$\rho\left(\frac{dq_\theta}{dt} + \frac{q_r q_\theta}{r}\right) = \rho \cdot X_\theta - \frac{1}{r}\frac{\partial p}{\partial \theta} + \mu\left(\nabla^2 q_\theta + \frac{2}{r^2}\frac{\partial q_r}{\partial \theta} - \frac{q_\theta}{r^2}\right)$$

$$\rho\frac{dq_z}{dt} = \rho X_z - \frac{\partial p}{\partial z} + \mu\nabla^2 q_z$$

where $\qquad \frac{d}{dt} = \frac{\partial}{\partial t} + q_r\frac{\partial}{\partial r} + q_\theta\frac{1}{r}\frac{\partial}{\partial \theta} + q_z\frac{\partial}{\partial z},$

and $\qquad \bar{X} = (X_r, X_\theta, X_z)$

In the present case $\frac{\partial}{\partial t} \equiv 0$ and $q_r = q_\theta = 0$, $\bar{X} = 0$

Thus from the first two equations, we get

$$\frac{\partial p}{\partial r} = \frac{\partial p}{\partial \theta} = 0 \quad \Rightarrow p = p(z) \qquad (2)$$

The third equation gives.

$$0 = \frac{-\partial p}{\partial z} + \mu \nabla^2 q_z \qquad \Big| \because q_z = q_z(r) \text{ and } r \text{ is constant w.r.t. } t\,.$$

or
$$\frac{dp}{dz} = \mu \nabla^2 q_z = \mu\left(\frac{d^2 q_z}{dr^2} + \frac{1}{r}\frac{dq_z}{dr}\right) \qquad (3)$$

(In cylindrical co-ordinates $\nabla^2 \equiv \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2} + \frac{\partial^2}{\partial z^2}$)

since q_z is a function of r only (from (1)) and p is a function of z only (from (2)).

Equation (3) can be put as

$$\mu\left(r\frac{d^2 q_z}{dr^2} + \frac{dq_z}{dr}\right) = r\frac{dp}{dz}$$

i.e.
$$\frac{d}{dr}\left(r\frac{dq_z}{dr}\right) = \frac{r}{\mu}\frac{dp}{dz}$$

Integrating, w.r.t. r, we get.

$$r\,\frac{dq_z}{dr} = \frac{1}{\mu}\left(\frac{dp}{dz}\right)\frac{r^2}{2} + A$$

i.e.
$$\frac{dq_z}{dr} = \frac{1}{2\mu}\left(\frac{dp}{dz}\right)r + \frac{A}{r}$$

Integrating again, we get

$$q_z = \frac{1}{u\mu}\left(\frac{dp}{dz}\right)r^2 + A\log r + B \qquad (4)$$

where A and B are constants to be determined from the boundary conditions.

The first boundary condition is obtained from the symmetry of the flow such that

$$\frac{dq_z}{dr} = 0 \quad \text{on } r = 0 \tag{5}$$

and the second boundary condition is

$$q_z = 0, \text{ when } r = a \tag{6}$$

where a is the radius of the tube. Using these conditions, we get

$$A = 0,\ B = -\frac{1}{4\mu}\left(\frac{dp}{dz}\right)a^2 = \frac{1}{4\mu}\left(-\frac{dp}{dz}\right)a^2$$

Thus, the solution (4) becomes

$$q_z = \frac{1}{4\mu}\left(\frac{-dp}{dz}\right)(a^2-r^2) \tag{7}$$

This represents a paraboloid of revolution and thus the velocity profile is parabolic.

(i) The Max x Average Velocity : For extreme values of q_z, we have $\frac{dq_z}{dr} = 0$ | $\because$ q_z is a function of r only

From (7), it implies that r = 0 and thus

$$q_{max.} = \frac{a^2}{4\mu}\left(-\frac{dp}{dz}\right) \tag{8}$$

where $\frac{dp}{dz}$ is a negative constant.

From (7) and (8), the velocity distribution, in non dimensional from, is given by

$$\frac{q_z}{q_{max}} = 1 - \left(\frac{r}{a}\right)^2$$

The average velocity is defined by

$$q_0 = \frac{1}{\pi a^2}\int_0^{2\pi}\int_0^a q_z\, r\, dr\, d\theta$$

Using the value of q_z, we get

$$q_0 = \frac{a^2}{8\mu}\left(-\frac{dp}{dz}\right) = \frac{1}{2} q_{max}.$$

The average velocity is therefore half of the maximum velocity

The volume of fluid discharged over any section per unit time (i.e. volumetric flow) is defined as

$$Q = \int_0^a q_z . 2\pi \, r \, dr$$

Using (7), it is obtained to be

$$Q = \frac{\pi a^4}{8\mu}\left(-\frac{dp}{dz}\right) = \frac{1}{2}\pi a^2 \left[\frac{a^2}{4\mu}\left(\frac{-dp}{dz}\right)\right] = \frac{1}{2}\pi a^2 \; q_{max}. \qquad (9)$$

(ii) Shearing Stress : The shearing stress in Poiseuille's flow is given by

$$\sigma_{rz} = -\mu \frac{dq_z}{dr} = -\mu \frac{1}{4\mu}\left(\frac{dp}{dz}\right)(2r) = -\frac{r}{2}\left(\frac{dp}{dz}\right)$$

On the boundary of the tube, we have

$$(\sigma_{rz})_{r=a} = -\frac{a}{2}\left(\frac{dp}{dz}\right) = \frac{a}{2}\left(\frac{-dp}{dz}\right) = \frac{2\mu}{a}.q_{max}. \qquad (10)$$

The local frictional (skin) co-efficient C_f for laminar flow through a circular pipe is

$$C_f = \frac{(\sigma_{rz})_{r=a}}{\rho \, q_0^2/2} = \frac{2\mu}{a}\frac{q_{max}}{\rho \, q_0^2/2}$$

$$= \frac{4\mu}{\rho a}\frac{2q_0}{q_0^2} = \frac{8\mu}{\rho a}\frac{1}{q_0} = \frac{16}{R_e}$$

Where $R_e = 2aq_0/\nu$ is the Reynolds number. When R_e is less than the critical Reynolds number, which is 2300 in this flow problem, the flow is laminar but if $R_e > 2300$, the flow ceases to be laminar and becomes turbulent. Thus, in this problem, $R_e < 2300$.

8.1. Example. Establish the formula $\frac{1}{8}\frac{\pi a^4}{\mu l}(p_1-p_2)$ for the rate of steady flow of an incompressible liquid through a circular pipe of radius 'a', p_1 and p_2 being the pressures at two sections of the pipe distant l–apart. Also find the drag on the cylinder.

Solution. First we prove equation (9) and then we note that $\frac{dp}{dz}$ is the change in pressure per unit length and thus in the present case

$$\frac{dp}{dz} = \frac{p_2 - p_1}{l}$$

Therefore, from equation (9), we get

$$Q = \frac{\pi a^4}{8\mu}\left(\frac{p_1 - p_2}{l}\right)$$

Also, the drag on the cylinder is

$$F = 2\pi a l\,(\sigma_{rz})_{r=a}$$

$$= -\pi a^2 l\,\frac{dp}{dz}$$

$$= \pi a^2\,(p_1-p_2).$$

Hence the result.

9. Steady Flow Between co-axial Circular Cylinders

Let us consider the steady flow of a viscous fluid parallel to the axis in the annular space between two co-axial cylinders of radii r_1 and $r_2(r_2 > r_1)$. The velocity for such flow is

$$q_z = \frac{1}{4\mu}\left(\frac{dp}{dz}\right)r^2 + A\log r + B \qquad (1)$$

(from equation (4) of previous article) where A and B are constants to be determined from the boundary conditions, $\frac{dp}{dz}$ being the constant pressure gradient.

The boundary conditions are

$$q_z = 0 \text{ at } r = r_1 \text{ and } r = r_2 \qquad (2)$$

Applying (2) in (1), we get

$$A = \frac{1}{4\mu}\left(\frac{dp}{dz}\right)\frac{r_2^2 - r_1^2}{\log r_1 / r_2} = -\frac{1}{4\mu}\left(\frac{dp}{dz}\right)\frac{(n^2-1)r_1^2}{\log n},\ n = r_2/r_1$$

and

$$B = \frac{1}{4\mu}\left(\frac{dp}{dz}\right)\left[\frac{(n^2-1)r_1^2}{\log n}\log r_1 - r_1^2\right]$$

Thus the velocity distribution in the annular space between two co-axial cylinders is

$$q_z = -\frac{1}{4\mu}\left(\frac{dp}{dz}\right)\left[(r_1^2 - r^2) + \frac{(n^2-1)r_1^2}{\log n}\log\left(\frac{r}{r_1}\right)\right] \qquad (3)$$

The volumetric flow in this case is

$$Q = \int_0^{2\pi}\int_{r_1}^{r_2} q_z \, r \, dr \, d\theta$$

$$= \int_0^{2\pi}\int_{r_1}^{nr_1} -\frac{1}{4\mu}\left(\frac{dp}{dz}\right)\left[(r_1^2 - r^2) + \frac{(n^2-1)r_1^2}{\log n}\log\left(\frac{r}{r_1}\right)\right] r \, dr \, d\theta$$

$$= \frac{-2\pi}{4\mu}\left(\frac{dp}{dz}\right)\left[r_1^2\frac{r^2}{2} - \frac{r^4}{4} + \frac{(n^2-1)r_1^2}{\log n}\left(\frac{r^2}{2}\log\left(\frac{r}{r_1}\right) - \frac{r^2}{4}\right)\right]_{r_1}^{nr_1}$$

$$= \frac{-\pi}{2\mu}\left(\frac{dp}{dz}\right)\left[\frac{n^2 r_1^4}{2} - \frac{r_1^4}{2} - \frac{n^4 r_1^4}{4} + \frac{r_1^4}{4}\right.$$

$$\left. + \frac{(n^2-1)r_1^2}{\log n}\left\{\left(\log n - \frac{1}{2}\right)\frac{n^2 r_1^2}{2} + \frac{r_1^2}{4}\right\}\right]$$

$$= -\frac{\pi r_1^4}{8\mu}\left(\frac{dp}{dz}\right)\left[2n^2 - 2 - n^4 + 1 + \frac{n^2-1}{\log n}\left\{(2\log n - 1)n^2 + 1\right\}\right]$$

$$= \frac{-\pi r_1^4}{8\mu}\left(\frac{dp}{dz}\right)\left[2n^2 - n^4 - 1 + 2n^4 - 2n^2 - \frac{(n^2-1)^2}{\log n}\right]$$

$$= \frac{-\pi r_1^4}{8\mu}\left(\frac{dp}{dz}\right)\left[(n^4-1)-\frac{(n^2-1)^2}{\log n}\right] \qquad (4)$$

The average velocity q_0 in the annulus is given by

$$q_0 = \frac{Q}{\pi(n^2-1)r_1^2} = \frac{-r_1^2}{8\mu}\left(\frac{dp}{dz}\right)\left[(n^2+1)-\frac{n^2-1}{\log n}\right] \qquad (5)$$

The shearing stress on the inner and outer cylinders are

$$(\sigma_{rz})_{r=r_1} = \left(\mu\frac{dq_z}{dr}\right)_{r=r_1}$$

$$= -\mu\frac{1}{4\mu}\left(\frac{dp}{dz}\right)\left[-2r_1+\frac{(n^2-1)r_1^2}{r_1\log n}\right]$$

$$= -\frac{r_1}{4}\left(\frac{dp}{dz}\right)\left[\frac{(n^2-1)}{\log n}-2\right]$$

and

$$(\sigma_{rz})_{r=r_2} = -\left(\mu\frac{dq_z}{dr}\right)_{r=r_2}$$

$$= \frac{1}{4}\frac{dp}{dz}\left[-2r_2+\frac{(n^2-1)r_1^2}{r_2\log n}\right]$$

$$= -\frac{r_1}{4}\left(\frac{dp}{dz}\right)\left[2n-\frac{(n^2-1)}{n\log n}\right]$$

10. Steady Flow Between Concentric Rotating Cylinders (Couette's Flow)

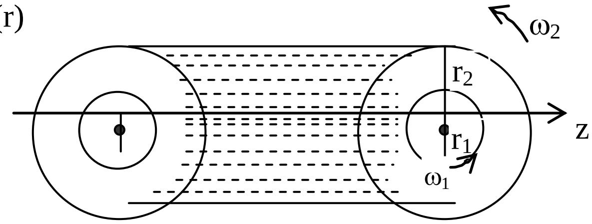
We consider the flow between two concentric rotating cylinders with radii r_1, r_2 ($r_2 > r_1$) having viscous fluid in between them. We assume that the flow is circular such that only the tangential component of velocity exists. Let w_1 and w_2 be the angular velocity of the inner and outer cylinders respectively.

The continuity equation in cylindrical co-ordinates (r, θ, z) reduces to

$\frac{\partial q_\theta}{\partial \theta} = 0, \quad \Rightarrow q_\theta = q_\theta(r)$

(1)

where $q_r = q_z = 0$



Now, the Navier-Stoke's equations for viscous in compressible fluid in cylindrical co-ordinates are

$$\rho\left(\frac{dq_r}{dt} - \frac{q_\theta^2}{r}\right) = \rho X_r - \frac{\partial p}{\partial r} + \mu\left(\nabla^2 q_r - \frac{q_r}{r^2} - \frac{2}{r^2}\frac{\partial q_\theta}{\partial \theta}\right)$$

$$\rho\left(\frac{dq_\theta}{dt} + \frac{q_r q_\theta}{r}\right) = \rho X_\theta - \frac{1}{r}\frac{\partial p}{\partial \theta} + \mu\left(\nabla^2 q_\theta + \frac{2}{r^2}\frac{\partial q_r}{\partial \theta} - \frac{q_\theta}{r^2}\right)$$

$$\rho\frac{dq_z}{dt} = \rho X_z - \frac{\partial p}{\partial z} + \mu \nabla^2 q_z$$

Here, $q_r = q_z = 0$; $\overline{X} = (X_r, X_\theta, X_z) = 0$, $q_\theta = q_\theta(r)$

From the last two equations, we have

$$\frac{\partial p}{\partial z} = 0, -\frac{1}{r}\frac{\partial p}{\partial \theta} + \mu\left(\nabla^2 q_\theta - \frac{q_\theta}{r^2}\right) = 0 \qquad (2)$$

and the first equation gives

$$\rho\frac{q_\theta^2}{r} = \frac{\partial p}{\partial r} \qquad (3)$$

The L.H.S. of (3) is a function of r and thus p is a function of r only. i.e. $\frac{\partial p}{\partial \theta} = 0$

$\therefore$ Equation (2) reduces to

$$\nabla^2 q_\theta - \frac{q_\theta}{r^2} = 0, \ \nabla^2 \equiv \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2} + \frac{\partial^2}{\partial z^2}$$

$$\Rightarrow \qquad \frac{d^2 q_\theta}{dr^2} + \frac{1}{r}\frac{dq_\theta}{dr} - \frac{q_\theta}{r^2} = 0$$

$$\Rightarrow \qquad \frac{d^2 q_\theta}{dr^2} + \frac{d}{dr}\left(\frac{q_\theta}{r}\right) = 0 \tag{4}$$

Integrating, we get

$$\frac{dq_\theta}{dr} + \frac{q_\theta}{r} = 2C_1$$

$$\Rightarrow \qquad r\frac{dq_\theta}{dr} + q_\theta = 2c_1 r \Rightarrow \frac{d}{dr}(r\,q_\theta) = 2c_1 r$$

Integrating, we get

$$r\,q_\theta = c_1 r^2 + c_2 \Rightarrow q_\theta = c_1 r + \frac{c_2}{r} \tag{5}$$

which is the general solution, where c_1 and c_2 are constants to be determined from the boundary conditions.

The boundary conditions are

$$\left.\begin{aligned} & q_\theta = r_1\,\omega_1, \text{ when } r = r_1 \\ \text{and} \qquad & q_\theta = r_2\,\omega_2, \text{ when } r = r_2 \end{aligned}\right\} \tag{6}$$

$$\left|\begin{aligned} & \because \text{ on the surface, } v = r\frac{d\theta}{dt} \Rightarrow v = r\omega \\ & l = r\theta \Rightarrow \frac{dl}{dt} = r\frac{d\theta}{dt} \text{ i.e. } v = r\,\omega \end{aligned}\right.$$

Using these in (5), we obtain

$$c_1 = \frac{\omega_1 r_1^2 - \omega_2 r_2^2}{r_1^2 - r_2^2}, \quad c_2 = \frac{r_1^2 r_2^2(\omega_1 - \omega_2)}{r_2^2 - r_1^2} \tag{7}$$

Thus the solution (5) in the present case is

$$q_\theta = \frac{1}{r_2^2 - r_1^2}\left[(r_2^2\omega_2 - r_1^2\omega_1)r - \frac{r_1^2 r_2^2(\omega_2 - \omega_1)}{r}\right] \tag{8}$$

In particular, if the inner cylinder is at rest i.e. $\omega_1 = 0$, $\omega_2 = w$(say), $r_1 = a$, $r_2 = b$, then the solution becomes

$$q_\theta = \frac{\omega b^2}{b^2 - a^2}\left(r - \frac{a^2}{r}\right) \qquad (9)$$

The radial pressure, given by (3), is

$$\frac{dp}{dr} = \rho \frac{q_\theta^2}{r} = \frac{\rho}{r}\left(c_1^2 r^2 + \frac{c_2^2}{r^2} + 2c_1 c_2\right) \qquad \mid \text{using (5)}$$

$$= \rho\left(c_1^2 r + \frac{c_2^2}{r^3} + \frac{2c_1 c_2}{r}\right)$$

Integrating w.r.t., we get

$$p = \rho\left[\frac{c_1^2 r^2}{2} - \frac{c_2^2}{2r^2} + 2c_1 c_2 \log r\right] + c_3 \qquad (10)$$

If $p = p_1$ when $r = r_1$, then

$$p_1 = \rho\left[\frac{c_1^2 r_1^2}{2} - \frac{c_2^2}{2r_1^2} + 2c_1 c_2 \log r_1\right] + c_3$$

$$\Rightarrow \qquad c_3 = p_1 - \rho\left[\frac{c_1^2 r_1^2}{2} - \frac{c_2^2}{2r_1^2} + 2c_1 c_2 \log r_1\right]$$

Hence the pressure is given by

$$p = p_1 + \rho\left[c_1^2\left(\frac{r^2 - r_1^2}{2}\right) - \frac{c_2^2}{2}\left(\frac{1}{r^2} - \frac{1}{r_1^2}\right) + 2c_1 c_2 \log \frac{r}{r_1}\right]$$

where c_1 and c_2 are given by (7).

The formula for shearing stress is

$$\sigma_{r\theta} = \mu\left[\frac{dq_\theta}{dr} - \frac{q_\theta}{r}\right] = \mu\left[r \frac{d}{dr}\left(\frac{q_\theta}{r}\right)\right]$$

$$= \mu\left[r \frac{d}{dr}\left(\frac{c_1 r + c_2/r}{r}\right)\right]$$

$$= \mu .\, r \frac{d}{dr}\left(c_1 + \frac{c_2}{r^2}\right) = \mu r\left(-\frac{2c_2}{r^3}\right)$$

$$= \frac{-2\mu c_2}{r^2} = \frac{-2\mu r_1^2 r_2^2 (\omega_1 - \omega_2)}{r^2 (r_2^2 - r_1^2)}$$

The expressions for the shearing stress on the outer and the inner cylinder are

$$(\sigma_{r\theta})_{r=r_2} = \frac{2\mu(\omega_2 - \omega_1) r_1^2}{r_2^2 - r_1^2}$$

$$(\sigma_{r\theta})_{r=r_1} = \frac{2\mu(\omega_2 - \omega_1) r_2^2}{r_2^2 - r_1^2}$$

11. Steady Flow in Tubes of Uniform Cross–Section

Here, we consider the incompressible unaccelerated flow through a tube of any uniform cross-section. We neglect body forces. Thus, we have

$$\frac{d\bar{q}}{dt} = \bar{0}, \bar{F} = \bar{0},\ \nabla \cdot \bar{q} = 0 \qquad (1)$$

and the Navier-Stoke's equations in vector form become

$$0 = -\frac{\nabla p}{\rho} + \frac{\mu}{\rho}\nabla^2 \bar{q}$$

i.e. $\nabla p = \mu \nabla^2 \bar{q}$ (2)

Let us work with fixed co-ordinate axis ox, oy, oz with oz taken parallel to the flow so that

$$\bar{q} = w\hat{k}, \qquad (3)$$

where $\bar{q}(u, v, w)$, $u = 0$, $v = 0$

From equation of continuity $\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0,$

we get $\frac{\partial w}{\partial z} = 0 \Rightarrow \quad w = w\,(x, y)$ (4)

Thus from equation (2), (3) & (4), we obtain

$$\frac{\partial p}{\partial x}\hat{i}+\frac{\partial p}{\partial y}\hat{j}+\frac{\partial p}{\partial z}\hat{k}=\mu\left(\frac{\partial^2 w}{\partial x^2}+\frac{\partial^2 w}{\partial y^2}\right)\hat{k}$$

$$\Rightarrow \qquad \frac{\partial p}{\partial x}=0,\ \frac{\partial p}{\partial y}=0 \qquad (5)$$

and
$$\frac{\partial p}{\partial z}=\mu\left(\frac{\partial^2 w}{\partial x^2}+\frac{\partial^2 w}{\partial y^2}\right) \qquad (6)$$

Equations (5) show that p is a function of z only, therefore, we can write

$$\frac{dp}{dz}=\mu\left(\frac{\partial^2 w}{\partial x^2}+\frac{\partial^2 w}{\partial y^2}\right) \qquad (7)$$

The L.H.S. of (7) is a function of z only while R.H.S. is a function of x, y only. Thus each side is a constant, say –P, the negative sign being taken since p decreases as z increases. Then the problem of solving the Navier-Stoke's equations reduces to the problem of solving the partial differential equation.

$$\frac{\partial^2 w}{\partial x^2}+\frac{\partial^2 w}{\partial y^2}=-\frac{P}{\mu} \qquad \left|\; P=\frac{-dp}{dz}\right. \qquad (8)$$

subject to the condition that w vanishes on the walls of the tube for a viscous fluid.

To obtain the solutions of (8), we first establish a uniqueness theorem. A form which is a little more general than that required here, is as follows :

11.1. Uniqueness Theorem. If

$$\frac{\partial^2 w}{\partial x^2}+\frac{\partial^2 w}{\partial y^2}=f(x,y)$$

at all points (x, y) of a region S in the plane ox, oy bounded by a closed curve c and if f(x, y) is prescribed at each point (x, y) of S and w at each point of C, then any solution w = w (x, y) satisfying these conditions is unique.

Proof. The given equation is

$$\frac{\partial^2 w}{\partial x^2}+\frac{\partial^2 w}{\partial y^2}=f(x,y) \qquad (9)$$

Let $w = w_1\,(x, y)$ and $w = w_2(x, y)$ be two solutions satisfying equation (9) in the region S together with the prescribed boundary conditions on C

i.e. $\quad w_1 = w_2$ on C

We are to prove that $w_1 = w_2$ in S.

For this, we write

$$W = w_1 - w_2 \tag{10}$$

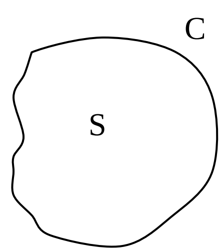
Then,
$$\frac{\partial^2 W}{\partial x^2} + \frac{\partial^2 W}{\partial y^2} = \left(\frac{\partial^2 w_1}{\partial x^2} + \frac{\partial^2 w_1}{\partial y^2}\right) - \left(\frac{\partial^2 w_2}{\partial x^2} + \frac{\partial^2 w_2}{\partial y^2}\right)$$

$$= f(x, y) - f(x, y) = 0$$

$$\Rightarrow \qquad \frac{\partial^2 W}{\partial x^2} + \frac{\partial^2 W}{\partial y^2} = 0 \text{ in S} \tag{11}$$

Also, on curve C, $\quad W = 0, \qquad (12)$

Since $w_1 = w_2$ on C.



Now, consider

$$I = \iint_S \left[\left(\frac{\partial W}{\partial x}\right)^2 + \left(\frac{\partial W}{\partial y}\right)^2\right] dxdy$$

$$= \iint_S \left[\left(\frac{\partial W}{\partial x}\right)^2 + \left(\frac{\partial W}{\partial y^2}\right)^2 + W\left(\frac{\partial^2 W}{\partial x^2} + \frac{\partial^2 W}{\partial y^2}\right)\right] \quad dx\, dy$$

| using (11)

$$= \iint_S \left[\frac{\partial}{\partial x}\left(W\frac{\partial W}{\partial x}\right) + \frac{\partial}{\partial y}\left(W\frac{\partial W}{\partial y}\right)\right] dxdy$$

$$= \oint_C \left(W \frac{\partial W}{\partial x} dy - W \frac{\partial W}{\partial y} dx \right), \text{ by Green's Theorem.}$$

$= 0$, as $W = 0$ on C.

Now, I = 0 $\Rightarrow \left(\frac{\partial W}{\partial x}\right)^2 + \left(\frac{\partial W}{\partial y}\right)^2 = 0$ in S which will be true only if $\frac{\partial W}{\partial x} = 0$, $\frac{\partial W}{\partial y} = 0$ at each point of S.

$\Rightarrow$ W = constant in S.

Since W = 0 on C, we infer from the continuity of W that W = 0 throughout S.

Hence $w_1 = w_2$ in S which establishes the uniqueness of the solution. Under the reference of the uniqueness theorem, we now find the solution of equation (8) for tubes having different types of uniform cross-section.

11.2. Tube having Uniform Elliptic Cross-Section : Suppose that the elliptic cross-section of the tube has the equation

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - 1 = 0 \qquad (13)$$

Then, we must solve

$$\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} = -\frac{P}{\mu} \qquad (14)$$

subject to the condition w = 0 on the cross section (13).

We first observe that the function

$$w = k\left(1 - \frac{x^2}{a^2} - \frac{y^2}{b^2}\right) \qquad (15)$$

satisfies the boundary condition, namely w = 0 on the elliptic cross-section. Regarding k as constant and on substituting w into the partial differential equation (14), we find

$$k\left(\frac{-2}{a^2} - \frac{2}{b^2}\right) = \frac{-P}{\mu}$$

$$\Rightarrow \qquad k = \frac{Pa^2b^2}{2\mu(a^2+b^2)} \tag{16}$$

Thus from equation (15) & (16), we get

$$w = \frac{Pa^2b^2}{2\mu(a^2+b^2)}\left(1-\frac{x^2}{a^2}-\frac{y^2}{b^2}\right) \tag{17}$$

The uniqueness theorems shows that w, given in (17) is the required solution.

The volume discharged through the tube per unit time is

$$Q = \iint_S w \; dx\, dy$$

$$= \frac{Pa^2b^2}{2\mu(a^2+b^2)}\left[\iint dxdy - \frac{1}{a^2}\iint x^2 dxdy - \frac{1}{b^2}\iint y^2 dxdy\right]$$

$$= \frac{Pa^2b^2}{2\mu(a^2+b^2)}\left[\pi ab - \frac{1}{a^2}\pi ab\frac{a^2}{4} - \frac{1}{b^2}\pi ab\frac{b^2}{4}\right]$$

$$= \frac{\pi P}{4\mu}\left(\frac{a^3b^3}{a^2+b^2}\right) \tag{18}$$

$$\text{Mean velocity} = \frac{Q}{\iint dxdy} = \frac{Q}{\pi ab} = \frac{P}{4\mu}\left(\frac{a^2b^2}{a^2+b^2}\right)$$

11.3. Remark (circular cross-section). When b = a, then the cross-section of the tube becomes a circle of radius a and, then

$$w = q_z = \frac{P}{2\mu}\frac{a^4}{2a^2}\left(1-\frac{x^2}{a^2}-\frac{y^2}{a^2}\right)$$

$$= \frac{P}{4\mu}(a^2-x^2-y^2) = \frac{P}{4\mu}(a^2-r^2),$$

$$| \; x^2 + y^2 = r^2$$

where $P = -\dfrac{dp}{dz}$

and $Q = \frac{\pi P}{4\mu}\left(\frac{a^6}{2a^2}\right) = \frac{\pi P a^4}{8\mu}$

mean velocity = $\frac{Q}{\pi a^2} = Pa^2/8\mu$.

These results have already been obtained.

11.4. Tube having Equilateral Triangular Cross-Section. Suppose that the cross-section of the tube is the equilateral triangle bounded by the lines

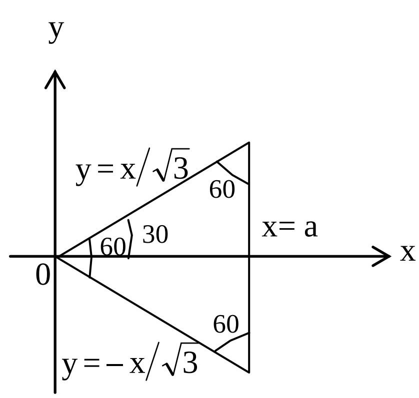
$$x = a,\ y = \pm\frac{x}{\sqrt{3}} \qquad (19)$$

If we take

$$w = k(x-a)\left(y^2 - \frac{1}{3}x^2\right) \qquad (20)$$

$$= k\left[y^2(x-a) - \frac{x^2}{3}(x-a)\right]$$

then w = 0 on the boundary of the tube.



Substituting for w in

$$\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} = -\frac{P}{\mu} \qquad (21)$$

we obtain

$$k\left[\left(-2x + \frac{2a}{3}\right) + (2x - 2a)\right] = \frac{-P}{\mu}$$

$$\Rightarrow \qquad k = \frac{3P}{4\mu a} \tag{22}$$

Thus, by the uniqueness theorem,

$$w = \frac{3P}{4\mu a}(x-a)\left(y^2 - \frac{1}{3}x^2\right) \tag{23}$$

is the unique solution

The volume discharged per unit time is

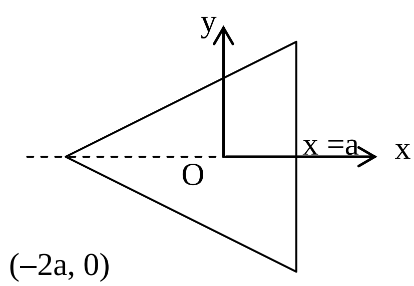
$$Q = \iint_S w\,ds = 2\int_0^a dx \int_0^{x/\sqrt{3}} w\,dy \qquad \text{| due to symmetry}$$

$$= \frac{3P}{2\mu a}\int_0^a dx \int_0^{x/\sqrt{3}} (x-a)\left(y^2 - \frac{x^2}{3}\right)dy$$

$$= \frac{Pa^4}{60\sqrt{3}\mu}$$

11.5. Remark. If we take the cross-section to be

$$(x–a)\ (x \pm \sqrt{3}y + 2a)$$



then $\qquad Q = \dfrac{27}{20\sqrt{3}}\dfrac{Pa^4}{\mu}$

| Replace a by 3a in the above example

12. Unsteady Flow Over a Flat Plate

So far we have discussed the examples of exact solutions of the Navier-Stokes equations for steady flows. Here, we consider the case of unsteady flow.

The simplest unsteady flow is that which results due to the impulsive motion of a flat plate in its own plane in an infinite mass of fluid which is otherwise at rest. This flow was first studied by Stokes and is generally known as **Stokes first problem.**

Let x-axis be taken in the direction of motion of the plate, which is suddenly accelerated from rest and moves with constant velocity U_0. Let y-axis be perpendicular to the plate. The motion is two-dimensional and the only non-zero component of velocity is u, where $\bar{q}$ = (u, v, w). Further, u is a function of y and t only. i.e. u ≡ u(y, t). The pressure in the whole space is constant. The Navier-Stokes equations in the absence of body forces, for the present case, become

$$\frac{\partial u}{\partial t} = \nu \frac{\partial^2 u}{\partial y^2}, \ \nu = \mu/\rho \tag{1}$$

The initial and boundary conditions are

$$u = 0 \text{ when } t = 0 \text{ for all } y \tag{2}$$

$$\left.\begin{array}{l} u = U_0 \text{ at } y = 0 \\ u = 0 \text{ at } y = \infty \end{array}\right\} \text{when } t > 0 \tag{3}$$

We observe that the partial differential equation (1) is the same as the equation of heat conduction, diffusion etc. It can be reduced to an ordinary differential equation if we make the following substitution (principle of similarity of flow)

$$\frac{u}{U_0} = f(\eta) \tag{4}$$

where

$$\eta = \frac{y}{2\sqrt{\nu t}} \tag{5}$$

is the similarity parameter.

We have,

$$\frac{\partial u}{\partial t} = \frac{\partial u}{\partial \eta}\frac{\partial \eta}{\partial t} = U_0 \frac{\partial f}{\partial \eta}\left(\frac{-y}{4\sqrt{\nu}t^{3/2}}\right)$$

$$\frac{\partial u}{\partial y} = \frac{\partial u}{\partial \eta}\frac{\partial \eta}{\partial y} = U_0 \frac{\partial f}{\partial \eta}\left(\frac{1}{2\sqrt{\nu t}}\right)$$

$$\frac{\partial^2 u}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial y}\right) = \frac{\partial}{\partial \eta}\left(\frac{\partial u}{\partial y}\right)\frac{\partial \eta}{\partial y} = U_0 \frac{\partial^2 f}{\partial \eta^2}\left(\frac{1}{4\nu t}\right)$$

Thus, in terms of the new variables, equation (1) reduces to

$$\frac{\partial f}{\partial \eta}(-2\eta) = \frac{\partial^2 f}{\partial \eta^2}$$

i.e. $$\frac{d^2 f}{d\eta^2} + 2\eta \frac{df}{d\eta} = 0 \quad (6)$$

and the corresponding boundary conditions are

$$f(0) = 1 \text{ and } f(\infty) = 0 \quad (7)$$

The second condition in (7) includes the initial condition (2).

The solution of (6) is
$$\left| \begin{array}{l} \dfrac{f''}{f'} = -2\eta \Rightarrow \log f' = -\eta^2 + \log C_1 \\ \Rightarrow f' = C_1 e^{-\eta^2} \\ \Rightarrow f = C_1 \int_0^{\eta} e^{-\eta^2} d\eta + C_2 \end{array} \right.$$

$$f(\eta) = C_1 \int_0^{\eta} e^{-\eta^2} d\eta + C_2 \quad (8)$$

Using the boundary conditions (7) in (8), the constants of integration C_1 and C_2 are obtained to be

$$C_2 = 1 \text{ and } C_1 = -\frac{1}{\int_0^{\infty} e^{-\eta^2} d\eta} = -\frac{2}{\sqrt{\pi}} \quad (9)$$

The velocity distribution, from equation (4), is therefore given by

$$\frac{u}{U_0} = f(\eta) = 1 - \frac{2}{\sqrt{\pi}} \int_0^{\eta} e^{-\eta^2} d\eta$$

$$= (1 - \text{erf } n) \quad (10)$$

the integral

$$\text{erf } \eta = \frac{2}{\sqrt{\pi}} \int_0^{\eta} e^{-\eta^2} d\eta \quad (11)$$

is called the **error function** or the probability integral and tables for it are readily available.

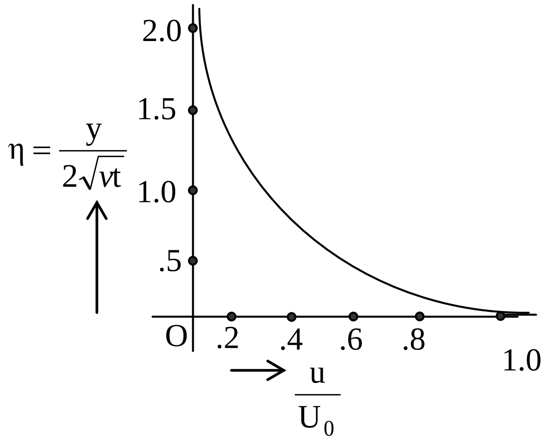
The velocity distribution (10) is tabulated as follows.

η	erf η	$\frac{u}{U_0}$
0	0	1
0.01	0.01128	0.98872
0.05	0.05637	0.94363
0.1	0.11246	0.88754
0.2	0.22270	0.77730
0.4	0.42839	0.57161
0.6	0.60386	0.39613
0.8	0.74210	0.25790
1.0	0.84270	0.15720
1.2	0.91031	0.08969
1.4	0.95229	0.04771
1.6	0.97635	0.02365
1.8	0.98909	0.01091
2.0	0.99532	0.00468
2.4	0.99931	0.00069
2.8	0.99992	0.00008
∞	1.00000	0

We observe that the velocity decreases continuously and tends to its limiting value zero as η tends to infinity. However, for all practical purposes, this value is reached at about η = 2.0 and therefore the corresponding value of y, which we shall denote by δ, from (5), is

$$\delta \simeq 4\sqrt{\nu t} \qquad (12)$$

Thus distance is a measure of the extent to which the momentum has penetrated the body of the fluid. It is proportional to the square root of the product of kinematic viscosity and time. If νt is small, then δ will be small and once again we shall have a boundary layer flow.



UNIT-V

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1. Dynamical Similarity

We have observed that due to non-linear character of the fundamental equations governing the flow of a viscous compressible fluid, there are no known general methods for solving them. Only in few particular cases and that too under restricted conditions, exact solutions of these equations, for all ranges of viscosity, exist and a few of them have already been considered. However, attempts have been made to simplify these equations for two extreme cases of viscosity, very large and very small, and we have well established theories for these cases which are respectively known as "Theory of slow motion" and "Theory of boundary layers". But the cases of moderate viscosities cannot be interpreted from these two theories. Further, even in these two extreme cases, we find great mathematical difficulties and therefore most of the research on the behaviour of viscous fluids have been carried out by experiments.

In practical cases, such as designing of ships, aircrafts, underwater projects etc, it is usually necessary to carry out experiments on models and to relate their behaviour to that of the actual object (prototype). In fact, the model and the prototype should be what is called as **dynamical similar.** Mathematically speaking, two physical systems are equivalent if the governing equations and the boundary conditions of the two systems are the same. Such systems are called **dynamically similar system.** One obvious condition is that the model should be geometrically similar to the prototype which means that we can obtain the actual object from the model by enlarging or contracting its size in every direction in the same proportion. This eliminates the consideration of boundary conditions in the discussion of dynamical similarity and so we have to consider only the governing equations. In short, we can say that two fluid motions are dynamically similar if with geometrically similar boundaries, the flow patterns are geometrically similar. Further, two geometrically similar flows are dynamically similar if forces acting at every point are similar i.e. the forces are acting in same direction having same ratio in magnitude.

We now discuss the conditions under which the fluid motions are dynamically similar. In other words, we have to find out those parameters which characterize a flow problem. There are two methods for finding out these parameters (i) inspection analysis (ii) dimensional analysis. In the first case, we reduce the fundamental equations to a non-dimensional form and obtain the non-dimensional parameters from the resulting equations. This procedure should always be used when the basic differential equations for a problem are available. In the second case, we form non-dimensional parameters from the physical quantities occurring in a problem, even when the knowledge of the governing equations is missing. We discuss these two methods with particular reference to the flow of a viscous compressible fluid.

1.1. Remark. (i) Some authors do not differentiate between the two methods and study both of them under the head of dimensional analysis.

(ii) In two dynamically similar systems, usually, all the non-dimensional numbers cannot be matched and so strictly speaking, perfect dynamical similarity is rare. So, many times we match only the important non-dimensional numbers.

1.2. Inspection Analysis, Reynolds Number. We know that the Navier-Stokes equation of motion of a viscous incompressible fluid in the x-direction is

$$\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}+w\frac{\partial u}{\partial z}=X-\frac{1}{\rho}\frac{\partial p}{\partial x}+\nu\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}\right) \qquad (1)$$

Suppose L, U, P denote a characteristic length, velocity and pressure respectively. Then the length, velocities and pressure in (1) may be expressed in terms of these standards. Thus, we write

$$x = Lx', \; y = Ly', \; z = Lz' \qquad (2)$$

$$u = Uu', \; v = Uv', \; w = Uw' \qquad (3)$$

$$p = P\,p' \qquad (4)$$

where all primed quantities are pure numbers having no dimensions. Then, since L/U is the characteristic time, we get

$$\frac{\partial u}{\partial t} = \frac{\partial (Uu')}{\partial (LU^{-1}t')} = \frac{U^2}{L}\frac{\partial u'}{\partial t'}$$

$$u\frac{\partial u}{\partial x} = (Uu')\frac{\partial (Uu')}{\partial (Lx')} = \frac{U^2}{L}u'\frac{\partial u'}{\partial x'} \text{ etc}$$

$$\frac{1}{\rho}\frac{\partial p}{\partial x} = \frac{1}{\rho}\frac{\partial (Pp')}{\partial (Lx')} = \frac{P}{\rho L}\frac{\partial p'}{\partial x'}$$

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 (Uu')}{\partial (Lx')^2} = \frac{U}{L^2}\frac{\partial^2 u'}{\partial x'^2} \text{ etc.}$$

Substituting these results in (1) and simplifying, we obtain

$$\frac{\partial u'}{\partial t'} + u'\frac{\partial u'}{\partial x'} + v'\frac{\partial u'}{\partial y'} + w'\frac{\partial u'}{\partial z'} = \frac{LX}{U^2} - \frac{P}{\rho U^2}\frac{\partial p'}{\partial x'}$$

$$+ \frac{\nu}{UL}\left(\frac{\partial^2 u'}{\partial x'^2} + \frac{\partial^2 u'}{\partial y'^2} + \frac{\partial^2 u'}{\partial z'^2}\right) \qquad (5)$$

The L.H.S. of (5) is entirely dimensionless, so R.H.S. must be also dimensionless. Thus, it follows that the three quantities

$$\frac{\nu}{UL}, \quad \frac{P}{\rho U^2}, \quad \frac{LX}{U^2} \qquad (6)$$

must be dimensionless quantities.

In order to produce a faithful model of a given incompressible viscous flow, it is essential to keep these three numbers constant. Based on these numbers we have the following definitions.

1.3. Reynolds Number. The first non-dimensional number in (6) ensures dynamical similarity at corresponding points near the boundaries where viscous effects supervene. Its reciprocal is called the Reynolds number and is denoted by R_e so that

$$R_e = \frac{UL}{\nu}$$

This is named after Osborne Reynolds who first introduced this number while discussing boundary layer theory. This is most important of viscous force over the inertia force. It can be easily seen from the equation of motion that the inertia forces $\left(\text{terms like } \rho u \frac{\partial u}{\partial x}\right)$ are of the order $\rho U^2/L$ and the viscous forces $\left(\text{terms like } \mu \frac{\partial^2 u}{\partial x^2}\right)$ are of the order $\mu U/L^2$.

Therefore,

$$\frac{\text{inertia forces}}{\text{viscous forces}} = \frac{\rho U^2/L}{\mu U/L^2} = \frac{\rho UL}{\mu} = \frac{UL}{\nu} = R_e$$

Thus, Reynolds number is the ratio of the inertia force to the viscous force. It is infact a parameter for viscosity. If R_e is small, the viscous forces will be predominant and the effect of viscosity will be felt in the whole flow field. On the other hand, if R_e is large the inertial forces will be predominant and in such a case the effect of viscosity can be considered to be confined in thin layer, known as **boundary layer,** adjacent to the solid boundary. When R_e is enormously large, the flow ceases to be laminar and becomes turbulent. The Reynolds number at which the transition, from laminar to turbulent, takes place is known as **critical Reynolds number.**

Further, we can write Re as $R_e = L/(\nu/U)$, where ν/U represents the viscous dissipation length. Thus, in other words, the Reynolds number is the ratio of length of the body to the viscous dissipation length.

1.4. Pressure co-efficient. The second non-dimensional number in (6) ensures dynamical similarity in two fluids at points where viscosity is unimportant. Such points would occur at stations remote from the boundaries. This number is called pressure co-efficient and is denoted by C_P. Thus $C_P = \frac{P}{\rho U^2}$ from the equation of motion, we note that the pressure forces $\left(\text{terms like } \frac{\partial p}{\partial x}\right)$ are of order P/L. Thus, we can write

$$\frac{\text{Pressure forces}}{\text{Inertia forces}} = \frac{P/L}{\rho U^2/L} = \frac{P}{\rho U^2} = C_P$$

i.e. C_P gives the relative importance of the pressure force to the inertia force. Usually, it is taken as unity.

1.5. Force Coefficient. The third non-dimensional number in (6) tells how to scale body forces. This number is called force co-efficient, denoted by C_F which is similar to C_P.

Thus $C_F = \dfrac{\text{body forces}}{\text{Inertia forces}} = \dfrac{\rho X}{\rho U^2/L} = \dfrac{LX}{U^2}$

If C_F is small, the body forces can be neglected as compared to the inertia forces. Reciprocal of this number is rather more important and is called **Froude number,** denoted by F_r. Thus

$$F_r = \frac{1}{C_F} = \frac{U^2}{LX}$$

This number is particularly used in cases when body forces are the gravitational forces. Thus,

$$F_r = \frac{\text{inertia forces}}{\text{gravity forces}} = \frac{\rho U^2/L}{\rho g} = \frac{U^2}{gL}$$

It is important only when there is a free surface, e.g. in an open channel problem. In such cases too the force due to gravity may be neglected in comparison to the inertia force if F_r is large i.e. if

$$\frac{F_r}{R_e} = \frac{\text{inertia force}}{\text{gravity force}} \times \frac{\text{viscous force}}{\text{intertia force}} = \frac{\text{viscous force}}{\text{gravity force}} >> 1.$$

1.6. Dimensional Analysis. In the previous case, we reduced the governing equations of a viscous compressible fluid to a non-dimensional form and obtained the dimensionless parameters. An alternative method, with which the non-dimensional parameters may be formed from the physical quantities occurring in a flow problem is known as dimensional analysis. In dimensional analysis of any problem, we write the dimensions of each physical quantity in terms of fundamental units. Then, by dividing and rearranging the different units, we get some non-dimensional (universal) numbers. Thus, dimensional analysis can put the quantities influencing a physical phenomenon into a useful form for the interpretation of data. It is not a tool for solving problems explicitly but a powerful method for establishing and the grouping of the relevant variables that are likely to appear if the analytic solution is at all possible. The major advantage of the use of dimensional analysis is most apparent where complete analytic solution of the physical problem is not possible.

There are, generally, three accepted methods of dimensional analysis due to Buckingham, Rayleigh and Bridgeman. We shall discuss Buckingham's Pi-theorem here as it is the simplest one among the three methods.

1.7. Buckingham π-theorem. The π-theorem makes use of the following assumptions

(i) It is possible to select always m independent fundamental units in a physical phenomenon (in mechanics, m = 3 i.e. length, time, mass or force)

(ii) There exist quantities, say $Q_1, Q_2, \ldots, Q_n$ involved in a physical phenomenon whose dimensional formulae may be expressed in terms of m fundamental units

(iii) There exists a functional relationship between the n dimensional quantities $Q_1, Q_2, \ldots, Q_n$, say

$$\phi(Q_1, Q_2, \ldots, Q_n) = 0 \tag{1}$$

(iv) Equation (1) is independent of the type of units chosen and is dimensionally homogeneous i.e. the quantities occurring on both sides of the equation must have the same dimensions.

Statement :- If $Q_1, Q_2, \ldots, Q_n$ be n physical quantities involved in a physical phenomenon and if there are m(< n) independent fundamental units in this system, then a relation

$$\phi(Q_1, Q_2, \ldots, Q_n) = 0$$

is equivalent to the relation

$$f(\pi_1, \pi_2, \ldots, \pi_{n-r}) = 0,$$

where $\pi_1, \pi_2, \ldots, \pi_{n-r}$ are the dimensionless power products of $Q_1, Q_2, \ldots, Q_n$ taken r + 1 at a time, r being the rank of the dimensional matrix of the given physical quantities.

Proof. Let $Q_1, Q_2, \ldots, Q_n$ be n given physical quantities and let their dimensions be expressed in terms of m fundamental units $u_1, u_2, \ldots, u_m$ in the following manner

$$[Q_1] = \left\lfloor u_1^{a_{11}} u_2^{a_{21}} \ldots u_m^{a_{m1}} \right\rfloor$$

$$[Q_2] = \left\lfloor u_1^{a_{12}} u_2^{a_{22}} \ldots u_m^{a_{m2}} \right\rfloor$$

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$$[Q_n] = \left\lfloor u_1^{a_{1n}} u_2^{a_{2n}} \ldots u_m^{a_{mn}} \right\rfloor$$

so that a_{ij} is the exponent of u_i in the dimension of Q_j . The matrix of dimensions i.e. the dimensional matrix of the given physical quantities is written as

$$\begin{array}{cccc} & Q_1: & Q_2: & Q_n: \\ \end{array}$$

$$\begin{pmatrix} u_1: & a_{11} & a_{12}\ldots\ldots\ldots a_{1n} \\ u_2: & a_{21} & a_{22}\ldots\ldots a_{2n} \\ & \ldots\ldots\ldots\ldots\ldots\ldots\ldots & \\ & \ldots\ldots\ldots\ldots\ldots\ldots\ldots & \\ u_m: & a_{m_1} & a_{m_2}\ldots\ldots\ a_{mn} \end{pmatrix}$$

This m×n matrix is usually denoted by A.

Now, let us form a product π of powers of $Q_1, Q_2, \ldots\ldots, Q_n$, say

$$\pi = Q_1^{x_1}\ Q_2^{x_2}\ldots\ldots Q_n^{x_n}$$

then $[\pi] =$

$$\left[\left(u_1^{a_{11}}u_2^{a_{21}}\ldots.u_m^{a_{m1}}\right)^{x_1}\left(u_1^{a_{12}}u_2^{a_{22}}\ldots.u_m^{a_{m2}}\right)^{x_2}\ldots\ldots\ldots\left(u_1^{a_{1n}}u_2^{a_{2n}}\ldots.u_m^{a_{mn}}\right)^{x_n}\right]$$

In order that the product π is dimensionless, the powers of $u_1, u_2, \ldots, u_m$ should be zero i.e. M^0, L^0, T^0 etc. Thus, we must have

$$a_{11}x_1 + a_{12}\,x_2 + \ldots\ldots\ldots.. + a_{1n}\,x_n = 0$$

$$a_{21}x_1 + a_{22}\,x_2 + \ldots\ldots\ldots.. + a_{2n}\,x_n = 0$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$a_{m1} + x_1 + a_{m2}\,x_2 + \ldots\ldots + a_{mn}\,x_n = 0$$

This is a set of m homogeneous equations in n unknowns and in matrix form can be written as

$$AX = 0,\ X = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$$

Now, from matrix algebra, we know the result that if there are m homogeneous equations in n unknowns, then the number of independent solutions will be n−r, where r is the rank of the matrix of co-efficients, and any other solution can be expressed as a linear combination of these linearly independent solutions. Further there will be only r independent equations in the set of equations.

Thus if r is the rank of the dimensional matrix A, then the number of linearly independent solutions of the matrix equation AX = 0 are n–r. So, corresponding to each independent solution of X, we will have a dimensionless product π and therefore the number of dimensionless products in a complete set will be n–r

Therefore, $\phi(Q_1, Q_2, \ldots\ldots\ldots, Q_n) = 0$

$\Rightarrow$ $f(\pi_1, \pi_2, \ldots\ldots\ldots\ldots, \pi_{n-r}) = 0$

Hence the theorem.

1.8. Method for π-products. To find out the π-products in a complete set, we adopt the following steps.

(i) Write down the dimensional matrix of n physical quantities, involving in a physical phenomenon, having in independent fundamental units.

(ii) Find the rank of the dimensional matrix. If the rank is r(say), then the number of π's will be n–r.

(iii) Select r quantities out of the n physical quantities as base quantities, keeping in view that these r quantities should have different dimensions and the dimension of any of the fundamental unit should not be zero in all of them.

(iv) Express $\pi_1, \pi_2, \ldots, \pi_{n-r}$ a power products of these r quantities raised to arbitrary integer exponents and one of the excluded, but different in different π's, (n–r) quantities.

(v) Equate to zero the total dimension of each fundamental unit in each π-product to get the integer exponents.

Thus, the Pi-theorem allows us to take n quantities and find the minimum number of non-dimensional parameters $\pi_1, \pi_2, \ldots, \pi_{n-r}$ as associated with these n quantities.

1.9. Application of π-theorem to Viscous Compressible Fluid Flow. We now follow the above mentioned fire steps to find out π-products and see the application of π-theorem and see the application of π-theorem to the simple case of viscous compressible fluid flow. Suppose that in the considered fluid flow, the physical quantities involved are

L, U, ρ, X, P, μ

and the fundamental units in which the dimensions of all these quantities can be expressed are mass [M], length [L] and time [T]. The above six quantities have dimensions as follows

Quantity	**Dimensions**	
L-length	[L]	
U-velocity	$[LT^{-1}]$	
ρ-density	$[ML^{-3}]$	
X-force per unit mass	$[LT^{-2}]$	$\rightarrow$ force $[MLT^{-2}]$
P-pressure force per unit area)	$[ML^{-1}\,T^{-2}]$	
μ-viscosity	$[ML^{-1}\,T^{-1}]$	

(i) The dimensional matrix for the present problem is

$$\begin{array}{c} \\ M: \\ L: \\ T: \end{array} \begin{pmatrix} L & U & e & X & P & \mu \\ 0 & 0 & 1 & 0 & 1 & 1 \\ 1 & 1 & -3 & 1 & -1 & -1 \\ 0 & -1 & 0 & -2 & -2 & -1 \end{pmatrix}$$

(ii) The rank of the above matrix is 3, so the number of independent dimensionless products will be 6–3 = 3.

(iii) Let us take L, U, ρ as base quantities.

(iv) Let
$$\pi_1 = L^{x_1} U^{x_2} \rho^{x_3} X$$
$$\pi_2 = L^{x_4} U^{x_5} \rho^{x_6} P$$
$$\pi_3 = L^{x_7} U^{x_8} \rho^{x_9} \mu$$

(v) Now,

$$[\pi_1] = [(L)^{x_1}(LT^{-1})^{x_2}(ML^{-3})^{x_3}(LT^{-2})]$$

$$= [L^{x_1+x_2-3x_3+1}M^{x_3}T^{-x_2-2}]$$

$$[\pi_3] = [(L)^{x_4}(LT^{-1})^{x_5}(ML^{-3})^{x_6}(ML^{-1}T^{-2})]$$

$$= [L^{x_4+x_5-3x_6-1}M^{x_6+1}T^{-x_5-2}]$$

$$[\pi_3] = [(L)^{x_7}(LT^{-1})^{x_8}(ML^{-3})^{x_9}(ML^{-1}T^{-1})]$$

$$= [L^{x_7+x_8-3x_9-1}M^{x_9+1}T^{-x_8-1}]$$

If π_1, π_2, π_3 are dimensionless, then we must have

$$\begin{array}{r|r|r}
x_1+x_2-3x_3+1=0 & x_4+x_5-3x_6-1=0 & x_7+x_8-3x_9-1=0 \\
x_3=0 & x_6+1=0 & x_9+1=0 \\
-x_2-2=0 & -x_5-2=0 & -x_8-1=0
\end{array}$$

Solving these equations, we get

$$\begin{array}{l|l|l}
x_1=1 & x_4=0 & x_7=-1 \\
x_2=-2 & x_5=-2 & x_8=-1 \\
x_3=0 & x_6=-1 & x_9=-1
\end{array}$$

Thus, we get

$$\pi_1 = L^1\,U^{-2}\,\rho^0\,X = \frac{LX}{U^2}$$

$$\pi_2 = L^0\,U^{-2}\,\rho^{-1}\,P = \frac{P}{\rho U^2}$$

$$\pi_3 = L^{-1}\,U^{-1}\,\rho^{-1}\,\mu = \frac{\mu/\rho}{LU} = \frac{\nu}{LU}$$

which are the same dimensionless quantities obtained in equation (6) of the inspection analysis

1.10. Remark. If we include the energy equation and equation of state in our study, then, in the general case of viscous compressible fluid dynamics, there are 9 physical quantities and the fundamental units in which the dimensions of all these quantities can be expressed are length, mass, time and temperature (Q) and thus there are 9–4 = 5 non-dimensional numbers.

2. Prandtl's Boundary Layer (case of small viscosity)

The simple problems of fluid motion which can be considered are divided into two classes according as the corresponding Reynolds number is small or large. In the case of small Reynolds number, viscosity is predominant and the inertia terms in the equations may be regarded as negligible. The case of large Reynolds number in which the frictional terms are small and inertia forces are predominant, was investigated by the German Scientist Ludwig Prandtl in 1904. He made an hypothesis that for fluids with very small viscosity i.e. large Reynolds number, the flow about a solid boundary can be divided into the following two regions.

(i) A thin layer in the neighbourhood of the body, known as the boundary layer, in which the viscous effect may be considered to be confined. The smaller the viscosity i.e. the larger the Reynolds number, the thinner is this layer. Its thickness is denoted by δ. In such layer, the velocity gradient normal to the wall of the body is very large.

(ii) The region outside this layer where the viscous effect may be considered as negligible and the fluid is regarded as non-viscous.

On the basis of this hypothesis, Prandtl simplified the Navier-Stokes equations to a mathematical tractable form which are termed as Prandtl boundary layer equations and thus he succeeded in giving a physically penetrating explanation of the importance of viscosity in the assessment of frictional drag. The theory was first developed for laminar flow of viscous incompressible fluids but was, later on, extended to include compressible fluids and turbulent flow. However, we shall consider only the case of incompressible fluids.

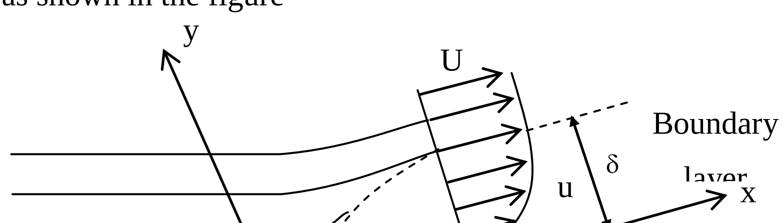
In the discussion of unsteady flow over a flat plate, we had obtained that

$$\delta \simeq 4\sqrt{\nu t}$$

i.e. the boundary layer thickness is proportional to the square root of kinematic viscosity. The thickness is very small compared with a linear dimension L of the body i.e. $\delta \ll L$.

2.1. Boundary Layer equation in Two-dimensions. The viscosity of water, air etc is very small. The Reynolds number for such fluids is large. This led Prandtl to introduce the concept of the boundary layer. We now discuss the mathematical procedure for reducing Navier-Stokes equations to boundary layer equations. The procedure is known as order of magnitude approach.

Let us consider a flow around a wedge submerged in a fluid of very small viscosity as shown in the figure



At the stagnation point O, the thickness of the boundary layer is zero and it increases slowly towards the rear of the wedge. The velocity distribution and the pattern of streamlines deviate only slightly from those in the potential flow. We take the x-axis along the wall of the wedge and y-axis perpendicular to it, so that the flow is two-dimensional in the xy-plane. Within a very thin boundary layer of thickness δ, a very large velocity gradient exists i.e. the velocity u parallel to the wall in the boundary layer increases rapidly from a value zero at the wall to a value U of the main stream at the edge of the boundary layer.

The Navier–Stokes equations, in the absence of body forces, for two dimensional flow, are

$$\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}=\frac{-1}{\rho}\frac{\partial p}{\partial x}+\nu\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}\right) \tag{1}$$

$$\frac{\partial v}{\partial t}+u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}=-\frac{1}{\rho}\frac{\partial p}{\partial y}+\nu\left(\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}\right) \tag{2}$$

The equation of continuity is

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0 \tag{3}$$

In studying the unsteady flow over a flat plate, we found that the thickness of the boundary layer δ is proportional to the square root of the kinematic viscosity *ν* which is indeed very small. For this reason δ < < x except near the stagnation point 0 where the boundary layer begins. In order to compare the order of magnitude of the individual terms in the above equations, we put them in non-dimensional form by introducing the non-dimensional notations

$$x^* = \frac{x}{l}, y^* = \frac{y}{\delta}, u^* \frac{u}{U}, v^* = \frac{v}{V}\; t^* = \frac{t}{l/U}, \; p^* = \frac{p}{p_\infty} \qquad (4)$$

where l, δ, U, V and p_∞ are certain reference values of the corresponding quantities x, y, u , v and p respectively. The non-dimensional quantities are all of order unity. The continuity equation in non-dimensional form is

$$\frac{U}{l}\frac{\partial u^*}{\partial x^*} + \frac{V}{\delta}\frac{\partial v^*}{\partial y^*} = 0 \qquad (5)$$

Integrating, we get

$$\frac{U}{l}\int_0^1 \frac{\partial u^*}{\partial x^*} dy^* + \frac{V}{\delta}\int_0^1 \frac{\partial v^*}{\partial y^*} dy^* = 0$$

or
$$\frac{V}{U} = -\frac{\delta}{l}\int_0^1 \frac{\partial u^*}{\partial x^*} dy^*, \text{ where } (v^*)_{y^*=1} = 1 \qquad (6)$$

Since the integral in (6) is of the order of unity, the ratio $\frac{V}{U}$ is of order $\frac{\delta}{l}$.

Therefore V < < U.

We now obtain the non-dimensional form of (1) using (4) such that

$$\frac{U^2}{l}\frac{\partial u^*}{\partial t^*} + \frac{U^2}{l}u^*\frac{\partial u^*}{\partial x^*} + \frac{UV}{\delta}v^*\frac{\partial u^*}{\partial y^*} = \frac{-p_\infty}{\rho l}\frac{\partial p^*}{\partial x^*} + \frac{\nu U}{l^2}\left(\frac{\partial^2 u^*}{\partial x^{*2}} + \frac{l^2}{\delta^2}\frac{\partial^2 u^*}{\partial y^{*2}}\right)$$

or

$$\frac{\partial u^*}{\partial t^*} + u^*\frac{\partial u^*}{\partial x^*} + \frac{V}{U}\;\frac{l}{\delta}v^*\frac{\partial u^*}{\partial y^*} = -\frac{p_\infty}{\rho U^2}\frac{\partial p^*}{\partial x^*} + \frac{l}{R_e}\left(\frac{\partial^2 u^*}{\partial x^{*2}} + \frac{l^2}{\delta^2}\frac{\partial^2 u^*}{\partial y^{*2}}\right) \qquad (7)$$

$$1 \qquad\quad 1 \qquad \delta \;\; \frac{1}{\delta} \qquad 1 \qquad\qquad\qquad \delta^2 \qquad 1 \qquad\quad \frac{1}{\delta^2}$$

The order of the terms involved are indicated.

Reynolds number, $R_e = \frac{lU}{\nu} \Rightarrow \frac{1}{R_e} = \frac{\nu}{lU} = 0(\delta)^2$ as δ is proportional to $v^{1/2}$.

Similarly, the non-dimensional form of (2) is

$$\frac{UV}{l}\frac{\partial v^*}{\partial t^*} + \frac{UV}{l}u^*\frac{\partial v^*}{\partial x^*} + \frac{V^2}{\delta}v^*\frac{\partial v^*}{\partial y^*}$$

$$= -\frac{p_\infty}{\rho\delta}\frac{\partial p^*}{\partial y^*} + \nu\left(\frac{V}{l^2}\frac{\partial^2 v^*}{\partial x^{*2}} + \frac{V}{\delta^2}\frac{\partial^2 v^*}{\partial y^{*2}}\right)$$

or
$$\frac{V}{U}\frac{\partial v^*}{\partial t^*} + \frac{V}{U}u^*\frac{\partial v^*}{\partial x^*} + \frac{V^2}{U^2}\frac{l}{\delta}v^*\frac{\partial v^*}{\partial y^*}$$

$$\delta \qquad \delta \qquad \delta^2 \quad \frac{1}{\delta}$$

$$= \frac{-p_\infty}{\rho U^2}\frac{l}{\delta}\frac{\partial p^*}{\partial y^*} + \frac{\nu V l}{l^2 U^2}\left(\frac{\partial^2 v^*}{\partial x^{*2}} + \frac{l^2 \partial^2 v^*}{\delta^2 \partial y^{*2}}\right)$$

$$= \frac{-p_\infty}{\rho U^2}\frac{l}{\delta}\frac{\partial p^*}{\partial y^*} + \frac{1}{R_e}\frac{V}{U}\left(\frac{\partial^2 v^*}{\partial x^{*2}} + \frac{l^2}{\delta^2}\frac{\partial^2 v^*}{\partial y^{*2}}\right)$$

$$\delta^2 \quad \delta \quad 1 \qquad \frac{1}{\delta^2} \tag{8}$$

We neglect the terms of the order of δ and higher as δ is small. We then revert back to the dimensional variables to obtain

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \nu\frac{\partial^2 u}{\partial y^2} \tag{9}$$

$$\frac{\partial p}{\partial y} = 0 \quad \Rightarrow p = p(x) \tag{10}$$

and
$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \tag{11}$$

Equations (9–11) are known as Prandtl's boundary layer equations with boundary conditions

$$\left.\begin{array}{l} u = v = 0, \quad y = 0 \\ u = U(x,t), y \to \infty \end{array}\right\} \tag{12}$$

Since p is independent of y, for given x, p has the same value through the boundary layer from y = 0 to y = δ. Thus, in boundary layer theory, there are only two variable terms u and v instead of three u, v and p in the Navier-Stokes equations. This is a great simplification in the solution of the differential equations.

Now, U is the velocity outside the boundary layer. The Euler's equation in the main stream (potential flow of non-viscous fluid) is obtained from (9) by taking *v*= 0 and

$$v = 0,\ \frac{\partial u}{\partial y} = 0 \ \text{ for } y \geq \delta$$

Thus, we get

$$\frac{\partial U}{\partial t} + U\frac{\partial U}{\partial x} = -\frac{1}{\rho}\frac{dp}{dx} \tag{13}$$

From (9) and (13), we obtain

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = \frac{\partial U}{\partial t} + U\frac{\partial U}{\partial x} + \nu\frac{\partial^2 u}{\partial y^2} \tag{14}$$

and
$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \tag{15}$$

Although these equations are obtained for a rectilinear flow but they hold for curved flow if the curvature of the boundary is small in comparison to the boundary layer thickness.

The integration of (14) and (15) can be simplified if we can reduce the number of variables by introducing the stream function ψ.

where
$$u = \frac{\partial \psi}{\partial y}, \qquad v = -\frac{\partial \psi}{\partial x} \tag{16}$$

The continuity equation is automatically satisfied. The boundary layer equation (14) in terms of ψ is

$$\frac{\partial^2 \psi}{\partial t \partial y} + \frac{\partial \psi}{\partial y}\frac{\partial^2 \psi}{\partial x \partial y} - \frac{\partial \psi}{\partial x}\frac{\partial^2 \psi}{\partial y^2} = \nu\frac{\partial^3 \psi}{\partial y^3} + U\frac{\partial U}{\partial x} + \frac{\partial U}{\partial t} \tag{17}$$

The boundary conditions (12) reduce to

$$\left.\begin{aligned} &\frac{\partial \psi}{\partial x} = \frac{\partial \psi}{\partial y} = 0,\ y = 0 \\ &\frac{\partial \psi}{\partial y} = U(x, t),\ y \to \infty \end{aligned}\right\} \tag{18}$$

The exact solution of (17) was given by H. Blasius in 1908, for the case of steady flow $(\partial/\partial t = 0)$ past a flat plate (U = constant).

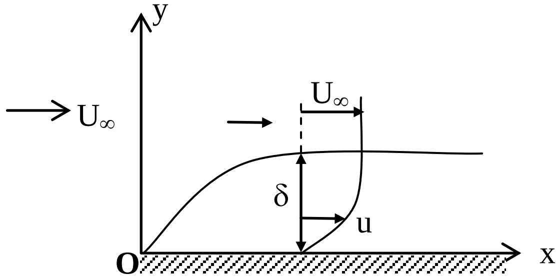
3. The Boundary Layer Along a Flat Plate (Blasius Solution or Blasius – Topfer for Solution)

Let us consider the steady flow of an incompressible viscous fluid past a thin semi-infinite flat plate which is placed in the direction of a uniform velocity U_∞. The motion is two-dimensional and can be analysed by using the Prandtl boundary layer equations. We choose the origin of the co-ordinates at the leading edge of the plate, x-axis along the direction of the uniformal stream and y-axis normal to the plate. The Prandtl boundary layer equations, for this case, are

$$u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}=\nu\frac{\partial^2 u}{\partial y^2} \quad (1)$$

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0 \quad (2)$$

where u, v are the velocity components and ν is the kinematic viscosity.



The boundary conditions are

$$\left.\begin{aligned} u = v = 0 \qquad & \text{when } y = 0 \\ u = U_\infty \qquad & \text{when } y \to \infty \end{aligned}\right\} \quad (3)$$

In this problem, the parameters in which the results are to be obtained, are U_∞, ν, x, y. So, we may take

$$\frac{u}{U_\infty} = F(x, y, \nu, U_\infty) = F(\eta) \quad (4)$$

Further, according to the exact solution of the unsteady motion of a flat plate, we have

$$\delta \sim \sqrt{\nu t} \sim \sqrt{\frac{\nu x}{U_\infty}} \quad (5)$$

where x is the distance travelled in time t with velocity U_∞. Hence the non-dimensional distance parameter may be expressed as

$$\eta = \frac{y}{\delta} = \frac{y}{\sqrt{\nu x / U_\infty}} = y\sqrt{\frac{U_\infty}{\nu x}} \qquad (6)$$

Thus, it can be seen that η in (4) is a function of x, y, ν, U_∞ as in (6)

The stream function ψ is given by

$$\psi = \int u\, dy \qquad \left|\; u = \frac{\partial\psi}{\partial y}, \qquad v = -\frac{\partial\psi}{\partial x}\right.$$

$$= \int U_\infty F(\eta) \frac{dy}{d\eta} d\eta$$

$$= U_\infty \sqrt{\frac{\nu x}{U_\infty}} \int F(\eta)\, d\eta = \sqrt{\nu x U_\infty}\;\; f(\eta) \qquad (7)$$

The velocity components in terms of η are (dash denotes derivative w.r.t. η)

$$u = \frac{\partial\psi}{\partial y} = \frac{\partial\psi}{\partial\eta}\frac{\partial\eta}{\partial y} = \sqrt{\nu x U_\infty}\sqrt{\frac{U_\infty}{\nu x}}\, f'(\eta) = U_\infty f'(\eta) \quad (8)$$

$$-v = \frac{\partial\psi}{\partial x} = \frac{1}{2}\sqrt{\frac{\nu U_\infty}{x}}\; f(\eta) + \sqrt{\nu x U_\infty}\;\, f'(\eta) y \sqrt{\frac{U_\infty}{\nu}} \left(-\frac{1}{2x^{3/2}}\right)$$

$$\Rightarrow \qquad v = -\frac{1}{2}\sqrt{\frac{\nu U_\infty}{x}}\; f(\eta) + \frac{1}{2} y \frac{U_\infty}{x} f'(\eta)$$

$$= \frac{1}{2}\sqrt{\frac{\nu U_\infty}{x}} \left(\sqrt{\frac{U_\infty}{\nu x}}\;\; y\, f'(\eta) - f(\eta)\right)$$

$$= \frac{1}{2}\sqrt{\frac{\nu U_\infty}{x}}\; (\eta f'(\eta) - f(\eta)) \qquad (9)$$

Also,

$$\frac{\partial u}{\partial x} = \frac{\partial^2\psi}{\partial x \partial y} = U_\infty\;\, f''(\eta)\frac{\partial\eta}{\partial x}$$

$$= -\frac{1}{2} U_\infty\, f''(\eta).\; y\sqrt{\frac{U_\infty}{\nu}}\;\; \frac{1}{x^{3/2}}$$

$$= -\frac{1}{2}\frac{U_\infty}{x}\eta\, f''(\eta) \tag{10}$$

$$\frac{\partial u}{\partial y} = U_\infty \frac{\partial}{\partial y}(f''(\eta)) = U_\infty \sqrt{\frac{U_\infty}{\nu x}}\, f''(\eta) \tag{11}$$

$$\frac{\partial^2 u}{\partial y^2} = \frac{U_\infty^2}{\nu x}\, f'''(\eta) \tag{12}$$

Using these values of u, v and their derivatives in (1), we obtain

$$U_\infty f'(\eta)\left(-\frac{1}{2}\frac{U_\infty}{x}\eta\, f''(n)\right) + \frac{1}{2}\sqrt{\frac{\nu U_\infty}{x}}\,(\eta f'(\eta) - f(\eta))U_\infty\sqrt{\frac{U_\infty}{\nu x}}\, f''(\eta)$$

$$= \nu\,\frac{U_\infty^2}{\nu x}\, f'''(\eta)$$

or $$-\frac{U_\infty^2}{2x}\eta\, f'\, f'' + \frac{U_\infty^2}{2x}(\eta f' - f)f'' = \frac{U_\infty^2}{x} f'''$$

or $$-\eta\, f'\, f'' + \eta\, f'\, f'' - f\, f'' = 2f'''$$

or $$2\, f''' + f\, f'' = 0$$

i.e. $$2\frac{d^3 f}{d\eta^3} + f\,\frac{d^2 f}{d\eta^2} = 0 \tag{13}$$

The boundary conditions (3) in terms of f and η are obtained as follows

$u = 0$ when $y = 0$ implies $f'(\eta) = 0$ when $\eta = 0$

and

$v = 0 \;\Rightarrow \eta f'(\eta) - f(\eta) = 0 \;\Rightarrow f(\eta) = 0$

Therefore,

$$f(\eta) = f'(\eta) = 0 \text{ when } \eta = 0 \tag{14}$$

$u = U_\infty$ when $y \to \infty$ implies that $U_\infty f'(\eta) = U_\infty$ when $\eta \to \infty$

Therefore,

$$f'(\eta) = 1 \text{ when } \eta \to \infty \tag{15}$$

Thus we have reduced the partial differential equation (1) to ordinary differential equation (13), known as Blasius equation, where η is the similarity parameter.

The third order non-linear differential equation (13) has no closed form solution, however, Blasius obtained the solution in the form of power series expansion about $\eta = 0$.

Let us consider

$$f(\eta) = c_0 + c_1\eta + \frac{c_2}{\lfloor 2}\eta^2 + \frac{c_3}{\lfloor 3}\eta^3 + \ldots\ldots. \tag{16}$$

$$f'(\eta) = c_1 + c_2\eta + \frac{c_3}{\lfloor 2}\eta^2 + \frac{c_4}{\lfloor 3}\eta^3 + \ldots\ldots. \tag{17}$$

$$f''(\eta) = c_2 + c_3\eta + \frac{c_4}{\lfloor 2}\eta^2 + \frac{c_5}{\lfloor 3}\eta^3 + \ldots\ldots. \tag{18}$$

$$f'''(\eta) = c_3 + c_4\eta + \frac{c_5}{\lfloor 2}\eta^2 + \frac{c_6}{\lfloor 3}\eta^3 + \ldots\ldots. \tag{19}$$

The constants c_i's are determined from the boundary conditions (14), (15) and the differential equation (13). From (14), we get

$$c_0 = c_1 = 0$$

From (13), we have

$$0 = (2c_3 + 2c_4\eta + c_5\eta^2 + \ldots..) + (c_0 + c_1\eta + \frac{c_2}{\lfloor 2}\eta^2 + \ldots.)\,(c_2 + c_3\eta + \frac{c_4}{\lfloor 2}\eta^2 + \ldots)$$

i.e.

$$(2c_3 + c_0 c_2) + (2c_4 + c_0 c_3 + c_1 c_2)\eta + \left(c_5 + \frac{c_0 c_4}{\lfloor 2} + c_1 c_3 + \frac{c_2^2}{2}\right)\eta^2 + \ldots. = 0$$

i.e.

$$2c_3 + 2c_4\eta + \left(c_5 + \frac{c_2^2}{2}\right)\eta^2 + \ldots. = 0$$

Equating the co-efficients to zero, we get

$$c_3 = c_4 = c_6 = c_7 = c_9 = c_{10} = 0$$

$$c_5 = -\frac{c_2^2}{\lfloor 2}, \quad c_8 = \frac{11}{4}c_2^3, \quad c_{11} = -\frac{375}{8}c_2{}^4$$

The solution (16) is

$$f(\eta) = \frac{c_2}{2}\eta^2 - \frac{c_2^2}{2}\frac{\eta^5}{\lfloor 5} + \frac{11}{4}c_2^3\frac{\eta^8}{\lfloor 8} - \frac{375}{8}c_2^4\frac{\eta^{11}}{\lfloor 11} + \ldots \qquad (20)$$

The constant c_2 is determined by the condition (15) i.e.

$$\frac{df}{d\eta} = 1 \text{ as } n \to \infty$$

We write (20) as

$$f(\eta) =$$

$$c_2^{1/3}\left[\frac{(c_2^{1/3}\eta)^2}{\lfloor 2} - \frac{1}{2}\frac{(c_2^{1/3}\eta)^5}{\lfloor 5} + \frac{11}{4}\frac{(c_2^{1/3}\eta)^8}{\lfloor 8} - \frac{375}{8}\frac{(c_2^{1/3}\eta)^{11}}{\lfloor 11} + \ldots\right]$$

$$= c_2^{1/3}F(c_2^{1/3}\eta) \qquad (21)$$

Therefore,

$$f'(\eta) = c_2^{2/3}\, F'(c_2^{1/3}\eta)$$

Thus, $\lim\limits_{\eta\to\infty} c_2^{2/3}F'(c_2^{1/3}\eta) = \lim\limits_{\eta\to\infty} f'(\eta) = 1$

Therefore,

$$c_2 = \left[\frac{1}{\lim\limits_{\eta\to\infty} f'(c_2^{1/3}\eta)}\right]^{3/2} \qquad (22)$$

where c_2 is determined numerically by Howarth (1938) as 0.33206. Thus $f(\eta)$ in (20) is completely obtained which helps in finding u and v from (8) and (9). Hence the Blasius solution.

The shearing stress τ_0 on the surface of the plate can be calculated from the results of the Blasius solution. Thus, we have

$$\tau_0 = \mu\left(\frac{\partial u}{\partial y}\right)_{y=0} = \frac{\mu U_\infty f''(0)}{\sqrt{\nu x/U_\infty}}$$

$$= \mu \frac{U_\infty C_2}{\sqrt{\nu x/U_\infty}} = \frac{0.332}{\sqrt{R_{e_x}}} \rho U_\infty^2 \tag{23}$$

where $R_{e_x} = xU_\infty / \nu$ is the Reynolds number.

The frictional drag coefficients or local skin friction coefficients C_f is

$$C_f = \frac{\tau_0}{\frac{1}{2}\rho U_\infty^2} = \frac{0.664}{\sqrt{R_{e_x}}} \tag{24}$$

The total frictional force F per unit width for one side of the plate of length *l* is given by

$$F = \int_0^l \tau_0 d_x = 0.664\ \rho U_\infty^2 \sqrt{\frac{\nu l}{U_\infty}} \tag{25}$$

Equation (25) shows that frictional force is proportional to the 3/2th power of the free stream velocity U_∞.

The average skin-friction co-efficient of the drag co-efficient is obtained as

$$C_F = \frac{F}{\frac{1}{2}\rho U_\infty^2 l} = \frac{0.664 P U_\infty^2 \sqrt{\nu l/U_\infty}}{\frac{1}{2} P U_\infty^2 l} = \frac{1.328}{\sqrt{R_{e_l}}} \tag{26}$$

Where $R_{e_l} = \frac{l U_\infty}{\nu}$.

3.1. Characteristic Boundary Layer Parameters : (i) Boundary Layer Thickness. The boundary layer is the region adjacent to a solid surface in which viscous forces are important. According to the boundary conditions (3), the velocity u in the boundary layer does not reach the value U_∞ of the free stream until $y \to \infty$, because the influence of viscosity in the boundary layer decreases asymptotically outwards. Hence it is difficult to define an exact thickness of the boundary layer. However, at certain finite value of η, the velocity in the boundary layer asymptotically blends into the free stream velocity of the potential flow. If an arbitrary limit of the boundary layer at $u = 0.9975\ U_\infty$ is considered, the thickness of the boundary layer is found to be

$$\delta = 5.64\sqrt{\frac{\nu x}{U_\infty}} = \frac{5.64x}{\sqrt{R_{e_x}}} \tag{27}$$

(ii) Displacement Thickness : The boundary layer thickness being somewhat arbitrary so more physically meaningful thickness is introduced. This thickness is known as displacement thickness, which is defined as

$$U_\infty \delta_1 = \int_{y=0}^{\infty} (U_\infty - u)dy \tag{28}$$

where the right-hand size signifies the decrease in total flow caused by the influence of the friction and the left-hand side represents the potential flow that has been displaced from the wall. Hence the displacement thickness δ_1 is that distance by which the external potential field of flow is displaced outwards due to the decrease in velocity in the boundary layer.

i.e. $$\delta_1 = \int_0^\infty \left(1 - \frac{u}{U_\infty}\right)dy \tag{29}$$

Using the expressions for $\frac{u}{U_\infty}$ and y from (8) and (6) respectively, we find δ_1 for the flow on a flat plate, as

$$\delta_1 = \sqrt{\frac{\nu x}{U_\infty}} \int_0^\infty (1 - f')d\eta$$

$$= \sqrt{\frac{Ux}{U_\infty}} \lim_{\eta \to \infty} [\eta - f(\eta)]$$

$$= 1.7208\sqrt{\frac{\nu x}{U_\infty}} = \frac{1.7208\,x}{\sqrt{R_{e_x}}} \tag{30}$$

(iii) Momentum Thickness : Analogous to the displacement thickness, another thickness, known as momentum thickness (δ_2), may be defined in accordance with the momentum law. This is obtained by equating the loss of momentum flow as a consequence of the wall friction in the boundary layer to the momentum flow in the absence of the boundary layer. Thus

$$\rho\delta_2 U_\infty^2 = \rho\int_{y=0}^{\infty} u(U_\infty - u)dy$$

or $$\delta_2 = \int_0^\infty \frac{u}{U_\infty}\left(1-\frac{u}{U_\infty}\right)dy \tag{31}$$

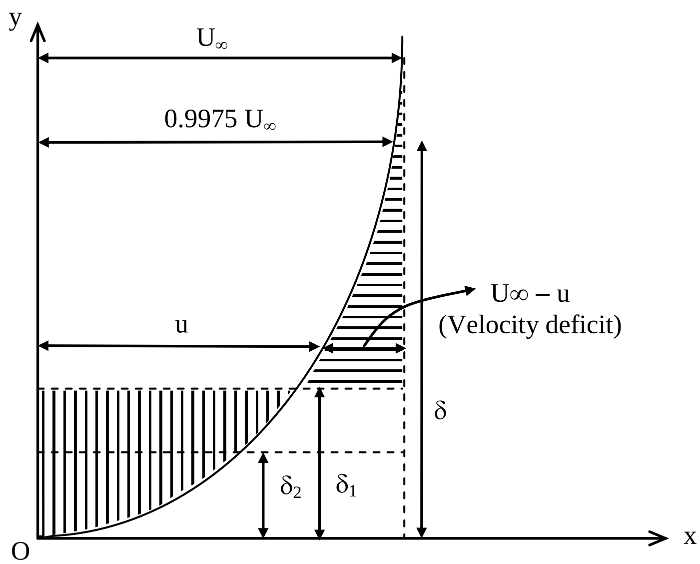
Again, using (8) and (6), we obtain δ_2 for the case of the flow on a flat plate, as

$$\delta_2 = \sqrt{\frac{\nu x}{U_\infty}}\int_0^\infty f'(1-f')\,d\eta$$

$$= 0.664\sqrt{\frac{\nu x}{U_\infty}} = \frac{0.664\,x}{\sqrt{R_{e_x}}} \tag{32}$$

Comparison among various thicknesses of the boundary layer is shown in the figure. We note that

$$\delta_2 < \delta_1 < \delta\,.$$



4. Integral Methods for the Approximate Solution of Boundary Layer Equations (Karman Integral Conditions)

We have observed that the solution of the steady boundary layer equations is very difficult. The solution obtained in the previous case is also a very special case. For engineering problems, it is often acceptable if an approximate solution can be obtained. One of the most useful methods is the Von Karman-Pohlhausen method based on the integral theorem. The basic concept of this method is that the solutions satisfy the differential equations only on the average, i.e., it is not anticipated that the solution satisfies the boundary layer equations at every point (x, y) but the momentum integral equation and the boundary conditions must be satisfied. The momentum integral equation is obtained by integrating the boundary layer equations with respect to y over the boundary layer thickness or by the momentum law.

4.1. Momentum Integral Equation for the Boundary Layer (Von Karman Integral Relation). The Prandtl's boundary layer equations are

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -\frac{1}{\rho}\frac{dp}{dx} + \nu\frac{\partial^2 u}{\partial y^2} \quad (1)$$

and

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \quad (2)$$

Integrating (1) w.r.t. y from y = 0 to y = $\delta(x)$, the outer edge of the boundary layer, we get

$$\frac{\partial}{\partial t}\int_0^\delta u\,dy + \int_0^\delta u\frac{\partial u}{\partial x}dy + \int_0^\delta v\frac{\partial u}{\partial y}dy = -\frac{1}{\rho}\int_0^\delta \frac{dp}{dx}dy + \frac{\mu}{\rho}\int_0^\delta \frac{\partial^2 u}{\partial y^2}dy \quad (3)$$

Let us simplify the third term on L.H.S. of (3). We have

$$\int_0^\delta v\frac{\partial u}{\partial y}dy = \int_0^\delta \frac{\partial(uv)}{\partial y}dy - \int_0^\delta u\frac{\partial v}{\partial y}dy$$

$$= \int_0^\delta d(uv) - \int_0^\delta u\frac{\partial v}{\partial y}dy$$

$$= [uv]_0^\delta - \int_0^\delta u\frac{\partial v}{\partial y}dy$$

$$= U\int_0^\delta \frac{\partial v}{\partial y}dy - \int_0^\delta u\frac{\partial v}{\partial y}dy \quad (4)$$

where u = U at y = δ.

Replacing $\frac{\partial v}{\partial y}$ by $-\frac{\partial u}{\partial x}$, from the continuity equation, we get

$$\int_0^{\delta} v\frac{\partial u}{\partial y}dy = -U\int_0^{\delta}\frac{\partial u}{\partial x}dy + \int_0^{\delta} u\frac{\partial u}{\partial x}dy \qquad (5)$$

Using (5) in (3), we find

$$\frac{\partial}{\partial t}\int_0^{\delta} u\,dy + \int_0^{\delta} u\frac{\partial y}{\partial x}dy - U\int_0^{\delta} u\frac{\partial u}{\partial x}dy + \int_0^{\delta} u\frac{\partial u}{\partial x}dy = -\frac{1}{\rho}\int_0^{\delta}\frac{dp}{dx}dy + \frac{\mu}{\rho}\int_0^{\delta}\frac{\partial^2 u}{\partial y^2}dy$$

$$\Rightarrow \qquad \frac{\partial}{\partial t}\int_0^{\delta} u\,dy + 2\int_0^{\delta} u\frac{\partial u}{\partial x}dy - U\int_0^{\delta}\frac{\partial u}{\partial x}dy = -\frac{1}{\rho}\int_0^{\delta}\frac{dp}{dx}dy + \frac{\mu}{\rho}\left[\frac{\partial u}{\partial y}\right]_0^{\delta}$$

$$\Rightarrow \qquad \frac{\partial}{\partial t}\int_0^{\delta} u\,dy + \int_0^{\delta}\frac{\partial u^2}{\partial x}dy - U\int_0^{\delta}\frac{\partial u}{\partial x}dy = -\frac{1}{\rho}\int_0^{\delta}\frac{dp}{dx}dy + \frac{\mu}{\rho}\left(\frac{\partial u}{\partial y}\right)_{y=\delta} - \frac{\mu}{\rho}\left(\frac{\partial u}{\partial y}\right)_{y=0}$$

$$= -\frac{1}{\rho}\frac{dp}{dx}\delta + \left(\frac{\tau}{\rho}\right)_{y=\delta} - \left(\frac{\tau}{\rho}\right)_{y=0}$$

$$= -\frac{\delta}{\rho}\frac{dp}{dx} - \frac{\tau_0}{\rho} \qquad (6)$$

where $\tau = \mu\dfrac{\partial u}{\partial y} = 0$ at $y = \delta$.

$= \tau_0$ at $y = 0$

i.e. τ_0 is the shear stress on the wall.

Let us further simplify the second and third terms on the L.H.S. of (6). For this we use the Leibnitz rule according to which

$$\frac{d}{dx}\int_{a(x)}^{b(x)} f(x,y)dy = \int_{a(x)}^{b(x)}\frac{\partial f}{\partial x}dy - f(a,y)\frac{da}{dx} + f(b,y)\frac{db}{dx}$$

$$\therefore \qquad \int_0^{\delta}\frac{\partial u^2}{\partial x}dy = \frac{d}{dx}\int_0^{\delta} u^2 dy - U^2\frac{d\delta}{dx}$$

and

$$U\int_0^{\delta}\frac{\partial u}{\partial x}dy = U\frac{d}{dx}\int_0^{\delta} u\,dy - U^2\frac{d\delta}{dx}$$

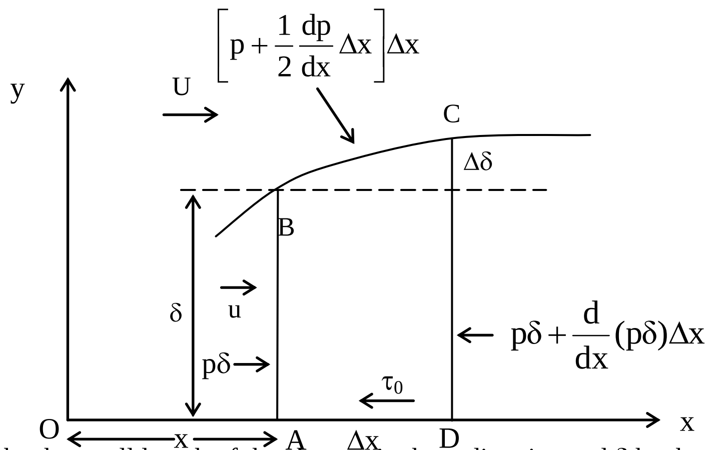
Thus, equation (6) reduces to

$$\frac{\partial}{\partial t}\int_0^{\delta} u dy + \frac{d}{dx}\int_0^{\delta} u^2 dy - U\frac{d}{dx}\int_0^{\delta} u\,dy = -\frac{\delta}{\rho}\frac{dp}{dx} - \frac{\tau_0}{\rho} \qquad (7)$$

This is one form of the Von Karman integral relation and is also called the momentum integral equation of the boundary layer.

4.2. Von Karman Integral Relation by Momentum Law. The Von Karman integral equation of the boundary layer represents the relation between the overall rate of flux of momentum across a section of the boundary and the surface forces due to the wall shearing stress and the pressure gradient. The Von Karman integral equation which we just obtained, can be derived from the momentum theorem of fluid mechanics.

Let us consider an element of the boundary layer, ABCDA of unit length perpendicular to the xy-plane as shown in the figure.



Let AD = Δx be the small length of the element in the x-direction and δ be the thickness of the boundary layer at a distance x from the leading edge of the plate. We assume that the velocity of the boundary layer flow at the outer edge of the boundary layer is the same as that of the potential flow, i.e., u = U at y = δ.

The rate of mass flow across AB into the element is

$$\int_0^\delta \rho u \, dy$$

The corresponding rate of mass flow across DC out of the element is

$$\int_0^\delta \rho u \, dy + \frac{\partial}{\partial x}\left[\int_0^\delta \rho u \, dy\right]\Delta x$$

The net rate of flow across AB and DC is

$$\frac{d}{dx}\left[\int_0^\delta \rho u \, dy\right]\Delta x \qquad (1)$$

Since there is no flow across the surface of the plate AD, so by continuity equation, the rate of mass flow out of the element across BC must be

$$-\frac{d}{dx}\left[\int_0^\delta \rho u \, dy\right]\Delta x \qquad (2)$$

Similarly, the net rate of change of momentum across AB and DC of the element, in the x-direction, becomes

$$\frac{d}{dx}\left[\int_0^\delta \rho u^2 \, dy\right]\Delta x \qquad (3)$$

The rate of change of momentum across BC is

$$-U\ \frac{d}{dx}\left[\int_0^{\delta}\rho u\,dy\right]\Delta x \qquad (4)$$

where U is the velocity across BC in the x-direction. Total outward flux of momentum becomes

$$\left[\frac{d}{dx}\int_0^{\delta}\rho u^2 dy - U\frac{d}{dx}\int_0^{\delta}\rho u\,dy\right]\Delta x \qquad (5)$$

The time rate of increase of momentum within the element is

$$\left[\frac{\partial}{\partial t}\int_0^{\delta}\rho u\,dy\right]\Delta x \qquad (6)$$

The forces acting on the fluid due to the shearing stress at the wall is

$$-\mu\left(\frac{\partial u}{\partial u}\right)_{y=0}\Delta x = -\tau_0\,\Delta x, \text{ in the x-direction} \qquad (7)$$

and due to the difference of pressure along AB and CD is

$$p\delta - \left[p\delta + \frac{d}{dx}(p\delta)\Delta x\right] + \left[p + \frac{1}{2}\frac{dp}{dx}\Delta x\right]\Delta x\frac{d\delta}{dx} = -\delta\frac{dp}{dx}\Delta x \qquad (8)$$

where we have neglected terms of order $\Delta x \Delta\delta$.

Now, according to the momentum law, we have

Rate of change of momentum in x-direction. = Total force in x-direction

$$\Rightarrow \qquad \left[\frac{\partial}{\partial t}\int_0^{\delta}\rho u\,dy\right]\Delta x + \left[\frac{d}{dx}\int_0^{\delta}\rho u^2\,dy - U\frac{d}{dx}\int_0^{\delta}\rho u\,dy\right]\Delta x$$

$$= -\delta\ \frac{do}{dx}\Delta x - \tau_0\,\Delta x \qquad (9)$$

Dividing both sides of (9) by $\rho\Delta x$, we get

$$\frac{\partial}{\partial t}\int_0^{\delta}u\,dy + \frac{d}{dx}\int_0^{\delta}u^2 dy - U\frac{d}{dx}\int_0^{\delta}u dy = -\frac{\delta}{\rho}\frac{dp}{dx} - \frac{\tau_0}{\rho} \qquad (10)$$

which is the required Von Karman integral equation, being the same as obtained by integrating Prandtl's boundary layer.

4.3. Other Forms of the Von Karman Integral Equation. It is often convenient to have the integral equation in terms of displacement and momentum thicknesses. The momentum integral equation of the boundary layer is

$$\frac{\partial}{\partial t}\int_0^\delta u\,dy+\frac{d}{dx}\int_0^\delta u^2dy-U\frac{d}{dx}\int_0^\delta udy=-\frac{\delta}{\rho}\frac{dp}{dx}-\frac{\tau_0}{\rho} \qquad (1)$$

Also, the Euler's equation in the main stream is

$$\frac{\partial U}{\partial t}+U\frac{\partial U}{\partial x}=-\frac{1}{\rho}\frac{dp}{dx} \qquad (2)$$

where U is the velocity of the potential flow, $\frac{dp}{dx}$ is the pressure gradient, ρ is the density , δ is the thickness of the boundary layer and τ_0 is the shearing stress at the wall. For a steady flow, we obtain from (1) and (2)

$$\frac{d}{dx}\int_0^\delta u^2\,dy-U\frac{d}{dx}\int_0^\delta u\,dy-\delta U\frac{dU}{dx}=\frac{\tau_0}{\rho}$$

or

$$\frac{d}{dx}\int_0^\delta u^2dy-\frac{d}{dx}\int_0^\delta U\,u\,dy+\frac{dU}{dx}\int_0^\delta udy-U\frac{dU}{dx}\int_0^\delta dy=-\frac{\tau_0}{\rho}$$

or

$$\frac{d}{dx}\int_0^\delta u(U-u)\,dy+\frac{dU}{dx}\int_0^\delta (U-u)\ =\frac{\tau_0}{\rho} \qquad (3)$$

The displacement and momentum thicknesses are defined by

$$\delta_1=\int_0^\delta\left(1-\frac{u}{U}\right)dy \quad \Rightarrow U\delta_1=\int_0^\delta\left(1-\frac{u}{U}\right)dy \qquad (4)$$

and

$$\delta_2=\int_0^\delta\frac{u}{U}\left(1-\frac{u}{U}\right)dy \quad \Rightarrow U^2\delta_2=\int_0^\delta u(U-u)dy \qquad (5)$$

Thus, equation (3) reduces to

$$\frac{d}{dx}(U^2\delta_2)+\frac{dU}{dx}U\delta_1=\frac{\tau_0}{\rho}$$

or

$$U^2\frac{d\delta_2}{dx}+2U\delta_2\frac{dU}{dx}+U\delta_1\frac{dU}{dx}=\frac{\tau_0}{\rho}$$

or

$$\frac{d\delta_2}{dx}+\frac{1}{U}\left(2\delta_2+\delta_1\right)\frac{dU}{dx}=\frac{\tau_0}{\rho U^2} \qquad (6)$$

This is the Von Karman momentum integral equation in terms of displacement and momentum thicknesses.

4.4. Application of the Momentum Integral Equation to Boundary Layers (Von Karman Pohlhausen Method). Pohlhausen introduced a fourth degree polynomial for the velocity in terms of a non-dimensional parameter $\eta = y/\delta$, $0 \le \eta \le 1$ such that

$$\frac{u}{U} = f(\eta) = a\eta + b\eta^2 + c\eta^3 + d\eta^4 \qquad (1)$$

The constants a, b, c d are to be determined from the boundary conditions

$$u = 0,\ v = 0,\ \frac{\partial^2 u}{\partial y^2} = \frac{1}{\mu}\frac{dp}{dx} = -\frac{U}{\nu}\frac{dU}{dx},\ \text{at } y = 0 \qquad (2)$$

$$u = U,\ \frac{\partial u}{\partial y} = 0, \frac{\partial^2 u}{\partial y^2} = 0,\ \text{at } y = \delta \qquad (3)$$

The first two conditions in (2) and the first condition in (3) are satisfied by all exact solutions of the boundary layer equations. The second condition in (3) is meant for continuous flow on the outer boundary of the layer. The third condition in (2) is obtained from Prandtl's boundary layer equation i.e.

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -\frac{1}{\rho}\frac{dp}{dx} + \nu\frac{\partial^2 u}{\partial y^2}$$

When the flow is steady and u = 0 = v on y = 0, then

$$\frac{\partial^2 u}{\partial y^2} = \frac{1}{\nu\rho}\frac{dp}{dx} = \frac{1}{\mu}\frac{dp}{dx} = -\frac{U}{\nu}\frac{dU}{dx}$$

The point where $\frac{\partial^2 u}{\partial y^2} = 0$ is called a point of inflexion of the velocity profile in the boundary layer. From (1), we get

$$\frac{\partial^2 u}{\partial y^2} = \frac{1}{\delta^2}\frac{d^2 u}{d\eta^2} = \frac{U}{\delta^2}(2b+6c\eta+12d\eta^2) = 0$$

$$\Rightarrow \qquad 6d\eta^2 + 3c\eta + b = 0$$

This gives two values of η. One of the points is near the wall and other is in the upper region of the boundary layer. For this reason, the boundary condition.

$$\frac{\partial^2 u}{\partial y^2} = 0 \text{ at } y = \delta \text{ is imposed.}$$

Let us now use the conditions (2) and (3) in (1) to find out

$$\eta = 0,\ \frac{\partial^2 u}{\partial y^2} = -\frac{U}{\nu}\frac{dU}{dx} \quad \Rightarrow \frac{2bU}{\delta^2} = -\frac{U}{\nu}\frac{dU}{dx}$$

$$\Rightarrow \qquad b = -\frac{\delta^2}{2\nu}\frac{dU}{dx} = -\frac{\lambda}{2}, \lambda = \frac{\delta^2}{\nu}\frac{dU}{dx}$$

$$\left.\begin{aligned}
&\eta = 1,\ u = U \quad \Rightarrow a - \frac{\lambda}{2} + c + d = 1 \\
&\eta = 1,\ \frac{\partial u}{\partial y} = 0 \quad \Rightarrow a - \lambda + 3c + 4d = 0 \\
&\eta = 1,\ \frac{\partial^2 u}{\partial y^2} = 0 \Rightarrow -\lambda + 6c + 12d = 0
\end{aligned}\right\} \qquad (4)$$

Solving (4), we get

$$a = 2 + \frac{\lambda}{6},\ b = \frac{-\lambda}{2},\quad c = -2 + \frac{\lambda}{2}, d = 1 - \frac{\lambda}{6} \qquad (5)$$

Therefore, the velocity in (1) has the expression

$$\frac{u}{U} = f(\eta) = \left(2 + \frac{\lambda}{6}\right)\eta - \frac{\lambda}{2}\eta^2 + \left(-2 + \frac{\lambda}{2}\right)\eta^3 + \left(1 - \frac{\lambda}{6}\right)\eta^4$$

$$= 2\eta - 2\eta^3 + \eta^4 + \lambda\left(\frac{\eta}{6} - \frac{\eta^2}{2} + \frac{\eta^3}{2} - \frac{\eta^4}{6}\right)$$

$$= F(\eta) + \lambda\, G(\eta) \qquad (6)$$

where $\qquad F(\eta) = 2\eta - 2\eta^3 + \eta^4$

$$G(\eta) = \frac{\eta}{6}(1 - 3\eta + 3\eta^2 - \eta^3) = \frac{\eta}{6}(1-\eta)^3$$

$$\lambda = \frac{\delta^2}{\nu}\frac{dU}{dx} \qquad (7)$$

The velocity profile expressed in terms of η in (6) constitute a one-parameter family of curves with a dimensionless parameter λ which depends mainly on the pressure gradient of the flow. λ may be written as

$$\lambda = \frac{\delta^2}{\nu}\frac{dU}{dx} = -\frac{dp}{dx}\left(\frac{\delta}{\mu U/\delta}\right)$$

which can be interpreted physically as the ratio of the pressure force to viscous force. This is known as the **shape factor.**

We shall now calculate the limits of λ. From (6), we get

$$f(\eta) = (2\eta - 2\eta^3 + \eta^4) + \frac{\lambda}{6}\eta(1-\eta)^3$$

Therefore,

$$\frac{df}{d\eta} = (2-6\eta^2 + 4\eta^3) + \frac{\lambda}{6}[(1-\eta)^3 - 3\eta(1-\eta)^2]$$

$$= (2-6\eta^2 + 4\eta^3) + \frac{\lambda}{6}(1-u\eta)(1-\eta)^2$$

and
$$\frac{d^2f}{d\eta^2} = (1-\eta)\left[12\left(\frac{\lambda}{6}-1\right)\eta - \lambda\right]$$

when
$$\eta = 0, \left(\frac{df}{d\eta}\right)_0 = 2 + \frac{\lambda}{6}$$

$\therefore$
$$\left(\frac{df}{d\eta}\right)_0 = 0 \quad \Rightarrow \lambda = -12$$

This is taken as lower limit of λ. The upper limit of λ can be determined from the condition of zero curvature of the velocity, i.e. $\frac{d^2f}{d\eta^2} = 0$ which gives $\eta = \frac{\lambda}{12}\Big/\left(\frac{\lambda}{6}-1\right)$. It is seen that for $\lambda \le 12$, $\eta \ge 1.0$ and for $\lambda > 12$, $\eta < 1.0$. Hence for $\lambda > 12$, the point of inflexion occurs within $\eta = 1.0$ i.e. the velocity profile in the boundary layer becomes greater than the velocity in the potential flow. This is not justified physically. Therefore, we take $\lambda \le 12$. So, the limits of λ are $-12 \le \lambda \le 12$.

For $\lambda = 0$, the velocity profile corresponds to the Blasius solution.

With the aid of the approximate Pohlhausen's velocity profile (6), we find the displacement and momentum thicknesses. These are defined by

$$\delta_1 = \int_0^{\delta}\left(1-\frac{u}{U}\right)dy \tag{8}$$

$$\delta_2 = \int_0^{\delta}\frac{u}{U}\left(1-\frac{u}{U}\right)dy \tag{9}$$

Using (6) and $\eta = \frac{y}{\delta}$, we have

$$\frac{\delta_1}{\delta} = \int_0^1 [1-F(\eta) - \lambda\, G(\eta)]\, d\eta$$

$$= \int_0^1 \left[1-2\eta+2\eta^3-\eta^4-\frac{\lambda}{6}\eta(1-\eta)^3\right]d\eta$$

$$= \left[\eta-\eta^2+\frac{\eta^4}{2}-\frac{\eta^5}{5}-\frac{\lambda}{6}\frac{(1-\eta)^5}{5}+\frac{\lambda}{6}\frac{(1-\eta)^4}{4}\right]_0^1$$

$$= \frac{3}{10} - \frac{\lambda}{120} \tag{10}$$

and

$$\frac{\delta_2}{\delta} = \int_0^1 [F(\eta)+\lambda\, G(\eta)]\, [1-F(\eta)-\lambda G(\eta)]d\eta$$

$$= \frac{37}{315} - \frac{\lambda}{945} - \frac{\lambda^2}{9072} \tag{11}$$

The shearing stress τ_0 at the wall is given by

$$\tau_0 = \mu\left(\frac{\partial u}{\partial y}\right)_{y=0} = \frac{\mu U}{\delta}\left(2 + \frac{\lambda}{6}\right) \tag{12}$$

Let us multiply each side of the momentum integral equation i.e.

$$\frac{d\delta_2}{dx} + \frac{1}{U}(2\delta_2 + \delta_1)\frac{dU}{dx} = \frac{\tau_0}{\rho U^2}$$

by $\dfrac{U\delta_2}{\nu}$ to find out

$$\frac{U\delta_2}{\nu}\frac{d\delta_2}{dx} + \left(2 + \frac{\delta_1}{\delta_2}\right)\frac{\delta_2^2}{\nu}\frac{dU}{dx} = \frac{\tau_0\delta_2}{\mu U} \tag{13}$$

To simplify (13), we use the parameters

$$z = \frac{\delta_2^2}{\nu},\ K = zU' = \frac{\delta_2^2}{\nu}\frac{dU}{dx} = \frac{\delta_2^2}{\nu}\frac{\lambda\nu}{\delta^2} = \left(\frac{\delta_2}{\delta}\right)^2 \lambda \tag{14}$$

With the help of (10), (11) and (12), we have

$$k = \left(\frac{\delta_2}{\delta}\right)^2 \lambda = \left(\frac{37}{315} - \frac{\lambda}{945} - \frac{\lambda^2}{9072}\right)^2 \lambda \tag{15}$$

$$f_1(k) = \frac{\delta_1}{\delta_2} = \frac{\delta_1}{\delta}.\frac{\delta}{\delta_2} = \frac{\dfrac{3}{10} - \dfrac{\lambda}{120}}{\dfrac{37}{315} - \dfrac{\lambda}{945} - \dfrac{\lambda^2}{9072}} \tag{16}$$

$$f_2(k) = \frac{\tau_0}{\mu}\frac{\delta_2}{U} = \frac{\mu U}{\mu\delta}\left(\frac{\lambda + 12}{6}\right)\frac{\delta_2}{U}$$

$$= \left(\frac{\lambda+12}{6}\right)\frac{\delta_2}{\delta} = \left(\frac{\lambda+12}{6}\right)\left(\frac{37}{315} - \frac{\lambda}{945} - \frac{\lambda^2}{9072}\right) \quad (17)$$

Using the values of z, k, $f_1(k)$, $f_2(k)$ in (13), and noting that $\frac{\delta_2}{\nu}\frac{d\delta_2}{dx} = \frac{1}{2}\frac{dz}{dx}$, the momentum integral equation takes the form

$$\frac{U}{2}\frac{dz}{dx} + (2+f_1(k))\,k = f_2(k)$$

or
$$\frac{dz}{dx} = \frac{F(k)}{U} \quad (18)$$

where $F(k) = 2f_2(k) - 2\,(2+f_1(k))\,k$ (19)

Equation (18) is a non-linear differential equation of the first order for z.

At the stagnation point x = 0 and U = 0. At this point $\frac{dz}{dx}$ cannot be infinite and so F(k) = 0. This gives the value of λ at the stagnation point. Thus, we have

$$2f_2(k) - 2(2 + f_1(k))\,k = 0 \quad (20)$$

Using the values from (15), (16) and (17), it is obtained that initial value λ_0 of λ at the stagnation point is $\lambda_0 = 7{\cdot}052$.

Further, we can determine F(k), in (19), numerically for different values of λ.

5. Separation of Boundary Layer

5.1. Physical Approach : The decelerated fluid particles in the boundary layer do not remain in the thin layer which adheres to the body along the whole wetted length of the wall. In some cases the thickness of the boundary layer increases considerably in the downstream direction and the flow in the boundary layer become reversed. The decelerated fluid particles then no longer remain in the boundary layer but forced outwards, which means that the boundary layer separates from the wall. Such phenomenon is known as **boundary layer separation** and the point at which the boundary layer separates is known as **point of separation.**

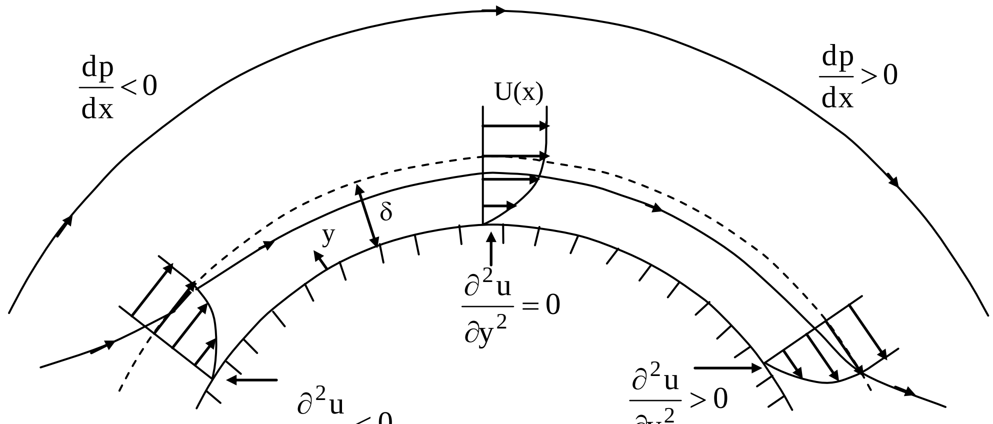
The phenomenon of boundary layer separation is primarily connected with the pressure distribution in the boundary layer and is very common in the flows about blunt bodies, such as circular and elliptic cylinders or spheres. The fluid flow in the boundary layer is determined by the following three factors.

(i) It is retarted due to viscosity because of no-slip condition at the wall.

(ii) It is pulled forward by the free stream velocity above the boundary layer

(iii) It is affected by the pressure gradient.

We have already observed that the pressure in the boundary layer is the same as it is outside the boundary layer. Let us consider a curved surface as shown in the figure



Upstream of the highest point the stream lines of the outer flow converge, resulting in an increase of the free stream velocity U(x) and a consequent fall of pressure with x $\left(\frac{dp}{dx} < 0\right)$ i.e. favourable pressure gradient. Downstream of the highest point the stream lines diverge, resulting in a decrease of U(x) and a rise of pressure with x $\left(\frac{dp}{dx} > 0\right)$. In the region with rising pressure, $\left(\frac{dp}{dx} > 0 \text{ i.e. adverse pressure gradient.}\right)$ along the wall, the retarded fluid particles with small momentum and energy cannot penetrate too far. Thus, the forward flow is brought to rest and thereafter a back flow sets in the direction of the pressure gradient. This causes a boundary layer separation and the point at which the forward flow is brought to rest is called the point of separation.

5.2. Analytical Approach. In this approach, the separation phenomenon may be explained by applying the Prandtl's boundary layer equations both outside the boundary layer and at the wall. Outside the boundary layer, the equation is

$$U\frac{dU}{dx} = -\frac{1}{\rho}\frac{dp}{dx} \tag{1}$$

and at the wall, i.e. at y = 0, we have u = v = 0, the equation is

$$\mu\left(\frac{\partial^2 u}{\partial y^2}\right)_0 = \frac{dp}{dx} \tag{2}$$

It may be noted that at the outer edge of the boundary layer both $\frac{\partial u}{\partial y}$ and $\frac{\partial^2 u}{\partial y^2}$ tend to zero, $\frac{\partial u}{\partial y}$ from the positive side whereas $\frac{\partial^2 u}{\partial y^2}$ from the negative side, as at the outer edge the maximum value of u i.e. U should occur and the boundary layer flow merges smoothly with the potential flow.

Since it is clear from equation (1) that the curvature of the velocity profiles in the immediate neighbourhood of the wall depends only on the pressure gradient, we consider the following three cases :

(i) $\frac{dp}{dx} = 0$ i.e. zero pressure gradient i.e. constant pressure :

In this case $\left(\frac{\partial^2 u}{\partial y^2}\right)_0 = 0$ and hence the velocity gradient $\frac{\partial u}{\partial y}$ decreases steadily from a positive value at the wall to zero at the outer edge of the boundary layer. The velocity profile must therefore have a steadily decreasing form (figure 1).

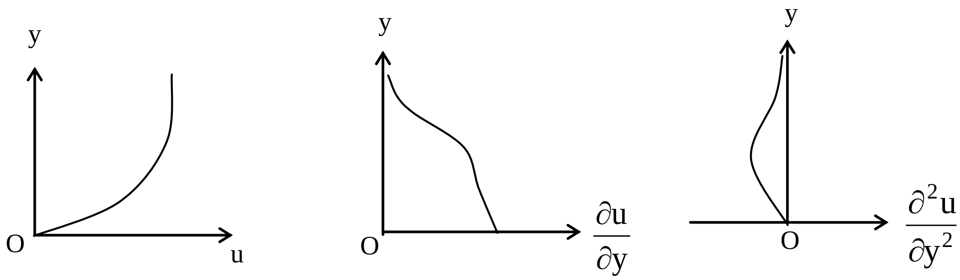


Figure 1

The point of inflexion occurs on the wall since $\left(\frac{\partial^3 u}{\partial y^3}\right)_0 = 0$ but $\left(\frac{\partial^4 u}{\partial y^4}\right) \neq 0$, which can easily be verified by differentiating the boundary layer equation w.r.t. y and evaluating the value at y = 0. The fluid particles continue to move forward and therefore separation of boundary layer does not occur.

(ii) $\frac{dp}{dx} < 0$ i.e. favourable pressure gradient :

For this case, from equation (2), we conclude that $\left(\frac{\partial^2 u}{\partial y^2}\right)_0 < 0$ and therefore it increases steadily to the value zero at the outer edge ($y = \delta$) of the boundary layer. The velocity gradient $\frac{\partial u}{\partial y}$ again decreases steadily from a positive value at the wall to the value zero at the outer edge of the boundary layer. The velocity profile does not have any point of inflexion (figure 2) and has a form similar to the case of zero pressure gradient. In this case also, the fluid particles continue to move forward and so there is no boundary layer separation.

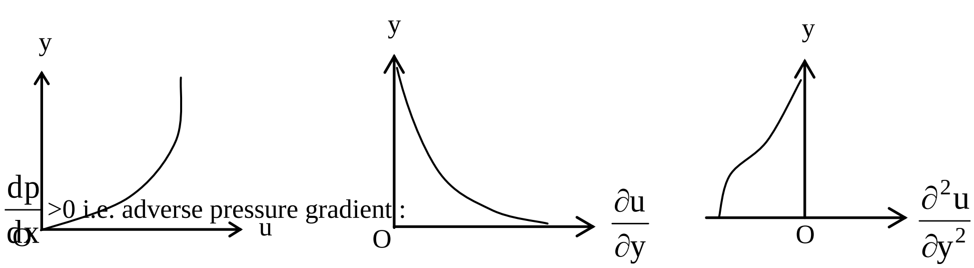


Figure 2

(iii) $\frac{dp}{dx} > 0$ i.e. adverse pressure gradient :

In this case $\left(\dfrac{\partial^2 u}{\partial y^2}\right)_0$ will be a positive quantity. In order to have a positive value of $\dfrac{\partial^2 u}{\partial y^2}$ at y = 0, the slope of the velocity gradient $\dfrac{\partial u}{\partial y}$ at y = 0 must be positive. But the boundary condition requires $\dfrac{\partial u}{\partial y} = 0$ at $y = \delta$. Therefore, the slope of the velocity gradient must change signs from positive to negative in the boundary layer which results in point of inflexion of the velocity profile in the boundary layer (fig. 3). The velocity gradient at the wall is much smaller compared to the case of zero pressure gradient.

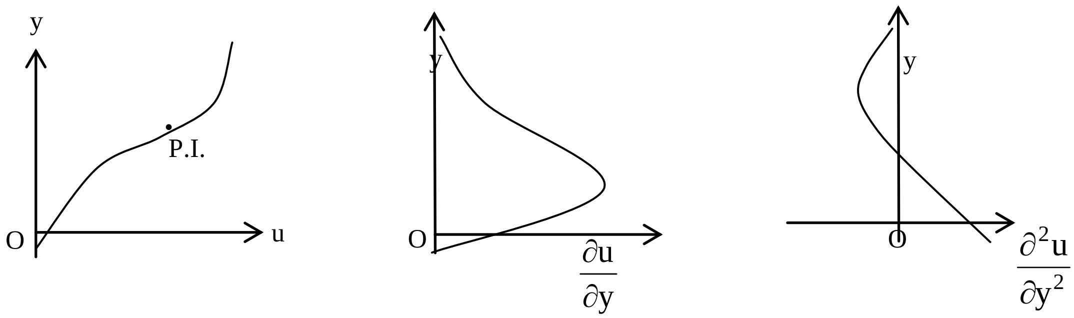


Figure 3.

As the adverse pressure gradient increases further, the velocity profile may become increasingly distorted until the velocity gradient at the wall $\left(\dfrac{\partial u}{\partial y}\right)_0$ is zero, as shown in figure 3. At this point, separation of flow from the wall begins. Further downstream, a back flow in the direction of the pressure gradient sets in.

It should be noted here that the type of velocity profile shown in figure 3 is naturally unstable and it frequently happens that the transition to turbulent flow in the boundary layer will take place before laminar separation can occur. Under such circumstances, the turbulent boundary layer will be maintained and separation of flow from the wall will be delayed.

Further, the point of separation is defined as the limit between forward and reverse flow in the layer in immediate neighbourhood of the boundary wall. In other words, the point of separation is the point at which $\left(\dfrac{\partial u}{\partial y}\right)_0 = 0$ i.e. $\tau_0 = 0$.