व्यावसायिक गणित

बी.कॉम I

दूरस्थ शिक्षा निदेशालय
महर्षि दयानन्द विश्वविद्यालय
रोहतक—124 001

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Developed & Produced by EXCEL BOOKS PVT LTD, A-45 Naraina, Phase 1, New Delhi-110028

विषय सूची

Paper-II: Business Mathematics

Max. Marks : 100

Time : 3 Hours

Note : Ten questions shall be set in the question paper covering the whole syllabi. The candidates shall be required to attempt any five questions in all.

Unit-I **Calculus** (Problems and theorems involving trigonometrical ratios are not to be done).

Differentiation: Partial derivatives up to second order; Homogeneity of functions and Euler's theorem; total differentials Differentiation of implicit function with the help of total differentials.

Manima and Minima; Cases of one variable involving second or higher order derivatives; Cases of two variables involving not more than one constraint.

Integration: Integration as anti-derivative process; Standard forms; Methods of integration-by substitution, by parts, and by use of partial fractions; Definite integration; Finding areas in simple cases; Consumers and producers surplus;

Nature of Commodities learning Curve; Leontiff Input-Output Model.

Unit-II **Matrices and Determinants:** Definition of a matrix; Types of matrices; Algebra of matrices; Properties of determinants; calculation of values of determinants upto third order; Adjoint of a matrix, through adjoint and elementary row or column operations; Solution of system of linear equations having unique solution and involving not more than three variables.

Unit-III **Linear Programming-Formulation of LPP:** Graphical method of solution; Problems relating to two variables including the case of mixed constraints; Cases having no solution, multiple solutions, unbounded solution and redundant constraints.

Simplex Method—Solution of problems up to three variables, including cases of mixed constraints; Duality; Transportation Problem.

Unit-IV **Compound Interest and Annuities:** Certain different types of interest rates; Concept of present value and amount of a sum; Types of annuities; Present value and amount of an annuity, including the case of continuous compounding; Valuation of simple loans and debentures; Problems relating to sinking funds.

अध्याय .1

अवकलन
(Differentiation)

अवकलन, संतत फलनों के अवकलज ;कमतपअंजपअमद्ध ज्ञात करने की विधि है, तथ अवकलज, फलन में होने वाली वृद्धि के संगत फलन को निरूपित करने वाले चर में अल्प वृद्धि के अनुपात की सीमा (या औसत परिवर्तन की दरद्ध को कहते है, अर्थात किसी संतत फलन में औसत परिर्वतन की दर ;अमतंहम तंजम वा बींदहमद्ध को व्यक्त करने के लिए, हमारे पास अवकलज की धारणा है जिसमें स्वतन्त्र चर में अनन्त अल्प वृद्धियों के संगत परतन्त्र चरों में भी अनन्त अल्प वृद्धियाँ होती है।

Differentiation is the technique of determining the derivatives of continuous functions and derivative is the limit of average rate of change in the dependent function following a change in the value of the variable. Very small changes in the value of independent variable is accompanied by a very small change in the value of dependent variable.

Example

1) Area of a circle depends upon its radius. If r is the radius, area is equal to πr^2 where π is a constant (= $\frac{22}{7}$) . So any change in the value of radius will result in a change in area.
2) Bill of a telephone call depends upon the duration of the call. Longer the time, greater will be the bill.
3) Total variable cost depends upon the number of units of a product.

In all these examples, the value of one variable, called dependent variable (Area, Bill, Cost) and represented by y depends upon the value of other variable called independent variable (radius, time, number of units) represented by x.

Mathematically we say that y is a function of x or

$$y = f(x).$$

The set of all permissible values of x is called **Domain** of the function and the set of corresponding values of y is called the **Range** of the function.

Derivative of a function –

ekuk y = f(x), x dk ,d lrar Qyu gSA ekuk x esa nh x;h vYi o`f) $\delta x$ rFkk y eas gksus okyh laxr vYi o`f) δy gSa] rks fHkUu $\frac{\delta y}{\delta x}$ dh lhek (vFkkZr o`f) vuqikr $\frac{\delta y}{\delta x}$) dks tc $\delta x \to 0$, y dk x ds lkis{k vody xq.kkad ;k vokyt (Differential coefficient or derivative) dgrs gSaA bl lhek dks ladsr $\frac{dy}{dx}$ ls fu:fir fd;k tkrk gS rFkk bls i<++++k tkrk gS "dy ckbZ dx vFkkZr dy by dx".

vr,o] ;fn y = f(x), rks

$$y + \delta y = f(x+\delta x)$$

$$\therefore \quad \delta y = f(x+\delta x) - y$$

;k $\delta y = f(x + \delta x) - f(x)$

nksuksa i{kksa dks δx ls Hkkx djus ij,

$$\frac{\delta y}{\delta x} = \frac{f(x+\delta x) - f(x)}{\delta x}$$

vr% vody xq.kkad dh ifjHkk"kk ls]

$$\frac{dy}{dx} = \lim_{\delta x \to 0} \frac{\delta y}{\delta x}$$

;k $$\frac{dy}{dx} = \lim_{\delta x \to 0} \frac{f(x+\delta h) - f(x)}{\delta x} \quad \ldots (1)$$

mijksDr (1) dks izFke fl)kUr ls Qyu dk vodyt (the derivative of a function from the first principles) ;k MsYVk fof/k (Delta method) dgk tkrk gSA

Theorem 1. The derivative of x^n is nx^{n-1} where n is fixed number, integer or rational.

Proof. Let $y = x^n$.

We have $y + \Delta y = (x + \Delta x)^n$.

$$\Rightarrow \quad \frac{\Delta y}{\Delta x} = \frac{(x+\Delta x)^n - x^n}{\Delta x}$$

As $\Delta x \to 0$, we obtain

$$\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{(x+\Delta x)^n - x^n}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{(x+\Delta x)^n - x^n}{(x+\Delta x)((x-\Delta x) - x}$$

Using $\lim_{x \to a} \frac{x^n - a^n}{x - a} = na^{n-1}$, we get

$$\frac{dy}{dx} = nx^{n-1}.$$

Hence $\frac{d}{dx}(x^n) = nx^{n-1}$, where n is an integer or rational.

Example 1. Find the derivatives of the functions

(i) x^{20} (ii) x^{-9} (iii) $x^{7/3}$ (iv) $x^{-2/3}$

Solution.

(i) Let $y = x^{20}$ $\therefore \frac{dy}{dx} = 20.\, x^{20-1} = 20.\, x^{19}$

(ii) Let $y = x^{-9}$ $\therefore \frac{dy}{dx} = (-9).\, x^{-9-1} = -9x^{-10}$

(iii) Let $y = x^{7/3}$ $\therefore \frac{dy}{dx} = \frac{7}{3}.\, x^{\frac{7}{3}-1} = \frac{7}{3} x^{4/3}$

(iv) Let $y = x^{-2/3}$ $\therefore \frac{dy}{dx} = -\frac{2}{3}.\, x^{-\frac{2}{3}-1} = -\frac{2}{3} x^{-\frac{5}{3}}$.

Theorem 2. The derivative of the constant function $f(x) = c$ is zero.

Proof. $\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}$

$$= \lim_{\Delta x \to 0} \frac{c - c}{\Delta x} = 0.$$

Hence, the derivative of a constant function is zero.

Theorem 3. The derivative of k . f(x) = k. $\frac{df}{dx}$.

Let y = kf (x) = kx^n

So $kf(x+h) = k(x+h)^n$

$$\therefore \frac{d}{dx} kf(x) = \lim_{h \to 0} k \left[\frac{f(x+h) - f(x)}{h} \right]$$

or

$$\frac{d}{dx}(kx^n) = \lim_{h \to 0} k \left[\frac{(x+h)^n - x^n}{h} \right]$$

$$= k.nx^{n-1} \quad \text{(From Theorem 1).}$$

$$\therefore \frac{dy}{dx}(kx^n)k = nx^{n-1}.$$

3. Derivative of Sum. The derivative of the sum of two functions is the sum of their derivative if these derivatives exist.

Solution. Let f and g be two differentiable functions and let f be the function defined by the relation

$$F(x) = f(x) + g(x)$$

$$\therefore \quad F'(x) = \lim_{h \to 0} \frac{F(x+h) - F(x)}{h}$$

$$= \lim_{h \to 0} \frac{[f(x+h) + g(x+h)] - [f(x) + g(x)]}{h}$$

$$= \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} + \lim_{h \to 0} \frac{g(x+h) - g(x)}{h}$$

$$= f'(x) + g'(x).$$

Hence $\frac{d}{dx}[f(x) + g(x)] = \frac{d}{dx}[f(x)] + \frac{d}{dx}[g(x)].$

Cor. 1. The above result can be extended to the sum of any number of finite functions.

Thus, $\frac{d}{dx}[f(x) + g(x) + h(x) + \ldots\ldots]$

$$= f'(x) + g'(x) + h'(x) + \ldots..$$

i.e., the derivative of the sum of any number of functions is equal to the sum of their derivatives provided these derivatives exist.

Cor. 2. If f(x) and g(x) are differentiable function, then f(x) −g(x) is also differentiable and $\frac{d}{dx}[f(x) - g(x)] = \frac{d}{dx} f(x) - \frac{d}{dx} g(x)$.

Example 2. Differentiate the following using first principle.

(i) $y = 3x^3$ (ii) $y = 2x^2 - 7x + 6$

(iii) $y = \frac{1}{x}$ (iv) $y = 2x^{-2}$

Solution.

(i) $y = 3x^3$ …(1)

Put (x + h) for x in equation (1)

$f(x+h) = 3(x+h)^3$

$$\therefore \quad \frac{dy}{dx} = \lim_{h\to 0}\frac{3(x+h)^3 - 3x^3}{h}$$

$$= \lim_{h\to 0}\frac{3[x^3 + 3x^2h + 3xh^2 + h^3 - x^3]}{h}$$

$$= \lim_{h\to 0}\frac{3h(3x^2 + 3xh - h^2)}{h}$$

$$= \lim_{h\to 0} 3(3x^2 - 3xh - h^2)$$

$$= 3(3x^2)$$

$$= 9x^2 \ .$$

(ii) Let $y=f(x) = 2x^2 - 7x+6$ ….(1)

put (x + h) for x in equation (1)

$f(x+h) = 2(x+h)^2 - 7(x+h) + 6$

$$\therefore \quad \frac{dy}{dx} = \lim_{h\to 0}\frac{[2(x+h)^2 - 7(x+h) + 6] - [2x^2 - 7x + 6]}{h}$$

$$= \lim_{h\to 0}\frac{[2x^2 + 2h^2 + 4xh - 7x - 7h + 6 - 2x^2 + 7x - 6]}{h}$$

$$= \lim_{h\to 0}\frac{(2h^2 + 4xh - 7h)}{h}$$

$$= 4x-7$$

(iii) Let $y = \frac{1}{x}$ and suppose that x increases by a very small amount δ x and y also increases by a very small amount δy. Therefore.

$$y + \delta y = \frac{1}{x + \delta x}$$

$$\therefore \quad \delta y = \frac{1}{x+\delta x} - \frac{1}{x} = \frac{(-\delta x)}{x(x+\delta x)}$$

$$\therefore \quad \frac{\delta y}{\delta x} = -\frac{1}{x^2 + x.\delta x}$$

$$\therefore \quad \frac{dy}{dx} = \lim_{\delta x\to 0} -\frac{1}{x^2 + x.\delta x}$$

$$= -\frac{1}{x^2}$$

(iv) Let $y = 2x^{-2} = \frac{2}{x^2}$

$$\therefore \quad y + \delta y = \frac{2}{(x+\delta x)^2} \quad \text{or} \quad \delta y = \frac{2}{(x+\delta x)^2} - \frac{2}{x^2}$$

$$= 2\left[\frac{x^2 - (x+\delta x)^2}{x^2(x+\delta x)^2}\right] = 2\frac{[-2x\delta x - (\delta x)^2]}{x^2(x+\delta x)}$$

$$\therefore \quad \frac{\delta y}{\delta x} = \frac{2[-2x - \delta x]}{x^2(x+\delta x)^2} \quad \text{or} \quad \frac{dy}{dx} = \lim_{\delta x \to 0}\left[\frac{2(-2x-\delta x)}{x^2(x-\delta x)^2}\right]$$

$$= -\frac{4x}{x^4} = -\frac{4}{x^3}$$

Example 3. Differentiate the following functions w.r.t. x

(i) $3x^4$ (ii) $\frac{x^5}{2}$ (iii) $\frac{7}{x}$ (iv) $\frac{1}{x^6}$

(v) $\frac{1}{4x^3}$ (vi) $3x - x^4$ (vii) $4x^3 - \frac{1}{x^2}$ (viii) $7x^{11} + 3x^4 - 4$

Solution. (i) $\frac{d}{dx}(3x^4) = 3\frac{d}{dx}(x^4) = 3.4x^3 = 12x^3$.

(ii) $\frac{d}{dx}\left(\frac{x^5}{2}\right) = \frac{1}{2}\frac{d}{dx}(x^5) = \frac{1}{2}.5x^4 = \frac{5}{2}x^4$

(iii) $\frac{d}{dx}\left(\frac{7}{x}\right) = 7\frac{d}{dx}\left(\frac{1}{x}\right) = 7\frac{d}{dx}(x^{-1}) = -7x^{-2}$

(iv) $\frac{d}{dx}\left(\frac{1}{x^6}\right) = \frac{d}{dx}(x^{-6}) = -6x^{-7}$

(v) $\frac{d}{dx}\left(\frac{1}{4x^3}\right) = \frac{1}{4}\frac{d}{dx}(x^{-3}) = -\frac{1}{4}.3x^{-4} = -\frac{3}{4}x^{-4}$

(vi) $\frac{d}{dx}(3x - x^4) = \frac{d}{dx}(3x) - \frac{d}{dx}(x^4) = 3 - 4x^3$

(vii) $\frac{d}{dx}\left(4x^3 - \frac{1}{x^2}\right) = \frac{d}{dx}(4x^3) - \frac{d}{dx}(x^{-2}) = 12x^2 + 2x^{-3}$

(viii) $\frac{d}{dx}(7x^{11} + 3x^4 - 4) = \frac{d}{dx}(7x^{11}) + \frac{d}{dx}(3x^4) - \frac{d}{dx}(4)$
$= 77x^{10} + 12x^3$

4. Differentiation of product of two functions

Let $y = f(x).\ \phi(x)$, then

$$\frac{d}{dx}[f(x).\phi(x)] = f(x).\frac{d}{dx}\phi(x) + \phi(x).\frac{d}{dx}f(x).$$

Let $y = f(x).\phi(x)$

Therefore, $y + \delta y = \{f(x+\delta x)\}.\{\phi(x+\delta x)\}$

or $\delta y = \{f(x+\delta x)\}.\{\phi(x+\delta x)\} - f(x).\phi(x)$ [Add and subtract $f(x+\delta x).\phi(x)$

$= f(x+\delta x)[\phi(x+\delta x) - \phi(x)] + \phi(x)[f(x+\delta x) - f(x)]$ subtract $f(x + \delta x).\phi(x)$

or $$\frac{\delta y}{\delta x} = f(x+\delta x).\frac{\phi(x+\delta x) - \phi(x)}{\delta x} + \phi(x).\frac{f(x+\delta x) - f(x)}{\delta x}$$

or $$\frac{dy}{dx} = f(x).\frac{d}{dx}\phi(x) + \phi(x).\frac{d}{dx}f(x).$$

So differentiation of product of two functions = First function × Derivative of second function + second function × Derivative of first function.

5. Differentiation of quotient of two functions

Let $y = \frac{f(x)}{\phi(x)}$. Then

$$\frac{dy}{dx} = \frac{d}{dx}\left[\frac{f(x)}{\phi(x)}\right] = \frac{\phi(x)\frac{d}{dx}f(x) - f(x)\frac{d}{dx}\phi(x)}{[\phi(x)]^2}$$

$$y = \frac{f(x)}{\phi(x)}$$

Therefore $$y+\delta y = \frac{f(x+\delta x)}{\phi(x+\delta x)} \quad \text{or} \quad \delta y = \frac{f(x+\delta x)}{\phi(x+\delta x)} - \frac{f(x)}{\phi(x)}$$

$$= \frac{f(x+\delta x).\phi(x) - f(x).\phi(x+\delta x)}{\phi(x+\delta x).\phi(x)}$$

or $$\frac{\delta y}{\delta x} = \frac{f(x+\delta x).\phi(x) - f(x).\phi(x+\delta x)}{\phi(x+\delta x).\phi(x).\delta x}$$

$$= \frac{\phi(x)\left[\frac{f(x+\delta x) - f(x)}{\delta x}\right] - f(x)\left[\frac{\phi(x+\delta x) - \phi x}{\delta x}\right]}{\phi(x).\phi(x+\delta x)} \quad \begin{bmatrix}\text{In Numerator} \\ \text{Add \& Subtract} \\ f(x).\phi(x)\end{bmatrix}$$

or $$\frac{dy}{dx} = \frac{\phi(x).\frac{d}{dx}f(x) - f(x)\frac{d}{dx}\phi(x)}{[\phi(x)]^2} \quad [\because \delta x \to 0]$$

i.e., the derivative of the quotient of two functions is the fraction having as its denominator, the square of the original denominator and as its numerator, the denominator times the derivative of the numerator minus the numerator times the derivative of the denominator, if these derivatives exits.

In words :

$$\text{Derivative of}\left(\frac{\text{Numerator}}{\text{Denominator}}\right)$$

$$= \frac{\text{Denom.} \times \text{Derivative of num.} - \text{Num.} \times \text{Derivative of denom.}}{(\text{Denom.})^2}$$

Example 4. Differentiate the following w.r.t. x

(i) $(2x^3 + 3)(3x^2 + 2x + 9)$ (ii) $(x+\frac{1}{x})(x^2 - \frac{1}{x^2})$

(iii) $(2x^3 - 3x^2 + 1)(3x^4 + 5x^3 + 2)$ (iv) $(2x+1)(3x^2 + 7x + 2)(5x^3 + 2x - 4)$

Solution. (i) Let $y = (2x^3+3)(3x^2+ 2x + 9)$

Then $\frac{dy}{dx} = \frac{d}{dx}(2x^3 + 3)(3x^2 + 2x+9)$

$= (2x^3 + 3)\frac{d}{dx}(3x^2 + 2x+9) + (3x^2+2x+9)\frac{d}{dx}(2x^3+3)$

$= (2x^3+3)(6x+2) + (3x^2+ 2x+9)(6x^2)$

$= 30x^4 + 16x^3 + 54x^2 + 18x+6$

(ii) Let $y = \left(x+\frac{1}{x^2}\right)\left(x-\frac{1}{x^2}\right)$

Then $\frac{dy}{dx} = \frac{d}{dx}\left(x+\frac{1}{x}\right)\left(x^2-\frac{1}{x^2}\right) = \left(x+\frac{1}{x}\right)\frac{d}{dx}\left(x^2-\frac{1}{x^2}\right) + \left(x^2-\frac{1}{x^2}\right)\frac{d}{dx}\left(x+\frac{1}{x}\right)$

$= \left(x+\frac{1}{x}\right)(2x+2x^{-3}) + \left(x^2-\frac{1}{x^2}\right)(1-x^{-2})$

$= \left(x+\frac{1}{x}\right)\left(2x+\frac{2}{x^3}\right) + \left(x^2-\frac{1}{x^2}\right)\left(1-\frac{1}{x^2}\right)$

(iii) Let $y = (2x^3 - 3x^2 + 1)(3x^4+5x^3+2)$

$\therefore \frac{dy}{dx} = (2x^3-3x^3+1).\frac{d}{dx}(3x^4+ 5x^3+2) + (3x^4+5x^3+2).\frac{d}{dx}(2x^3-3x^2+1)$

$= (2x^3-3x^2+1)(12x^3+15x^2) + (3x^4+5x^3+2)(6x^2-6x)$

$= (24x^6-36x^5+ 12x^3+ 30x^5-45x^4+ 15x^2) + (18x^6 +30x^5+12x^2-18x^5-30x^4-12x)$

$= 42x^6 + 6x^5-75x^4+12x^3+27x^2-12x.$

(iv) Let $y = (2x+1)(3x^2+7x+2)(x^3+2x-4)$

$\therefore \frac{dy}{dx} = (3x^2+7x+2)(5x^3+2x-4)\frac{d}{dx}(2x+1) + (2x+1)(5x^3+2x-4)\frac{d}{dx}(3x^2+7x+2)$

$+ (2x+1)(3x^2+7x+2)\frac{d}{dx}(5x^3+2x-4)$

$= (3x^2+7x+2)(5x^3+2x-4) : 2 + (2x+1)(5x^3+2x-4)(6x+7) + (2x+1)(3x^2+7x+2)(15x+2)$

$= 2(15x^5 + 6x^3 - 12x^2 + 35x^4 + 14x^2 - 28x + 10x^3 + 4x - 8) +$

$+ (2x+1)\left[(30x^4 + 35x^3 + 12x^2 + 14x - 24x - 28) + (45x^3 + 105x^2 + 30x + 6x^2 + 14x + 4)\right]$

$= 30x^5 + 70x^4 + 32x^3 + 4x^2 - 48x - 16 + (2x+1)(30x^4 + 80x^3 + 123x^2 + 34x - 24)$

$= 30x^5 + 70x^4 + 32x^3 + 4x^2 - 48x - 16 + 60x^5 + 210x^4 + 326x^3 + 191x^2 - 14x - 24$

$= 90x^5 + 280x^4 + 358x^3 + 195x^2 - 62x - 40$

Example 5. Differentiate the following w.r.t. x

(i) $\frac{3x^2+1}{x^3+2x}$ (ii) $\frac{5x+6}{x^{-2}}$ (iii) $\frac{2x^3+x+7}{3x^2+5}$

(iv) $\dfrac{\left(x^3-\dfrac{1}{x^3}\right)}{\left(x-\dfrac{1}{x}\right)}$ (v) $\dfrac{x^2+1}{x+2}$, $x \neq -2$.

Solution. (i) Let $y = \dfrac{3x^2+1}{x^3+2x}$

Then $\dfrac{dy}{dx} = \dfrac{(x^3+2x)\dfrac{d}{dx}(3x^2+1)-(3x^2+1)\dfrac{d}{dx}(x^3+2x)}{(x^3+2x)^2}$

$= \dfrac{(x^3+2x(6x)-(3x^2+1)(3x^2+2)}{(x^3+2x)^2}$

$= \dfrac{-3x^4+3x^2-2}{(x^3+2x)^2}$

(ii) Let $y = \dfrac{5x+6}{x^{-2}}$

Then $\dfrac{dy}{dx} = \dfrac{(x^{-2})\dfrac{d}{dx}(5x+6)-(5x+6)\dfrac{d}{dx}(x^{-2})}{(x^{-2})^2}$

$= \dfrac{5x^{-2}-(5x+6)(-2x^{-3})}{x^{-4}}$

$= \dfrac{5x^{-2}+2x^{-2}(5x+6)}{x^{-4}} = \dfrac{5x^{-2}+10x^{-2}+12x^{-3}}{x^{-4}}$

$= \dfrac{15x^{-2}+12x^{-3}}{x^{-4}} = \dfrac{x^{-2}(15+12x^{-1})}{x^{-4}}$

$= \dfrac{15+12x^{-1}}{x^{-2}}$

$= x^2\,(15+12x^{-1})$

$= 15x^2 + 12x$

(iii) Let $y = \dfrac{2x^3+x+7}{3x^2+5}$

Then $\dfrac{dy}{dx} = \dfrac{(3x^2+5)\dfrac{d}{dx}(2x^3+x+7)-(2x^3+x+7)\dfrac{d}{dx}(3x^2+5)}{(3x^2+5)^2}$

$= \dfrac{(3x^2+5)(6x^2+1)-(2x^3+x+7)(6x)}{(3x^2+5)^2}$

$= \dfrac{6x^4+27x^2-42x+5}{(3x^2+5)^2}$

(iv) Let $y = \dfrac{\left(x^3 - \frac{1}{x^3}\right)}{\left(x - \frac{1}{x}\right)}$

or $y = \dfrac{\left(x - \frac{1}{x}\right)\left(x^2 + 1 + \frac{1}{x^2}\right)}{\left(x - \frac{1}{x}\right)}$

$= x^2 + 1 + \dfrac{1}{x^2}$

$\therefore \dfrac{dy}{dx} = \dfrac{d}{dx}(x^2) + \dfrac{d}{dx}(i) + \dfrac{d}{dx}\left(\dfrac{1}{x^2}\right)$

$= 2x - \dfrac{2}{x^3}$

(v) Let $y = \dfrac{x^2+1}{x+2}$, $x \neq -2$.

Then $\dfrac{dy}{dx} = \dfrac{(x+2)\frac{d}{dx}(x^2+1) - (x^2+1)\frac{d}{dx}(x+2)}{(x+2)^3}$

$= \dfrac{(x+2)(2x) - (x^2+1).1}{(x+2)^2} = \dfrac{2x^2 + 4x - x^2 - 1}{(x+2)^2}$

$= \dfrac{x^2 + 4x - 1}{(x+2)^2}$.

6. Differentiation of function of a function (Chain rule)

If y is a function of u and u is a function of x, then y is called ‘a function of a function’ or ‘a composite function’.

If y = f(u) and u = g(x), then y is a function of x .

y = f{g(x)} = (fog)(x).

Theorem (Chain rule) **:** If f and g are differentiable functions, then fog is also differentiable, and

$(fog)'(x) = f'(g(x))g'(x).$

$= \lim_{h \to 0} \dfrac{f(u+k) - f(u)}{h}$

[where u = g(x) and u + k = g(x+h)
so that k = g(x+h) − g(x). As $h \to 0$, $k \to 0$]

$= \lim_{k \to 0} \left[\dfrac{f(u+k) - f(u)}{k} \times \dfrac{k}{h}\right]$

$= \lim_{k \to 0} \left[\dfrac{f(u+k) - f(u)}{k} \times \dfrac{g(x+h) - g(x)}{h}\right]$

$$= \lim_{k\to 0}\frac{f(u+k)-f(u)}{k}\times\lim_{h\to 0}\frac{g(x+h)-g(x)}{h}$$
$$= f'(u).g'(x) = f'(g(x)).g'(x).$$

Remember. This rule is called 'chain rule'.
It can be written as follows :
If y = f(u) and u = g(x)

Then $\frac{dy}{dx} = f'(u)g'(x) = \frac{dy}{du}\times\frac{du}{dx}$

i.e., $\frac{d}{dx}[f\{g(x)\}] = f'\{g(x)\}\times g'(x)$

This rule can be extended.
If y is a function of u, u is a function of v and v is a function of x, then

$$\frac{dy}{dx}=\frac{dy}{du}\times\frac{du}{dv}\times\frac{dv}{dx}.$$

Example 6. Differentiate the following
(i) $(3x^2+2x+4)^5$ (ii) $(2x^3-8)^{-3}$

Solution. (i) Let $y = (3x^2+2x+4)^5$
Put $3x^2+2x+4 = u$

$\therefore\quad y = u^5$

and $\frac{dy}{du} = 5.u^4.\frac{du}{dx}$

$= 5(3x^2+2x+4)^4.\ \frac{d}{dx}(3x^2+2x+4)$
$= 5(3x^2+2x+4)^4(6x+2)$
$= 10(3x+1)\ (3x^2+2x+4)^4$

(ii) Let $y = (2x^3-8)^{-3}$
Put $u = 2x^3-8$

$\therefore\quad y = u^{-3}$

and $\frac{dy}{du} = -3.u^{-4}.\frac{du}{dx}$

$= -3(2x^3-8)^{-4}.\ \frac{d}{dx}(2x^3-8)$
$= -3(2x^3-8)^{-4}.6x^2$
$= \frac{-18x^2}{(2x^3-8)^4}$

Example 7. Find $\frac{dy}{dx}$ for the following cases :

(i) $y = u^4$ and $u = 2x-1$ (ii) $y = 5u^8$ and $u = 4x + 3$
(ii) $y = 3u^3 + u$ and $u = 1-4x^2$ (iv) $y = \sqrt{(3x^2-4)}$

Solution.
(i) $y = u^4$ and $u = 2x-1$

$\therefore \quad \frac{dy}{du} = 4u^3$ and $\frac{du}{dx} = 2$

$\therefore \quad \frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$

$= 4u^3.2 = 8u^3$

$= 8(2x-1)^3$ [Putting $u = 2x - 1$]

(ii) Given $y = 5u^8$ and $u = 4x+3$

$\therefore \quad \frac{dy}{du} = 40u^7$ and $\frac{du}{dx} = 4$

$\therefore \quad \frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$

$= 40u^7.4 = 160u^7$

$= 160\,(4x+3)^7$ [Putting $u = 4x+3$]

(iii) Given $y = 3u^3 + u$ and $u = 1 - 4x^2$

$\therefore \quad \frac{dy}{du} = 9u^2 + 1$ and $\frac{du}{dx} = -8x$

$\therefore \quad \frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$

$= (9u^2+1)\,(-8x)$

$= -8x[9(1-4x^2)^2+1]$ [Putting $u = 1-4x^2$]

$= -8x(144x^4 - 72x^2 + 10)$

(iv) Let $u = 3x^2-4$, $\therefore y = \sqrt{u} = u^{1/2}$.

$\therefore \quad \frac{du}{dx} = 6x$ and $\frac{dy}{du} = \frac{1}{2}u^{-1/2}$

$\therefore \quad \frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$

$= (\frac{1}{2}u^{-1/2})(6x)$

$= 3x(3x^2-4)^{-1/2}$ [Putting $u = 3x^2-4$]

$= \frac{3x}{\sqrt{(3x^2-4)}}$

Example 8. Differentiate w.r.t. x

$$\frac{1}{\sqrt{(x^2+a^2)+\sqrt{(x^2+b^2)}}}$$

Solution. $y = \frac{1}{\sqrt{(x^2+a^2)}+\sqrt{(x^2+b^2)}}$

Rationalizing the denominator

$$y = \frac{\sqrt{(x^2+a^2)}-\sqrt{(x^2+b^2)}}{\sqrt{(x^2+a^2)}+\sqrt{(x^2+b^2)}} \cdot \frac{1}{\sqrt{(x^2+a^2)}-\sqrt{(x^2+b^2)}}$$

$$= \frac{\sqrt{(x^2+a^2)} - \sqrt{(x^2+b^2)}}{a^2-b^2}$$

$$\therefore \frac{dy}{dx} = \frac{1}{(a^2-b^2)}\frac{d}{dx}\left[\sqrt{(x^2+a^2)} - \sqrt{(x^2+b^2)}\right]$$

$$= \frac{1}{a^2-b^2}\left[\frac{2x}{2\sqrt{(x^2+a^2)}} - \frac{2x}{2\sqrt{(x^2+b^2)}}\right]$$

$$= \frac{x}{a^2-b^2}\left[\frac{1}{2\sqrt{(x^2+a^2)}} - \frac{1}{2\sqrt{(x^2+b^2)}}\right]$$

Example 9. If $y = \sqrt{\left[\frac{1-x}{1+x}\right]}$ prove that $(1-x^2)\frac{dy}{dx} + y = 0$.

Solution. $\frac{dy}{dx} = \frac{d}{dx}\sqrt{\left[\frac{1-x}{1+x}\right]}$

$$= \frac{\sqrt{(1+x)}\frac{d}{dx}\sqrt{(1-x)} - \sqrt{(1-x)}\frac{d}{dx}\sqrt{(1+x)}}{\left(\sqrt{(1+x)}\right)^2}$$

$$= \frac{\sqrt{(1+x)}[-\frac{1}{2}(1-x)^{-1/2}] - \sqrt{(1-x)}[\frac{1}{2}(1+x)^{-1/2}]}{\sqrt{(1+x)}}$$

$$= \frac{-\frac{\sqrt{(1+x)}}{2\sqrt{(1-x)}} - \frac{\sqrt{(1-x)}}{2\sqrt{(1+x)}}}{1+x}$$

$$= \frac{\frac{-1(1+x)-(1-x)}{2\sqrt{(1-x^2)}}}{1+x}$$

$$= \frac{-1-x-1+x}{2(1+x)\sqrt{(1-x^2)}} = \frac{-1}{(1+x)\sqrt{(1-x^2)}}$$

or $\frac{dy}{dx} = \frac{-\sqrt{(1-x^2)}}{(1+x)(1-x^2)}$

or $(1-x^2)\frac{dy}{dx} = \frac{-\sqrt{(1-x^2)}}{1+x}$

or $(1-x^2)\frac{dy}{dx} + \frac{\sqrt{(1-x^2)}}{1+x} = 0$

or $(1-x^2)\frac{dy}{dx} + \frac{\sqrt{(1+x)}\sqrt{(1-x)}}{1+x} = 0$

or $(1-x^2)\dfrac{dy}{dx}+\dfrac{\sqrt{(1-x)}}{\sqrt{1+x}}=0$

or $(-x^2)\dfrac{dy}{dx}+y=0$ $\quad\left[\because \dfrac{\sqrt{(1-x)}}{\sqrt{(1+x)}}=y\right]$

Exercise 1.1

(1) Find the derivatives of the following function using first principle :

(i) $y = 6x^3$ (ii) $y = ax^2 + bx + c$

(iii) $y = \dfrac{1}{\sqrt{x}}$ (iv) $y = x^{-3/2}$

(2) Differentiate the following w.r.t. x

(i) $\dfrac{1}{x^2}+\dfrac{1}{x}+1$ (ii) $3x^2 + 5x-1$

(iii) (x+a) (x+x) (x+c) (iv) $(3x^2+1)(x^3+2x)$

(v) $\dfrac{3x^2+5x}{7x+4}$ (vi) $\dfrac{(2x+1)(3x+1)}{4x+1}$

(vii) $\dfrac{5x^4-6x^2-7x+8}{5x-6}$ (viii) $(\sqrt{x}+2\sqrt[3]{x})(\sqrt[4]{x}-2\sqrt[5]{x})$

(ix) $(5x^3+6x^2+11x+7)^{11}$ (x) $\dfrac{1}{\sqrt[3]{2x^4+3x^3-5x+6}}$

(xi) $\dfrac{1}{\sqrt[3]{6x^5-7x^3+9}}$ (xi) $\dfrac{x^4+x^2+x}{x^2-x-1}$

(3) Calculate dy/dx for the following cases :

(i) $y = 3u^9$, $u = 2x-5$ (ii) $y = 4-3t^2$, $t = x^2-x+1$

(iii) $y = 1-\dfrac{1}{t}$, $t = \dfrac{1}{x^2}$ (iv) $y = 4u^2+\dfrac{1}{u}$, $u = 2x^2 + 1$

Differentiation of Logarithmic and Exponential Functions

Memorise the following formulae :-

1. $\lim\limits_{h\to 0}(1+h)^{1/h} = e$
2. $\log mn = \log m + \log n$
3. $\log\left(\dfrac{m}{n}\right) = \log m - \log n$
4. $\log m^n = n \log m$.
5. $\log_n m = \dfrac{\log m}{\log n}$ [Base changing formula]
6. $\log_e e = \log_a e = 1$.
7. $\log_a 1 = 0$ for $a \neq 0$.

Whenever no base is mentioned in the logarithmic function, it is understood to be the base e.

Caution. Log(m+n) ≠ log m+ log n.

Article 1. Derivative of Logarithmic Function

Let f be the function defined by the rule $f(x) = \log_a x$.

$$f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

$$= \lim_{h\to 0}\frac{1}{h}\,[\log_a (x+h) - \log_a x\,]$$

$$= \lim_{h\to 0}\frac{1}{h}\,\log_a\left(\frac{x+h}{x}\right)\quad [\because\ \log m - \log n = \log\frac{m}{n}\,]$$

$$= \lim_{h\to 0}\frac{1}{h}\log_a\left(1+\frac{h}{x}\right)$$

$$= \lim_{h\to 0}\frac{1}{h}.\frac{h}{x}.\log_a\left(1+\frac{h}{x}\right)$$

$$= \lim_{h\to 0}\frac{1}{h}.\frac{h}{x}\log_a\left(1+\frac{h}{x}\right)^{x/h}\qquad \text{[Note this step } \because n\log m = \log m^n]$$

$$= \lim_{h\to 0}\frac{1}{x}\log_a\left(1+\frac{h}{x}\right)^{x/h}$$

$$= \frac{1}{x}\lim_{h\to 0}\log_a\left(1+\frac{h}{x}\right)^{x/h}$$

$$= \frac{1}{x}\log_a e\qquad [\because\ \lim_{\theta\to 0}(1+\theta)^{1/\theta} = e\,]$$

Hence $\frac{d}{dx}(\log_a x) = \frac{1}{x}\log_a e$

Corollary. If a = e, then $\frac{d}{dx}(\log_a x) = \frac{d}{dx}$ $[\because \log_e e = 1]$

$$= \frac{1}{x}\log_e e = \frac{1}{x}.$$

Article 2. Derivatives of Exponential Functions

(a) Find the differential co-efficient of $f(x) = a^x\ (a > 0)$

(b) Find the differential co-efficient of $f(x) = e^x$.

Solution. (a) Here $f(x) = a^x$

$$\therefore\ f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(h)}{h} = \lim_{h\to 0}\frac{a^{x+h}-a^x}{h}$$

$$= \lim_{h\to 0}a^x\left(\frac{a^{x+h-x}-1}{h}\right) = \lim_{h\to 0}a^x\left(\frac{a^h-1}{h}\right)$$

$$= a^x \lim_{h\to 0} \frac{a^h - 1}{h} = a^x \log a \qquad \left[\because \operatorname{Lt}_{x\to 0} \frac{a^x - 1}{x} = \log a\right]$$

$$\therefore \quad \frac{d}{dx}(a^x) = a^x \log a \,;(a > 0)$$

(b) Here $f(x) = e^x$

Then $f(x+h) = e^{x+h} = e^x \cdot e^h$

$$\therefore \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h\to 0} \frac{e^x . e^h - e^x}{h}$$

$$= \lim_{h\to 0} e^x \left(\frac{e^h - 1}{h}\right) = e^x \lim_{h\to 0} \frac{e^h - 1}{h}$$

$$= e^x . 1 \qquad \left[\because \lim_{h\to 0} \frac{e^h - 1}{h} = 1\right]$$

$$= e^x$$

Hence $\frac{d}{dx}(e^x) = e^x$.

Example 10. Differentiate x log x w.r.t. x .

Solution. Let y = x log x

$$\therefore \frac{dy}{dx} = x \frac{d}{dx} (\log x) + \log x . \frac{d}{dx} (x)$$

$$= x \times \frac{1}{x} + \log x \times 1$$

$$= 1 + \log x$$

Example 11. Find $\frac{d}{dx}(2^x)$.

Solution. We know that $\frac{d}{dx}(a^x) = a^x . \log a$

$$\therefore \frac{d}{dx}(2^x) = 2^x . \log 2$$

Example 12. Differentiate log 5x w.r.t. x

Solution. Let 5x = u

So that y = log u and $\frac{du}{dx} = 5$.

We know that $\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$

Since y = log u

$$\therefore \frac{dy}{du} = \frac{1}{u}$$

Hence $\frac{dy}{dx} = \frac{1}{u} \times 5 = \frac{5}{u} = \frac{5}{5x} = \frac{1}{x}$

Logarithmic Differentiation

The process of taking logarithms before differentiation is called **Logarithmic Differentiation.** When the function consists of a single term which is a variable raised to a variable power, we first take logarithms and then differentiate.

Article. Differentiate u^v when u and v are both functions of x.

Solution. Let $y = u^v$.
Taking logarithms of both sides,

$$\log y = \log u^v = v \log u$$

Differentiating w.r.t. x

$$\frac{1}{y} \cdot \frac{dy}{dx} = v. \frac{1}{u} . \frac{du}{dx} + \log u . \frac{dv}{dx}$$

or
$$\frac{dy}{dx} = y\left[\frac{v}{u}.\frac{du}{dx} + \log u.\frac{dv}{dx}\right]$$

$$= u^v\left[\frac{v}{u}.\frac{du}{dx} + \log u.\frac{dv}{dx}\right]$$

Logarithmic differentiation can also be applied to a function which is again the product or quotient of two or more functions.

Example 13. Differentiate w.r.t. x

(i) x^x (ii) x^{x^x}

Solution. (i) Let $y = x^x$
Taking logarithms of both sides

$$\log y = x \log x$$

Differentiating w.r.t. x

$$\frac{1}{y} \cdot \frac{dy}{dx} = x. \frac{1}{x} + \log x. 1 = 1 + \log x$$

$$\therefore \quad \frac{dy}{dx} = y(1+\log x) = x^x (1+\log x).$$

(ii) Let $y = x^{x^x}$

$$\therefore \quad \log y = \log x^{x^x} = x^x \log x$$

Taking logarithms again

$$\log (\log y) = \log (x^x \log x) = \log x^x + \log (\log x)$$

or $\log (\log y) = x \log x + \log (\log x)$

Differentiating w.r.t. x

$$\frac{1}{\log y}.\frac{1}{y}.\frac{dy}{dx} = x \times \frac{1}{x} + \log x \times 1 + \frac{1}{\log x} \times \frac{1}{x}$$

$$= 1 + \log x + \frac{1}{x \log x}$$

or $\frac{dy}{dx} = y \log y \left(1 + \log x + \frac{1}{x \log x}\right)$

$= x^{(x^x)} . x^x \log x \left[1 + \log x + \frac{1}{x \log x}\right]$

$= x^{x^x} . x^x \left[\log x + (\log x)^2 + \frac{1}{x}\right]$

Example 14. If $x^y = y^x$, find $\frac{dy}{dx}$.

Solution. $x^y = y^x$

Taking logarithms, we get

$y \log x = x \log y$

Differentiating w.r.t. x ,

$$y \times \frac{1}{x} + \log x \times \frac{dy}{dx} = x \times \frac{1}{y} \times \frac{dy}{dx} + \log y \times 1$$

or $\left(\log x - \frac{x}{y}\right)\frac{dy}{dx} = \log y - \frac{y}{x}$

or $\left(\frac{y \log x - x}{y}\right)\frac{dy}{dx} = \frac{x \log y - y}{x}$

$\therefore$ $\frac{dy}{dx} = \frac{y(x \log y - y)}{x(y \log x - x)}$

Example 15. Differentiate e^{ax+b} w.r.t. x

Solution. Let $y = e^{ax+b}$

Put $ax + b = u$

$\therefore \frac{du}{dx} = a$

Now $y = e^u$

$\therefore \frac{dy}{dx} = e^u . \frac{du}{dx}$

$= e^{ax+b} . a$

Example 16. Differentiate $\log(1+x^2)$ w.r.t. x

Let $y = \log(1+x^2)$

and $u = 1+x^2$

$\therefore \frac{du}{dx} = 2x$

Now $y = \log u$

$\therefore \frac{dy}{dx} = \frac{1}{u} . \frac{du}{dx}$

$= \frac{1}{1+x^2} \times 2x = \frac{2x}{1+x^2}$

Example 17. Differentiate $\log(e^{mx} + e^{-mx})$ w.r.t. x

Let $y = \log(e^{mx} + e^{-mx})$

and $u = e^{mx} + e^{-mx}$

$\therefore \quad \frac{du}{dx} = e^{mx}.\ m + e^{-mx}\ (-m)$

Now $y = \log u$

$\therefore \quad \frac{dy}{dx} = \frac{1}{u}.\ \frac{du}{dx}$

$= \frac{1}{e^{mx} + e^{-mx}}\ [m.e^{mx} - m.e^{-mx}]$

Comment [JS1]: m

Differentiation of implicit functions

In such functions, y is not directly expressed in terms of x. Derivatives can be found by differentiating the given equation and then separating $\frac{dy}{dx}$.

Example 17. Find $\frac{dy}{dx}$ if $x^3 + y^3 = 3axy$.

Solution. $X^3 + y^3 = 3axy$

Differentiating with respect to x,

$$3x^2 + 3y^2.\frac{dy}{dx} = 3a\left[x.\frac{dy}{dx} + y\right]$$

$$= 3ax.\frac{dy}{dx} + 3ay$$

or $\quad 3y^2.\frac{dy}{dx} - 3ax.\frac{dy}{dx} = 3ay - 3x^2$

$$3.\frac{dy}{dx}(y^2 - ax) = 3(ay - 3x^2)$$

or $\quad \frac{dy}{dx} = \frac{ay - x^2}{y^2 - ax}$

Example 19. Find $\frac{dy}{dx}$ in terms of t when

$$y = \frac{2bt}{1+t^2} \text{ and } x = \frac{a(1-t^2)}{1+t^2}$$

Solution. $y = \frac{2bt}{1+t^2}$

$$\frac{dy}{dx} = \frac{(1+t^2).2b - 2bt.2t}{(1+t^2)^2}$$

$$= \frac{2b[1+t^2-2t^2]}{(1+t^2)^2} = \frac{2b(1-t^2)}{(1+t^2)^2}$$

and $x = \frac{a(1-t^2)}{1+t^2}$

$$\frac{dx}{dt} = \frac{(1+t^2).a(-2t) - a(1-t^2).2t}{(1+t^{2^2}}$$

$$= \frac{-2at - 2at^3 - 2at + 2at^3}{1+t^2)^2} = \frac{-4at}{(1+t^2)^2}$$

Now $\frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{dx} = \frac{2b(1-t^2)}{(1+t^2)^2} \times \frac{(1+t^2)^2}{-4at}$ - $\frac{b(1-t^2)}{2at}$.

Differentiation of a function w.r.t. another function

If f(x) and g(x) are two functions of x, then to find $\frac{dy}{dx}$ we put y = f(x) and z = g(x) and determine $\frac{dy}{dx}$ in the following way

$$\frac{dy}{dz} = \frac{\frac{dy}{dx}}{\frac{dz}{dx}}$$

Example 20. Differentiate $\frac{x^2}{1+x^2}$ w.r.t. x^2

Let $y = \frac{x^2}{1+x^2}$ and $z = x^2$

$$\therefore \frac{dy}{dx} = \frac{(1+x^2).2x - x^2(2x)}{(1+x^2)^2} = \frac{2x}{(1+x^2)^2}$$

and $\frac{dz}{dx} = 2x$

$$\therefore \frac{dy}{dz} = \frac{\frac{dy}{dx}}{\frac{dz}{dx}} = \frac{2x}{(1+x^2)} \times \frac{1}{2x} = \frac{1}{(1+x^2)^2}$$

Exercise 1.2

(1) If $y = x^y$ prove that $\frac{dy}{dx} = \frac{y^2}{x(1-y\log x)}$

(2) Differentiate the following w.r.t. x

(i) $e^x \log_e x$ (ii) $e^{\log(x+\sqrt{x^2+a^2}}$ (iii) $\log(\sqrt{x+a} + \sqrt{x+b})$

(iv) $\log \frac{\sqrt{1-x^2}}{x}$ (v) x^x (vi) $x^x + x^{\frac{1}{x}}$

(3) Find $\frac{dy}{dx}$ if $x^3 - xy^2 + 3y^2 + 2 = 0$

(4) If $y = x^{x^{x...\infty}}$, prove that $x.\frac{dy}{dx} = \frac{y^2}{1-y\log x}$

(5) Find $\frac{dy}{dx}$ if $(x+y)^{m+n} = x^m.y^n$

(6) Differentiate :

(i) x^2 w.r.t x^3 (ii) $\log x$ w.r.t. $\frac{1}{x}$

(iii) $\sqrt{x^3+1}$ w.r.t. $\sqrt[3]{x^2+1}$

(7) Differentiate w.r.t. x $\log\left[e^h\left(\frac{x-2}{x+3}\right)^{3/4}\right]$

Answers

Exercise 1.1

1. (i) $18x^2$ (ii) $2ax+b$ (iii) $-\frac{1}{2}.x^{-3/2}$ (iv) $-\frac{3}{2}.x^{-5/2}$

2. (i) $-\frac{2}{x^3}-\frac{1}{x^2}$ (ii) $6x+5$ (iii) $3x^2+2ax+2bx+2cx+ab+bc+ca$

(iv) $15x^4+21x^2+2$ (v) $\frac{21x^2+24x+20}{(7x+4)^2}$ (vi) $\frac{24x^2+12x+1}{(4x+1)^2}$

(vii) $\frac{75x^4-120x^3-30x^2+72x+2}{(5x-6)^2}$

(viii) $(x^{1/2}+2x^{1/3})\left(\frac{1}{4}x^{-3/4}-\frac{2}{5}x^{-4/5}\right)+(x^4-2x^{1/5})\left(\frac{1}{2}x^{-1/2}+\frac{2}{3}x^{-2/3}\right)$

(ix) $11(5x^3+6x^2+11x+7)^{10}(15x^2+12x+1)$

(x) $-\frac{8x^3+9x^2-5}{3.\sqrt[3]{(2x^4+3x^3-5x+6)^4}}$ (xi) $-\frac{1}{3}(6x^5-7x^3+9)^{-4/3}.30x^4-21x^2)$

(xii) $\frac{6x^5-5x^4-4x^3-4x^2-4x-1}{(x^2-x-1)^2}$

3. (i) $54(2x-5)^8$ (ii) $-6(x^2-x+1)(2x-1)$ (iii) $\frac{-2}{x^7}$

(iv) $84x\,[16(2x^2+1)^3-\frac{1}{(2x^2+1)^2}]$

Exercise 1.2

2. (i) $e^x \left(\frac{1}{x} + \log_e x\right)$ (ii) $1 + \frac{x}{\sqrt{x^2 + a^2}}$ (iii) $\frac{1}{2\sqrt{x-a}\sqrt{x-b}}$

(iv) $-\frac{1}{x(1-x^2)}$ (v) $x^x\,(1+\log x)$ (vi) $x^x\,(1+\log x) + x^{\frac{1}{x}}\left(\frac{1-\log x}{x^2}\right)$

3. $\frac{y^2 - 3x^2}{2y(3-x)}$

5. $\frac{y}{x}$

6. (i) $\frac{2}{3x}$ (ii) $-x$ (iii) $\frac{9x(x^2+1)^{2/3}}{4\sqrt{x^3+1}}$

7. $1 + \frac{3}{4(x-2)} - \frac{3}{4(x+3)}$

अध्याय .2

आंशिक अवकलन
(Partial Differentiation)

पिछले अध्याय में हमने एक चर के फलत का अवकलन किया था। यानि ल त्र ग़िद्ध की तरह के फलतों का अवकलन निकाला था। उसमें ल सिर्फ एक स्वतन्त्र चर ग का फलत था। लेकिन कई बार ऐसी स्थितियाँ भी आती हैं जब एक आश्रित चर कई स्वतन्त्र चरों का फलत है। ऐसी स्थितियों में हम आंशिक अवकलन निकालते हैं।

In previous chapter, we differentiated functions involving one variable only i.e. we differentiated function of the form y = f(x). In this y is a function of one independent variable x only. However many times there are situations when one dependent variable is a function of many independent variables. In such situations, we determine partial differentiation.
For example, demand for a product not only depends upon its price but also on the income of the individuals and price of the related goods. Consider another example. The volume V of a right circular cylinder is a function of its radius r, and height h.
Mathematically $V = \pi r^2 h$
Let height remains constant while radius changes. Since value of h does not change it can be considered a constant, first like π and differentiating V w.r.t r, we have

$$\frac{d}{dr}(V)_{\text{h constant}} = (\pi h).2r$$

This derivative gives the rate of change of v w.r.t r when h remains constant.
Similarly when r is constant and h varies then

$$\left(\frac{dv}{dh}\right)_{\text{(r constant)}} = (\pi r^2).1 = \pi r^2$$

This derivative gives the rate of change of V w.r.t. h when r remains constant.

This type of differentiation, when in one situation, only one independent variable changes while others remain constant, is called partial differentiation. It is denoted by

$$\frac{\partial u}{\partial x} \text{ or } \frac{\partial f}{\partial x} \text{ or } f_x$$

where u = f (x, y)
Similarly if x is held constant and y changes then we find out

$$\frac{\partial u}{\partial y} \text{ or } \frac{\partial f}{\partial y} \text{ or } f_y$$

$$\frac{\partial u}{\partial x} = \lim_{\delta x \to 0}\left[\frac{f(x+\delta x, y) - f(x, y)}{\delta x}\right] \text{ is the partial derivative of u w.r.t. x}$$

and $\frac{\partial u}{\partial y} = \lim_{\delta y \to 0}\left[\frac{f(x, y+\delta y) - f(x, y)}{\delta y}\right]$ is the partial derivative of u w.r.t y

Rules of partial differentiation

1. If $z = u \pm v$ where u, v are functions of x and y then

$$\frac{\partial z}{\partial x} = \frac{\partial u}{\partial x} \pm \frac{\partial v}{\partial x} \text{ and } \frac{\partial z}{\partial y} = \frac{\partial u}{\partial y} \pm \frac{\partial v}{\partial y}$$

2. If z = u v then

$$\frac{\partial z}{\partial x} = u.\frac{\partial v}{\partial x} + v.\frac{\partial u}{\partial x} \text{ and } \frac{\partial z}{\partial y} = u\frac{\partial v}{\partial y} + v\frac{\partial u}{\partial y}$$

3. If $z = \frac{u}{v}$ then

$$\frac{\partial z}{\partial x} = \frac{v\frac{\partial u}{\partial x} - u\frac{\partial v}{\partial x}}{v^2} \text{ and } \frac{\partial z}{\partial y} = \frac{v\frac{\partial u}{\partial y} - u\frac{\partial v}{\partial y}}{v^2}$$

4. If z = f(u) where u is a function of x and y, then

$$\frac{\partial z}{\partial x} = \frac{\partial z}{\partial u}.\frac{\partial u}{\partial x} \text{ and } \frac{\partial z}{\partial y} = \frac{dz}{du}.\frac{\partial u}{\partial y}$$

Example 1. Find the first order partial derivative of $x^3 + 6x^2y + 4xy^2 + y^3$

Solution Let $z = f(x, y) = x^3 + 6x^2 y + 4xy^2 + y^3$

$$\therefore \frac{\partial z}{\partial x} = 3x^2 + 12\,xy + 4y^2$$

and $\frac{\partial z}{\partial y} = 0 + 6x^2 + 8xy + 3y^2$

$= 6x^2 + 8xy + 3y^2$

Example 2. If $u = e^{xy}$, find $\frac{\partial u}{\partial x}$ and $\frac{\partial u}{\partial y}$

Solution $\frac{\partial u}{\partial x} = \frac{\partial}{\partial x}(e^{xy}) = e^{xy}.\ y = y.e^{xy}$

and $\frac{\partial u}{\partial y} = \frac{\partial}{\partial y}(e^{xy}) = e^{xy}.\ x = x.e^{xy}$

Example 3. If $u = x^2 y^3 z^4 + 6x + 7y + 9z$, find $\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial u}{\partial z}$

Solution $\frac{\partial u}{\partial x} = 2x.\ y^3z^4 + 6$

$\frac{\partial u}{\partial y} = x^2.\ 3y^2.\ z^4 + 7$

$\frac{\partial u}{\partial z} = x^2\ y^3.\ 4z^3 + 9$

nqljs Øe ds vakf'kd vodyt
(Partial derivatives of second order)

The derivatives, discussed above, are of first order. From them we can obtain second order derivatives.

For the function u = f(x, y), there are four second order derivatives

(i) $\frac{\partial^2 u}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right) = f_{xx}$

(ii) $\frac{\partial^2 u}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial y}\right) = f_{yy}$

(iii) $\frac{\partial^2 u}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial y}\right) = f_{yx}$

(iv) $\frac{\partial^2 u}{\partial y \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial x}\right) = f_{xy}$

Example 4. Find second order partial derivatives of

$u = 4x^2 + 9xy - 5y^2$

Solution. First we find first order derivatives

$$\frac{\partial u}{\partial x} = 8x + 9y$$

and $$\frac{\partial u}{\partial y} = 9x - 10y$$

Now $$\frac{\partial^2 u}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right) = \frac{\partial}{\partial x}(8x + 9y) = 8 \text{ (y is a constant)}$$

$$\frac{\partial^2 u}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial y}\right) = \frac{\partial}{\partial y}(9x - 10y) = -10 \text{ (x is a constant)}$$

$$\frac{\partial^2 u}{\partial x.\partial y} = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial y}\right) = \frac{\partial}{\partial x}(9x - 10y) = 9 \text{ (y is a constant)}$$

and $$\frac{\partial^2 u}{\partial y.\partial x} = \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial x}\right) = \frac{\partial}{\partial y}(8x + 9y) = 9 \text{ (x is a constant)}$$

Example 5. Find $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ when $z = \frac{x^2}{x - y + 1}$

Solution. $z = \frac{x^2}{x - y + 1}$

So $$\frac{\partial z}{\partial x} = \frac{(x - y + 1)\frac{\partial}{\partial x}(x^2) - x^2 \frac{\partial}{\partial x}(x - y + 1)}{(x - y + 1)^2}$$

$$= \frac{2x(x - y + 1) - x^2}{(x - y + 1)^2}$$

$$= \frac{2x^2 - 2xy + 2x - x^2}{(x - y + 1)^2} = \frac{x^2 - 2xy + 2x}{(x - y + 1)^2}$$

and $$\frac{\partial z}{\partial y} = \frac{(x-y+1)\frac{\partial}{\partial y}(x^2) - x^2\frac{\partial}{\partial y}(x-y+1)}{(x-y+1)^2}$$

$$= \frac{0 - x^2(-1)}{(x-y+1)^2} = \frac{x^2}{(x-y+1)^2}$$

Example 6. If $z = \log(x^2 - y^2)$ show that

$$\frac{\partial^2 z}{\partial x \partial y} = \frac{\partial^2 z}{\partial y \partial x}$$

Solution. $z = \log(x^2 - y^2)$

$$\frac{\partial z}{\partial x} = \frac{1}{x^2 - y^2} 2x = \frac{2x}{x^2 - y^2}$$

$$\frac{\partial z}{\partial y} = \frac{1}{x^2 - y^2}(-2y) = -\frac{2y}{x^2 - y^2}$$

Now $$\frac{\partial^2 z}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right) = \frac{\partial}{\partial x}\left(-\frac{2y}{x^2 - y^2}\right) = \frac{\partial}{\partial x}(-2y(x^2 - y^2)^{-1})$$

$$= (-2y)(-1)(x^2 - y^2)^{-2}(2x)$$

$$= \frac{4xy}{(x^2 - y^2)^2}$$

and $$\frac{\partial^2 z}{\partial y \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial x}\right) = \frac{\partial}{\partial y}\left(\frac{2x}{x^2 - y^2}\right) = 2x(x^2 - y^2)^{-1}$$

$$= 2x(-1)(x^2 - y^2)^{-2}(-2y)$$

$$= \frac{4xy}{(x^2 - y^2)^2}$$

Hence the result

Example 7. If $f(x, y, z) = x^3 + y^3 + z^3 - 3xyz$ show that

$$x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} + z\frac{\partial f}{\partial z} = 3f(x, y, z)$$

Solution. $f(x,y,z) = x^3 + y^3 + z^3 - 3xyz$

$$\therefore \quad \frac{\partial f}{\partial x} = 3x^2 - 3yz$$

$$\frac{\partial f}{\partial y} = 3y^2 - 3xz$$

$$\frac{\partial f}{\partial z} = 3z^2 - 3xy$$

Now $$x.\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} + z.\frac{\partial f}{\partial z} = x(3x^2 - 3yz) + y(3y^2 - 3xz) + z(3z^2 - 3xy)$$

$$= 3x^3 - 3xyz + 3y^3 - 3xyz + 3z^3 - 3xyz$$
$$= 3x^3 + 3y^3 + 3z^3 - 9xyz$$
$$= 3(x^3 + y^3 + z^3 - 3xyz)$$
$$= 3\, f(x, y, z)$$

Hence the result.

Homogeneous functions

A function f(x, y) is said to be a homogeneous function of order n if sum of powers of x and y in each term is equal to n.

In general, it is represented as

$$a_0x^n + a_1\, x^{n-1}y + a_2\, x^{n-2}\, y^2 + \ldots + a_{n-1}\, x.\, y^{n-1} + a_n\, y^n$$
$$x^n[a_0 + a_1 \left(\frac{y}{x}\right) + a_2\left(\frac{y}{x}\right)^2 + \ldots + a_{n-1} \left(\frac{y}{x}\right)^{n-1} + a_n\left(\frac{y}{x}\right)^n\,]$$

In other words, we can say that any function which can be expressed in the form $x^n\, f\left(\frac{y}{x}\right)$ is a homogeneous function of nth order.

For example the function $x^3 + 3x^2y - xy^2 - y^3$ is homogeneous function of degree 3 and the functions $\frac{x^3+y^3}{x-y}$ and $\frac{x^3+y^3}{x^2+y^2}$ one of degree 2 and 1 respectively.

Now $$\frac{x^3+y^3}{x-y} = \frac{x^3\left[1+\left(\frac{y}{x}\right)^3\right]}{x\left[1-\frac{y}{x}\right]} = \frac{x^2\left[1+\left(\frac{y}{x}\right)^3\right]}{\left(1-\frac{y}{x}\right)} \qquad \text{(Degree 2)}$$

and $$\frac{x^3+y^3}{x^2+y^2} = \frac{x^3\left[1+\left(\frac{y}{x}\right)^3\right]}{x^2\left[1+\left(\frac{y}{x}\right)^2\right]} = x\,\frac{\left[1+\left(\frac{y}{x}\right)^3\right]}{\left[1+\left(\frac{y}{x}\right)^2\right]} \qquad \text{(Degree 1)}$$

Theorem. If u is a homogeneous function of degree n in x and y, $\frac{\partial u}{\partial x}$ and $\frac{\partial u}{\partial y}$ are also homogeneous functions of degree (n–1) each in x and y.

Proof. Now $u = x^n\, f\left(\frac{y}{x}\right)$

$$\therefore \qquad \frac{\partial u}{\partial x} = n.\, x^{n-1}\, f\left(\frac{y}{x}\right) + x^n.\; f'\left(\frac{y}{x}\right).\, \left(\frac{-y}{x^2}\right)$$
$$= x^{n-1}\left[n.f\left(\frac{y}{x}\right) - \frac{y}{x} f'\left(\frac{y}{x}\right)\right]$$

$$= x^{n-1} g\left(\frac{y}{x}\right) \text{(say)}$$

which is homogeneous function of degree (n−1) in x and y

Also $\frac{\partial u}{\partial y} = x^n.f'\left(\frac{y}{n}\right).\frac{1}{x} = x^{n-1}f'\left(\frac{y}{x}\right)$

$$= x^{n-1} h\left(\frac{y}{x}\right) \text{(say)}$$

which is also a homogeneous function of degree n−1 in x and y.
Hence the result

Euler's Theorem on Homogeneous functions

If f(x, y) is a homogeneous function of degree n in x and y then

$$x.\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} = n.f$$

Proof. f (x, y) being a homogeneous function of degree n in x and y can be written as

$$f(x, y) = x^n f\left(\frac{y}{x}\right)$$

$\therefore$ $$\frac{\partial f}{\partial x} = n.\, x^{n-1}.\, f\left(\frac{y}{x}\right) + x^n.\, f'\left(\frac{y}{x}\right)\left(\frac{-y}{x^2}\right)$$

or

$$x.\frac{\partial f}{\partial x} = nx^n f\left(\frac{y}{x}\right) - yx^{n-1} f'\left(\frac{y}{x}\right) \qquad \ldots(1)$$

and $$\frac{\partial f}{\partial y} = x^n.\, f'\left(\frac{y}{x}\right).\, \frac{1}{x} = x^{n-1} f'\left(\frac{y}{x}\right)$$

or

$$y\frac{\partial f}{\partial y} = y\, x^{n-1} f'\left(\frac{y}{x}\right) \qquad \ldots(2)$$

Substituting the value of y.$\frac{\partial f}{\partial y}$ in equation (1) we get

$$x.\frac{\partial f}{\partial x} = n.\, x^n f\left(\frac{y}{x}\right) - y\frac{\partial f}{\partial y}$$

or

$$x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} = n.\, x^n f(y/x)$$

$$= n.\, f(x, y) \left[\because f(x, y) = x^n .f\left(\frac{y}{x}\right)\right.$$

Hence proved

Example 8. Show that $x^4 + x^3y + x^2y^2 + xy^3 + y^4$ is a homogeneous function in x and y.

Solution. Let $f(x, y) = x^4 + x^3 y + x^2 y^2 + xy^3 + y^4$

$$= x^4\left[1+\left(\frac{y}{x}\right)+\left(\frac{y^2}{x^2}\right)+\left(\frac{y^3}{x^3}\right)+\left(\frac{y^4}{x^4}\right)\right]$$

$$= x^4\left[1+\left(\frac{y}{x}\right)+\left(\frac{y}{x}\right)^2+\left(\frac{y}{x}\right)^3+\left(\frac{y}{x}\right)^4\right]$$

which shows that $x^4 + x^3 y + x^2y^2 + xy^3 + y^4$ is a homogeneous function in x and y of degree 4.

Example 9. Verify Euler's theorem for the function

$$u = \frac{x^4+y^4}{x+y}$$

Solution. $u = f(x, y) = \dfrac{x^4+y^4}{x+y}$

$$\therefore \qquad \frac{\partial u}{\partial x} = \frac{(x+y)\frac{\partial}{\partial x}(x^4+y^4)-(x^4+y^4)\frac{\partial}{\partial x}(x+y)}{(x+y)^2}$$

$$= \frac{4x^3(x+y)-(x^4+y^4)}{(x+y)^2}$$

$$= \frac{3x^4+4x^3y-y^4}{(x+y)^2}$$

Similarly we can find that

$$\frac{\partial u}{\partial y} = \frac{3y^4+4xy^3-x^4}{(x+y)^2}$$

Then

$$x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y} = \frac{3x^5+4x^4y-xy^4+3y^5+4xy^4-yx^4}{(x+y)^2}$$

$$= \frac{3x^5+3x^4y+3xy^4+3y^5}{(x+y)^2}$$

$$= \frac{3x^4(x+y)+3y^4(x+y)}{(x+y)^2}$$

$$= \frac{3(x^4+y^4)(x+y)}{(x+y)^2} = \frac{3(x^4+y^4)}{(x+y)}$$

$= \mathbf{3f}$

Thus Euler's Theorem is verified

Example 10. If $z = xy\, f(x/y)$, show that

$$x\frac{\partial z}{\partial x}+y\frac{\partial z}{\partial y} = 2z.$$

Solution. $z = F(x, y) = xy\, f(x/y)$

$$F(\lambda x, \lambda y) = \lambda x\ \lambda y\ f\left(\frac{\lambda x}{\lambda y}\right)$$
$$= \lambda^2\ xy\ f(x/y)$$
$$= \lambda^2\ F(x, y)$$

Hence z is a homogeneous function in x and y, of degree 2

By Euler's Theorem therefore,

$$x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = 2z.$$

Example. 11. If $u = \log \dfrac{x^2+y^2}{x+y}$

Prove that $x\dfrac{\partial u}{\partial x} + y\dfrac{\partial u}{\partial y} = 1.$

Solution. Here $u(x, y) = \log \dfrac{x^2+y^2}{x+y}$

$$\Rightarrow \qquad u(\lambda x, \lambda y) = \log\left(\frac{\lambda^2x^2+\lambda^2y^2}{\lambda x+\lambda y}\right)$$
$$= \log \frac{\lambda(x^2+y^2)}{(x+y)}$$

This shows that u (x, y) is not a homogeneous function of x and y.

However, if we put $z = e^u = \dfrac{x^2+y^2}{x+y}$

then clearly z is a homogeneous function of x and y, of degree 1.

So by Euler's Theorem

$$x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = z$$

or $$x\ \frac{\partial u}{\partial x}\frac{\partial Z}{\partial u} + y\frac{\partial u}{\partial y}\frac{\partial Z}{\partial u} = z$$

But $$\frac{dZ}{du} = e^u = z$$

So we get $$zx\frac{\partial u}{\partial x} + zy\frac{\partial u}{\partial y} = z$$

or $$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 1$$

Exercise 2.1

1. Compute $\dfrac{\partial z}{\partial x}$ and $\dfrac{\partial z}{\partial y}$ for functions

   (i) $z = (x+y)^2$ (ii) $z = \log(x+y)$

(iii) $z = \dfrac{x}{x^2 + y^2}$ (iv) $z = e^{x^y}$

2. If $v = r^m$ where $r^2 = x^2 + y^2 + z^2$, show that

$$\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial z^2} = m(m+1)\, r^{m-2}$$

3. If $u = f\,(ax^2 + 2hxy+by^2)$, $v = \phi\,(ax^2 + 2hxy + by^2)$

prove that $\dfrac{\partial}{\partial y}\left(u\dfrac{\partial v}{\partial x}\right) = \dfrac{\partial}{\partial x}\left(u\dfrac{\partial v}{\partial y}\right)$

4. If $u\,(1 - 2x\nu + y^2)^{-1/2}$ prove that

$$\frac{\partial}{\partial x}\left[(1-x^2)\frac{\partial u}{\partial x}\right] + \frac{\partial}{\partial y}\left(y^2\frac{\partial u}{\partial y}\right) = 0$$

5. If $u = \dfrac{1}{\sqrt{x^2 + y^2 + z^2}}$, show that

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 0.$$

6. Verify Euler's Theorem for the functions

(i) $z = (x^2 + xy + y^2)^{-1}$

(ii) $z = \dfrac{x^{3/4} + y^{3/4}}{x^{3/5} + y^{3/5}}$

(iii) $z = x^{-3} \log \dfrac{y}{x}$

7. Prove that $z = x\, f(y/x) + g(y/x)$

satisfies the relation $x^2\dfrac{\partial^2 z}{\partial x^2} + 2xy\dfrac{\partial^2 z}{\partial x \partial y} + y^2\dfrac{\partial^2 z}{\partial y^2} = 0,$

8. If u is homogeneous function of degree n in x and y, prove that

$$x^2\frac{\partial^2 z}{\partial x^2} + 2xy\frac{\partial^2 z}{\partial x \partial y} + y^2\frac{\partial^2 z}{\partial y^2} = n(n-1)z$$

9. If $u = \sqrt{x^2 + y^2 + z^2}$ prove that $\left(\dfrac{\partial u}{\partial x}\right)^2 + \left(\dfrac{\partial u}{\partial y}\right)^2 + \left(\dfrac{\partial u}{\partial z}\right)^2 = 1$

10. If $u = \log \dfrac{x^4 + y^4 + x^2y^2}{x + y + \sqrt{xy}}$ prove that $x\dfrac{\partial u}{\partial x} + y\dfrac{\partial u}{\partial y} = 3$

Total Differentials

Partial differentiation tells us about the change in value of dependent variable (u) when there is a change in the values of one of two independent variable (x, y).

Total differential tells us about the change in the value of u when values of both x and y change.

Let $u = f(x, y)$ …(1)

Further assume δx, δy and δu be the small changes in the values of x, y and u respectively.

So $u + \delta u = f\,(x + \delta x, y + \delta y)$ …(2)

Subtracting (1) from (2)

$$\delta u = f(x+\delta x,\ y+\delta y) - f(x,\ y)$$

or $$\delta u = f(x+\delta x,\ y+\delta y) - f(x,\ y+\delta y) + f(x,\ y+\delta y) - f(x,y)$$

$$= \frac{f(x+\delta x, y+\delta y) - f(x, y+\delta y)}{\delta x} \times \delta x$$

$$+ \frac{f(x, y+\delta y) - f(x, y)}{\delta y} \times \delta y$$

Let du, dx, dy be the limiting values of δu, δx and δy respectively

$$du = \frac{\partial u}{\partial x}.dx + \frac{\partial u}{\partial y}.dy \ ,$$

du is called the total differential of u.

Example 12. Find the total differential of the following functions

(i) $u = 5x^2 - 2y^2 + 3xy$ (ii) $u = \dfrac{x^2 - y^2}{x^2 + y^2}$

Solution. (i) $u = 5x^2 - 2y^2 + 3xy$

$$\frac{\partial u}{\partial x} = 10x + 3y$$

$$\frac{\partial u}{\partial y} = -4y + 3x$$

$\therefore$ $$du = \frac{\partial u}{\partial x}.dx + \frac{\partial u}{\partial y}.dy$$

$$= (10x+3y)\ dx + (3x - 4y)\ dy$$

(ii) $$u = \frac{x^2 - y^2}{x^2 + y^2}$$

$$\frac{\partial u}{\partial x} = \frac{(x^2+y^2).2x - (x^2-y^2)2x}{(x^2+y^2)^2} = \frac{4xy^2}{(x^2+y^2)^2}$$

$$\frac{\partial u}{\partial y} = \frac{(x^2+y^2).(-2y) - (x^2-y^2)(2y)}{(x^2+y^2)^2} = \frac{-4x^2y}{(x^2+y^2)^2}$$

$\therefore$ $$du = \frac{4xy^2}{(x^2+y^2)^2}dx - \frac{4x^2y}{(x^2+y^2)^2}.dy$$

Composite functions

In the relationship u = f(x, y), u is a function of x and y. But if both x and y are functions of another variable (say t) then u is called the composite function of t.

Differentiation of a composite function

Let $u = f(x, y)$, $x = \phi(t)$ and $y = \psi(t)$

Then $$\frac{du}{dt} = \frac{\partial u}{\partial x}.\frac{\partial x}{\partial t} + \frac{\partial u}{\partial y}.\frac{dy}{dt}$$

Here $\frac{du}{dt}$ is the total derivative of u w.r.t. t

If u = f(x, y) and y = $\phi(x)$ then u is a composite function of x. In this case

$$\frac{du}{dx} = \frac{\partial u}{\partial x}.\frac{\partial x}{\partial x} + \frac{\partial u}{\partial y}.\frac{dy}{dx}$$

$$= \frac{\partial u}{\partial x} + \frac{\partial u}{\partial y}.\frac{dy}{dx}$$

Example 13. Find the total derivative $\frac{du}{dt}$ if

$$u = x^2 + y^2, x = at^2, y = 2at$$

Solution. $u = x^2 + y^2$

$\therefore$ $\frac{\partial u}{\partial x} = 2x$ and $\frac{\partial u}{\partial y} = 2y$

Also $x = at^2$ and $y = 2\ at$

So $\frac{dx}{dt} = 2\ at$ and $\frac{dy}{dt} = 2a$

So substituting these values in the formula

$$\frac{du}{dt} = \frac{\partial u}{\partial x}.\frac{dx}{dt} + \frac{\partial u}{\partial y}.\frac{dy}{dt}$$

we get $\frac{du}{dt} = 2\ x \times 2\ at + 2y \times 2a$

$= 4\ x\ at + 4ya$

$= 4\ at.\ at^2 + 4a.\ 2at$ $[\because x = at^2, y = 2at]$

$= 4\ a^2\ t^3 + 8a^2\ t$

$= 4a^2t\ (t^2 + 2)$

Example 14. Find the differential coefficient of $x^2\ y$ w.r.t. x when x and y are connected by the relation $x^2 + xy + y^2 = 1$

Solution. Let $u = x^2y$

$\therefore$ $\frac{\partial u}{\partial x} = 2\ xy$ and $\frac{\partial u}{\partial y} = x^2$

Now $x^2 + xy + y^2 = 1$

Differentiating w.r.t. x we get

$$2x + x\ .\ \frac{dy}{dx} + y + 2y.\ \frac{dy}{dx} = 0$$

or $\frac{dy}{dx} = -\frac{2x + y}{x + 2y}$

So now $\frac{du}{dx} = \frac{\partial u}{\partial x} + \frac{\partial u}{\partial y}.\frac{dy}{dx} = 2xy + x^2\left(-\frac{2x + y}{x + 2y}\right)$

$$= 2\,xy - \frac{x^2(2x+y)}{x+2y}$$

$$= \frac{2x^2y + 4xy^2 - 2x^3 - x^2y}{x+2y} = \frac{x^2y + 4xy^2 - 2x^3}{x+2y}$$

$$= \frac{x(4y^2 + xy - 2x^2)}{x+2y}$$

Example 15. If f(x,y) = 0 prove that

$$\frac{\partial^2 f}{\partial x^2} + 2\frac{\partial^2 f}{\partial x \partial y}.\frac{dy}{dx} + \frac{\partial^2 f}{\partial y^2}\left(\frac{dy}{dx}\right)^2 + \frac{\partial f}{\partial y}.\frac{d^2y}{dx^2} = 0$$

(M.D.University, 1993)

Solution. Let f(x, y) be a composite function of x

$$\therefore \quad \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}.\frac{dy}{dx} = 0\,.$$

Differentiating this differential w.r.t. x , we get

$$\frac{d}{dx}\left(\frac{\partial f}{\partial x}\right) + \frac{d}{dx}\left(\frac{\partial f}{\partial y}.\frac{dy}{dx}\right) = 0$$

or $$\left[\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) + \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right).\frac{dy}{dx}\right] + \left[\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) + \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right).\frac{dy}{dx}\right] - \frac{dy}{dx} + \frac{\partial f}{\partial y}.\frac{d^2y}{dx^2} = 0$$

or $$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y \partial x}.\frac{dy}{dx} + \frac{\partial^2 f}{\partial x \partial y}.\frac{dy}{dx} + \frac{\partial^2 f}{\partial y^2}\left(\frac{dy}{dx}\right)^2 + \frac{\partial f}{\partial y}.\frac{d^2y}{dx^2} = 0$$

or $$\frac{\partial^2 f}{\partial x^2} + 2\frac{\partial^2 f}{\partial x \partial y}.\frac{dy}{dx} + \frac{\partial^2 f}{\partial y^2}.\left(\frac{dy}{dx}\right)^2 + \frac{\partial f}{\partial y}.\frac{d^2y}{dx^2} = 0$$

Hence proved.

Differentiation of implicit functions

For an implicit function, we have the following differentials :

If f(x, y) = 0 or f(x, y) = c, then

(i) $$\frac{dy}{dx} = -\frac{\frac{\partial f}{\partial x}}{\frac{\partial f}{\partial y}} = -\frac{f_x}{f_y}$$

(ii) $$\frac{d^2y}{dx^2} = \frac{[f_{xx}(f_y)^2 - 2f_xf_yf_{xy} + f_{yy}(f_x)^2}{(f_y)^3}\,.$$

Example 16. If $x^5 + y^5 - 5a^3xy = 0$ find $\frac{dy}{dx}$

Solution. Let $f(x,y) = x^5 + y^5 - 5a^3xy$

Then $f_x = 5x^4 - 5a^3y$ and $f_y = 5y^4 - 5a^3x$

$f_{xx} = 20x^3$ and $f_{yy} = 20y^3$

$f_{xy} = -5a^3$, $f_x{}^2 = 20x^3$, $f_y{}^2 = 20y^3$

Now $\frac{dy}{dx} = \frac{-f_x}{f_y} = -\frac{5x^4 - 5a^3y}{5y^4 - 5a^3x} = \frac{x^4 - a^3y}{y^4 - a^3x}$

Exercise 2.1

1. Find the total differentials of the following functions :
   (i) $u = 5x^2 - 2y^2 + 3xy$ (ii) $u = (x^2+y^2)(2x^2-y)$
   (iii) $u = \frac{x^3 + y^2}{x - y}$ (iv) $u = \log(x^2-2y)$
   (v) $z = u^3$ and $u = x^2+y^2$ (vi) $u = e^{x+y^2}$
2. Find $\frac{du}{dt}$ when $u = 3x^2 - 2xy + 5y$ and $x = 3t^2 + 2t$, $y = 5t+7$
3. Find $\frac{dz}{dt}$ if $z = e^{wxy}$ where $w = e^t$, $x = t^3$, $y = \frac{1}{t}$
4. Find $\frac{du}{dr}$ and $\frac{du}{ds}$ if $u = x^2 + xy + y^2$, $x = 2r+s$, $y = r-2s$
5. If $u = xyz$, $x = s^2t^2$, $y = st^2$, $z = st$ find $\frac{du}{ds}$
6. If $x^2+y^3 = 100$, find $\frac{dy}{dx}$
7. If $v = f(y-z, z-x, x-y)$ prove that $\frac{\partial v}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial v}{\partial z} = 0$
8. Find $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ for the following implicit functions :
   (i) $x^{2/3} + y^{2/3} = a^{2/3}$ (ii) $xy^2 + x^2y - x^3 - y^3 = 0$.

Answer

Exercise 2.1

1. (i) $2(x+y)$ (ii) $\frac{1}{x + y}$ (iii) $\frac{y^2 - x^2}{(x^2 + y^2)^2}, \frac{-2xy}{(x^2 + y^2)^2}$
   (iv) $y\, e^{xy}\, x^{y-1}$, $x^y \cdot e^{xy\log x}$

Exercise 2.1

1. (i) $(10x + 3y)\,dx + (3x-4y)dy$ (ii) $(8x^3+4xy^2-2xy)dx + (4x^2y-x^2-2y^3)dy$
   (iii) $\frac{(2x^3 + 3x^2y - y^2)dx + (2xy - y^2 + x^3)dy}{(x - y)^2}$ (iv) $\frac{2(xdx - dy)}{x^2 - 2y}$
2. $(18t^2 + 2t-14)(6t + 2) + 5(5-4t-6t^2)$
3. $wxy\, e^{wxy}\left(\frac{et}{w} + \frac{3t^2}{x} - \frac{1}{yt^2}\right)$ 4. $-3y$

5. $\frac{2s^2t}{x}+\frac{2st}{y}+\frac{s}{z}$

6. $-\frac{2x}{3y^2}$

8. (i) $-\frac{y^{1/3}}{x^{1/3}}, \frac{a^{2/3}}{3x^{4/3}y^{1/3}}$

(ii) $-\frac{y^2+2xy-3x^2}{2xy+x^2-3y^2}, \quad \frac{(2y-6x)(2xy+x^2-3y^2)^2+2(2x+2y)(y^2+2xy+3x^2)}{(2xy+x^2-3y^2)-(2x-6y)(y^2+2xy-3x^2)^2}{(2xy+x^2-3y^2)^3}$

अध्याय .3

उच्चष्ठ और निम्निष्ठ
(Maxima and Minima)

उच्च अवकलज

tSlkfd ge igys i<+ pqds gSa ;fn y = f(x) rks blds fy, ge $\frac{dy}{dx}$ fudky ldrs gaS tks fd izFke Øe dk vodyt dgykrk gSaA blh rjg ls ge f}rh;] r`rh; rFkk vU; mPp Øe ds vodyt Hkh fudky ldrs gSaA

As we have already studied that if y = f(x), we can calculate $\frac{dy}{dx}$ and it is called first order derivative. In the same way, we can calculate second, third and other higher order derivates. For example.

Let $y = 15x^4 - 3x^2$

Now $\frac{dy}{dx} = 60x^3 - 6x$

Second order derivative $= \frac{d}{dx}\left(\frac{dy}{dx}\right)$ or $\frac{d^2y}{dx^2} = \frac{d}{dx}(60x^3 - 6x) = 180x^2 - 6$

and

Third order derivative $= \frac{d}{dx}\left(\frac{d^2y}{dx^2}\right) = \frac{d}{dx}(180x^2 - 6) = 360\,x$

In the same way, we can obtain higher order derivatives.

Example 1. If $y = x^4 + x^3 + x^2 + x$, find $\frac{dy}{dx}, \frac{d^2y}{dx^2}, \frac{d^3y}{dx^3}, \frac{d^4y}{dx^4}$

Sol. $y = x^4 + x^3 + x^2 + x$ (gives)

Then $\frac{dy}{dx} = 4x^3 + 3x^2 + 2x+1$ …(i)

$\frac{d^2y}{dx^2} = 12x^2 + 6x + 2$ …(ii)

$\frac{d^3y}{dx^3} = 24x + 6$ …(iii)

तथा $\frac{d^4y}{dx^4} = 24$ …(iv)

Example 2. If $v = 14t^3 - 8t$, then find $\frac{d^2v}{dt^2}$ at t = 1

Sol. $v = 14t^3 - 8t$ (given)

Then $\frac{dv}{dt} = 42\, t^2 - 8$

तथा $\frac{d^2v}{dt^2} = 84t.$

इसलिए ए जब ज त्र 1, $\frac{d^2v}{dt^2} = 84 \times 1 = 84$.

vodyt dk fpUg (Sign of Derivative) ;fn x dk fn;k x;k Qyu y gS] rks x ds lkis{k $\frac{dy}{dx}$] y dh o`f) dh nj (rate of increase) n'kkZrk gSA

vr% ;fn $\frac{dy}{dx}$ /kukRed gS rks tSls&tSls x c<+rk gS oSls&oSls y c<+rk gS] rFkk] ;fn $\frac{dy}{dx}$ _.kkRed gS rks tSls&tSls x c<+rk gS oSls&oSls y ?kVrk gSA

blh izdkj] x ds lkis{k $\frac{d^2y}{dx^2}$ dk vody xq.kkad $\frac{dy}{dx}$ gS] rFkk ;fn $\frac{d^2y}{dx^2}$ /kukRed gS rks tSls&tSls x c<+rk gS oSls&oSls $\frac{dy}{dx}$ c<+rk gS] rFkk] ;fn $\frac{d^2y}{dx^2}$]_.kkRed gS rkS tSls&tSls x c<rk gS oSls&oSls $\frac{dy}{dx}$ घटता है।

If y is a function of x, then the derivative $\frac{dy}{dx}$ measures the rate of increase of y. So if $\frac{dy}{dx}$ is positive, y will increase with increase in the values of x and if $\frac{dy}{dx}$ is negative, y will decrease when x increases.

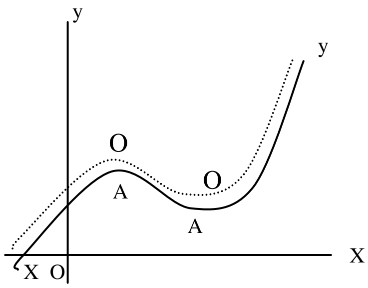
Similarly, $\frac{d^2y}{dx^2}$ is a derivative of $\frac{dy}{dx}$. So, if $\frac{d^2y}{dx^2}$ is positive then $\frac{dy}{dx}$ increase with increase in x and if $\frac{d^2y}{dx^2}$ is negative, value of $\frac{dy}{dx}$ decreases with increase in values of x.

mfPp"B rFkk fufEu"B fcUnq (Maximum and Minimum points)

संलग्न चित्र पर विचार कीजिए। वक्र के भाग ग। के लिए जैसे-जैसे ग बढ़ता है वैसे-वैसे ल बढ़ता है तथा इस प्रकार $\frac{dy}{dx}$ धनात्मक है।

बिन्दु । पर $\frac{dy}{dx}$ शून्य है। ।ठ के साथ ऋणात्मक है तथा पुनः यह ठ पर शुन्य है। वक्र के भाग ठल के लिए, $\frac{dy}{dx}$ पुनरू धनात्मक है। वक्र पर बिंदुओ । व ठ को वर्तन बिन्दु ;जनतदपदह चवपदजेद्ध कहते हैं। । की कोटि ;वतकपदंजमद्ध किसी सामीप्य कोटि ;दमपहीइवनतपदह वतकपदंजमद्ध से अधिक है, अतः । वक्र का महत्म बिन्दु ;उंगपउनउ

चवपदजद्ध कहलाता है, जबकि ठ की कोटि किसी सामीप्य कोटि से कम है, अत: ठ वक्र का निम्नतम बिन्दु ;उपदपउनउ चवपदजद्ध कहलाता हैं।



उच्चिष्ठ तथा निम्निष्ठ बिन्दु ;Maxima and Minima)

परिभाषा ;क्मपिदपजपवदद्ध माना ग का कोई फलन ल त्र ि;गद्धए ग त्र ं के सामीप्य में संतत है। अत: ि;द्ध का मान, ि;गद्ध का महत्तम या निम्नतम मान कहा जाता है जब ि;ीद्ध का मान ि;द्ध से छोटा या बड़ा हो चाहे वृद्धि ी का मान कोई भी हो, धनात्मक या ऋणात्मक दिया गया हो, इसको साधारणतया: छोटा ;अल्पद्ध लिया जाता है तथा यह शून्य के बराबर नहीं होता है।

Consider the graph given above. For part XA, y increases with increase in x, so $\frac{dy}{dx}$ is positive. At point A, $\frac{dy}{dx} = 0$ and for part AB, $\frac{dy}{dx}$ is negative and at B it is again zero. For BY, $\frac{dy}{dx}$ is again positive. The points A and B are called turning points. Ordinate at A is more than that at any neighbouring ordinate. So A is called maximum point while ordinates of B are less than any neighbouring ordinates. So B is called minimum point.

Maxima and Minima

Let y = f(x), a function of x, be continuous near x = a so value of f (a) is called the minimum or maximum value of f(x) accordingly as value of f (a+h) is less than or greater than f(a). Value of h (increase or decrease) is taken to be small in magnitude but it is never equal to zero.

mfPp"B rFkk fufEu"B ekuksa ds fy, vko';d izfrcU/k (Necessary condition for maximum or minimum values) x = ं पर ि;गद्ध के उच्चिष्ठ या निम्निष्ठ के लिए आवश्यक प्रतिबन्ध है:

$$f(a) = 0 \text{ or } \left[\frac{d}{dx}f(x)\right]_{\text{at } x=a} = 0 \quad \text{or} \quad \left(\frac{dy}{dx}\right)_{x=a} = 0$$

ल **के उच्चिष्ठ तथा निम्निष्ठ के लिए क्रिया विधि** (Working rule for maximum and minimum of y).

1. Find $\frac{dy}{dx}$ i.e. $f'(x)$ for the function f(x)
2. Put $f'(x) = 0$ and solve this equation to obtain various values of x (say $a_1, a_2, a_3 \ldots$) these are the only points at which f (x) will have minimum or maximum value
3. Find $\frac{d^2y}{dx^2}$ i.e. $f''(x)$ and find its value by substituting the values of $a_1, a_2, a_3 \ldots$ in $f''(x)$.

4. If $\frac{d^2y}{dx^2}$ is negative then f(x) is maximum at $x = a_1$. If $\frac{d^2y}{dx^2}$ is positive then f(x) is minimum at $x = a_1$. Similarly, we can check for other values $a_2, a_3 \ldots$

5. If $\frac{d^2y}{dx^2} = 0$ at x = a , find $\frac{d^3y}{dx^3}$ and put $x = a_1$ in it. If at $x = a_1$, $\frac{d^3y}{dx^3} \neq 0$, then f(x) is neither maximum nor minimum at $x = a_1$. If $\frac{d^3y}{dx^3} = 0$, then find $\frac{d^4y}{dx^4}$ and put $x = a_1$ in it. It $\frac{d^4y}{dx^4}$ is negative at $x = a_1$, then f(x) is maximum at $x = a_1$ and if it is positive, f(x) is minimum at $x = a_1$

Local Maxima and Local Minima

Definition 1. Let f be a real function and let x_0 be an interior point in the domain of f. We say that x_0 is a local maximum of f, if there is an open interval containing x_0 such that $f(x_0) > f(x)$ for ever x in that open interval

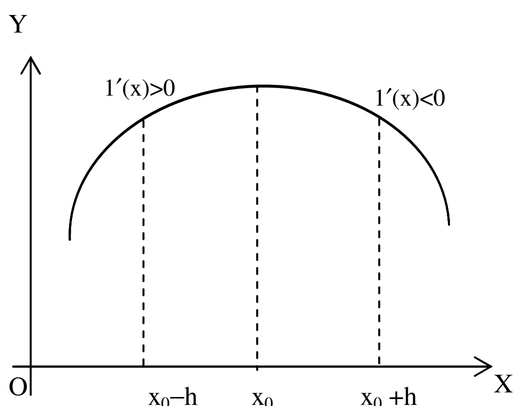


Fig. 3.1

If x_0 is a point of local maximum of f(x), then the graph of f(x) around x_0 will be a shown in Fig. Here f(x) is increasing in the interval (x_0-h, x_0) and decreasing in the interval (x_0, x_0+h).
$\therefore$ In (x_0-h, x_0), $f'(x) > 0$ and in (x_0, x_0+h), $f'(x) < 0$. This suggests that $f'(x_0)$ must be zero

Definition 2. Let f be a real valued function and x_0 be an interior point, in the domain of f. We say that x_0 is a local minimum of f if there is open interval containing x_0 such that $f(x_0) < f(x)$ for every x in that open interval.
If x_0 is a point of local minimum of f(x), then the graph of f(x) around x_0 will be as shown in Fig.

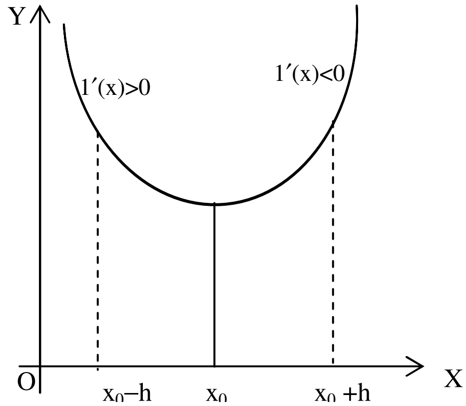


Fig. 3.2

Here f(x) is decreasing in $(x-h, x_0)$ and increasing in $(x_0, x_0 + h)$. So $f'(x) < 0$ in (x_0-h, x_0) suggesting that $f'(x_0)$ must be zero.

Note. The end points of the interval cannot be the points of local extremum.

Theorem (without proof) : Let f be a differentiable function. Then f′ vanishes at every local maximum and at every local minimum.

Note. When x_0 is a point of local minimum or local maximum, the tangent at x_0 parallel to the x-axis.

Working Rule. For finding the points of local maxima or point of local minima.

(1) First Derivative Test. Let f(x) be a differentiable function on I and let $x_0 \in I$. Then

(a) x_0 is a point of local maximum of f(x) if

(i) $f'(x_0) = 0$.

(ii) $f'(x) > 0$ at every point close to and to the left of x_0 ; and $f'(x) < 0$ at every point close to and to the right of x_0.

(b) x_0 is a point of local minimum of f(x) if

(i) $f'(x_0) = 0$.

(ii) $f'(x) < 0$ at every point close to and to the left of x_0 ; and $f'(x) > 0$ at every point close to and to the right of x_0.

(c) If $f'(x_0) = 0$, but $f'(x)$ does not change sign as x increase through x_0 is neither a point of local minimum nor a point of local maximum.

Remark. If $f'(x_0) = 0$ and x_0 is neither a point of local minimum nor a point of local maximum, then x_0 is called a point of inflexion.

Example 1. Find the local maximum and minimum for the following functions using the first derivative test only.

(i) $x^3 - 6x^2 + 9x + 15$ (ii) $\frac{x}{2} + \frac{2}{x}$, $x > 0$

(iii) $(x-3)^4$

Sol. (i) $f(x) = x^3 - 6x^2 + 9x + 15$

$\therefore$ $f'(x) = 3x^2 - 12x + 9 = 3(x^2 - 4x + 3)$

$= 3(x-1)(x-3)$

$f'(x) = 0 \quad 3(x-1)(x-3) = 0$

$\therefore$ $x = 1, 3$

Let us see whether x = 1 is a point of local maximum or minimum. Let us take x = 9 to the left of the point x = 1, x =1·1 to the right of the point x = 1.

$f'(9) = (\cdot9-1)(\cdot9-3) = 0\cdot63$, which is a positive number.

$f'(1\cdot1) = 3(1\cdot1-3) = -0.57$ which is a negative number.

∴ f′(x) change sign from positive to negative as x increases through x = 1.

Thus from the first derivative test x = 1 is a point of local maximum. Local maximum value = f(1) =1 − 6+9+15 = 19.

Again, let us see whether x = 3 is a point of local maximum or minimum. Let us take x = 2·9 to the left of the point x = 2·9 and x = 3·1 to the right of the point.

$f'(2\cdot9) = 3(2\cdot9-1)(2\cdot9-3) = -0\cdot57$ which is a negative number.

$f'(3\cdot1) = 3(3\cdot1-1)(3\cdot1-3) = 0\cdot63$ which is a positive number.

∴ f′(x) change sign from negative to positive as x increases through x = 3.

Thus from the first derivatives test x = 3 is point of local minimum.

Local minimum value = f(3) = 27 − 54 + 27 + 15 = 15.

(ii) $f'(x) = \frac{x}{2}+\frac{2}{x} \quad x > 0$

$f'(x) = \frac{1}{2}-\frac{2}{x^2}$

For local maxima or minima, f′(x) = 0

$\therefore \quad \frac{1}{2}-\frac{2}{x^2} = 0 \quad \Rightarrow x^2 = 4$ i.e., $x = \pm 2$

But x > 0, x = 2.

Let us see whether x = 2 is a point of local maximum or local minimum. Let us take x = 1·9 to the left of the point x = 2 and x = 2.1 to the right of the point x = 2.

$$f'(1\cdot9) = \frac{1}{2}-\frac{2}{(1\cdot9)^2} = \frac{1}{2}-\frac{2}{3\ 61} = 0\cdot5-0\cdot554 = -0\cdot054$$

$$f'(2\cdot1) = \frac{1}{2}-\frac{2}{(2\cdot1)^2} = \frac{1}{2}-\frac{2}{4\ 41} = 0\cdot5-0\cdot453 = -0\cdot047$$

∴ f′(x) changes sign from negative to positive as x increases through x = 2

Thus, from first derivative tests, x = 2 is a point of local minimum.

Local minimum value = $f(2) = \frac{2}{2}+\frac{2}{2} = 1+1 = 2$

(iii) $f'(x) = (x-3)^4$

$\therefore \quad f'(x) = 4(x-3)^3 \times \frac{d}{dx}(x-3) = 4(x-3)^3 . 1 = 4(x-3)^3$

For local maxima or minima f′(x) = 0

$\therefore \quad 4(x-3)^3 = 0 \quad \Rightarrow x-3 = 0.$ i.e., x = 3.

Let us see whether x = 3 is a point of local maximum or minimum. Let us take x = 2·9 to the left of the point x = 3 and x = 3·1 to the right of the point x = 3.

Now $f'(2\cdot9) = 4(2\cdot9-3)^3 = 4(-0\cdot1)^3 = -\cdot004$,
which is a negative number

$f'(3\cdot1) = 4(3\cdot1-3)^3 = 4(0\cdot1)^3 = \cdot004$

which is a positive number.

Since $f'(x)$ changes sign from negative to positive as x increases through x = 3.
Thus from the first derivative test x = 3 is a point of local minimum.
Local minimum value = $f(3) = (3-3)^4 = 0$.

Example 2. Examine $y = (x-2)^3 (x-3)^2$ for local maximum and minimum values. Also find the point of inflexion, if any.

Sol. $y = (x-2)^3 (x-3)^2$

$$\therefore \quad \frac{dy}{dx} = (x-2)^3 \cdot 2 (x-3) + (x-3)^2 \cdot 3 (x-2)^2$$
$$= (x-2)^2 (x-3) (2x-4 +3x-9)$$
$$= (x-2)^2 (x-3) (5x-13)$$

For local maximum or minimum, put $\frac{dy}{dx} = 0$

$\therefore \quad (x-2)^2 (x-3) (5x-13) = 0$ or x = 2,3, 13/5

(i) x = 2. Let us take 1.9 to the left of point x = 2 and 2.1 to the right of point x =2

So $f'(1.9) = (1.9-2)^2 (1.9-3)(9.5-13) = 1.2$
$= 0.01 \times (1.1)(3.5) = 0.0385$ (positive in sign)

and $f(2.1) = (2.1 \;\; 2)^2(2.1-3) (10.5-13) = 0.01(-0.9) (-2.5)$
$= 0.0225$ (positive in sign)

Since, there is no change in sign of $\frac{dy}{dx}$ as increases through 2. Hence x = 2 is a point of inflexion.

(ii) At x = 3 when x is slightly less than 3

sign of $\frac{dy}{dx}$ = (+) (–) (+) = negative

sign of $\frac{dy}{dx}$ when x is slightly more than 3 = (+) (+) (+) = positive.

Thus, $\frac{dy}{dx}$ changes sign from negative to positive as x increases through 3.

So f(x) has local minimum at x = 3.

Local minimum value $= (3-2)^3(3-3)^2 = 0$

(iii) At $x = \frac{13}{5}$

when x is slightly less than $\frac{13}{5}$. Sign of $\frac{dy}{dx}$ = (+) (–) (–) = positive

When x is slightly more than $\frac{13}{5}$. Sign of $\frac{dy}{dx}$ = (+) (–) (+) = negative

Thus sign of $\frac{dy}{dx}$ changes from positive to negative, as x increases through $\frac{13}{5}$.

So f(x) has local maxima at $x = \frac{13}{5}$ and the local maximum value

$$= \left(\frac{13}{5}-2\right)^3 \left(\frac{13}{5}-3\right)^2 = \frac{27}{125} \times \frac{4}{25} = \frac{108}{3125}.$$

Example 3. Find the maximum and minimum of value of

$f(x) = x^3 - 12x^2 + 36x + 17$ in $1 \le x \le 10$

Sol. We have

$$f(x) = x^3 - 12x^2 + 36x + 17$$

$\therefore \quad f'(x) = 3x^2 - 24x + 36 = 3(x^2-8x+12)$

i.e., $\quad f'(x) = 3(x-6)(x-2)$

$f'(x) = 0 \Rightarrow 3(x-6)(x-2) = 0$

$\Rightarrow \quad x = 6, \; x = 2.$

Now $\quad f(1) = (1)^2 - 12(1)^2 + 36(1) + 17$

$= 1-12+36+17 = 42$

$f(2) = (2)^3 - 12(2)^2 + 36(2) + 17$

$= 8-48+72+17 = 49$

$f(6) = (6)^3 - 12(6)^2 + 36(6) + 17$

$= 216-432+216+17 = 17$

$f(10) = (10^3-12(10)^2 + 36(10)+17$

$= 1000-1200+360+17 = 177.$

Thus, f has the maximum at x = 10 and the minimum at x = 6.
The maximum of f is 177 and the minimum of f is 17.

Example 4. Find the maximum and minimum value of the function

$f(x) = 2x^3-21x^2+36x+20$

Sol. $\quad f(x) = 2x^2- 21x^2 + 36x-20$

$f'(x) = 6x^2-42x+36 = 6(x^2-7x+6)$

$= 6(x-1)(x-6).$

For max. or min. value of f(x)

$f'(x) = 0 \Rightarrow 6(x-1)(x-6) = 0$

$\therefore \quad x = 1,6$

$f''(x) = 12x-42 = 6(2x-7)$

$\therefore \quad f'(1) = 6(2-7) = -30 = 0$

$f''(6) = 6(12-7) = 30 > 0.$

$\therefore$ f(x) has max. at x = 1 and min. at x = 6.

Max. value = $f(1) = 2(1) -21(1) + 36(1) - 20$

$= 2-21+36-20 = -3.$

Min. value $= f(6) = 2(216)- 21(36) + 36(6) - 20$

$= 432-756+216-20 = -228.$

Application of Maxima and Minima

Example 4. Find two positive numbers whose product is 64 and their sum is minimum.

Sol. Let x and y be the two positive numbers.

Then $xy = 64$ or $y = \frac{64}{x}$

Let S denote their sum.

$\therefore \quad S = x+y = x+\frac{64}{x}$

$\therefore \quad \frac{dS}{dx} = \frac{d}{dx}\left(x+\frac{64}{x}\right) = 1-\frac{64}{x^2}$

For max. or min. $\frac{dS}{dx} = 0$

$\therefore \quad 1-\frac{64}{x^2} = 0, \Rightarrow \frac{64}{x^2} = 1$ or $x^2 = 64$

$\therefore \quad x = +\sqrt{64} = 8$

Now $\frac{d^2S}{dx^2} = \frac{d}{dx}\left(\frac{dS}{dx}\right) = \frac{d}{dx}\left(1-\frac{64}{x^2}\right)$

$= 0 + \frac{128}{x^3} = \frac{128}{x^3}$

$\left(\frac{d^2S}{dx^2}\right)_{x=8} = \frac{128}{8^3} = \frac{128}{8\times8\times8} = \frac{1}{4} > 0$

$\therefore$ S is minimum when $x = 8$.

$\therefore$ The other number $y = \frac{64}{x} = \frac{64}{8} = 8$

Hence, the required numbers are 8 and 8.

Example 5. Find the dimensions of a rectangle, having perimeter 40 metres, which has maximum area. Also find the maximum area.

Sol. Let x and y be dimension of the rectangle.

Perimeter of Rectangle $= 2(x+y)$

$\therefore \quad 2x+2y = 40$

or $\quad x + y = 20$

Area of rectangle, $A = xy$

$\therefore \quad A = x(20-x) = 20x-x^2$

$\therefore \quad \frac{dA}{dx} = 20-2x$

Now for area to be max. or min.,

$$\frac{dA}{dx} = 0$$

$\Rightarrow \quad 2x = 20$ or $x = 10$

Again, $\frac{d^2A}{dx^2} = -2 = -$ve quantity

$\therefore$ A is max. when $x = 10$.

Now $\quad x = 10$ cm

$\therefore \quad y = 20-x = 20-10 = 10$ cm.

Thus, of all the rectangles each of which has perimeter 40 cm, The square having a side 10 cm has the maximum area and required area = 10×10 = 100 sq. cm.

Example 6. If 40 square feet of sheet metal are to be used in the construction of an open tank with a square base, find the dimensions so that the capacity is greatest possible.

Sol. Let each side of the base, depth and the volume (capacity) of the tank be x, h and v respectively.

Whole surface area = 40 sq. feet

Also whole surface are $a = x^2 + 4xh$

$$\therefore \quad x^2+4xh = 40$$

or
$$h = \frac{40-x^2}{4x}$$

Now
$$v = x \times x \times \frac{40-x^2}{4x} = \frac{1}{4}x(40-x^2) = 0$$

For max. or min.,

$$\frac{dv}{dx} = \frac{1}{4}(40-x^2) - \frac{1}{2}x^2 = \frac{1}{4}(40-3x^2) = 0$$

i.e., $40-2x^2 = 0$

or
$$x = \sqrt{\left(\frac{40}{3}\right)}$$

Again $\frac{d^2y}{dx^2} = -\frac{3}{2} = -ve$ when $x = \sqrt{\left(\frac{40}{3}\right)}$

and therefore, it gives a maximum.

Also
$$h = \frac{40-\frac{40}{3}}{4.\sqrt{\left(\frac{40}{3}\right)}} = \frac{1}{2}\sqrt{\left(\frac{40}{3}\right)}$$

$\therefore$ The required dimensions are

$$\sqrt{\left(\frac{40}{3}\right)} . \sqrt{\left(\frac{40}{3}\right)} \text{ and } \frac{1}{2}. \sqrt{\left(\frac{40}{3}\right)}.$$

(Important Formula for Geometrical Figures).

(1) For a rectangle − Area = Length × Breadth
Perimeter = 2(Length + Breadth)

(2) For a circle − Area = πr^2 and circumference = $2\pi r$
where $\pi = \frac{22}{7}$ or 3.1428 and r is the radius of the circle.

(3) For a square Area = a^2 and Perimeter = 4a
where a is the side of the square.

(4) For a sphere, Volume = $\frac{4}{3}\pi r^3$ and surface are a = $4\pi r^2$
where r is the radius of the sphere.

(5) For a right circular cylinder volume = $\pi r^2 h$.
Surface area = $2\pi rh + 2\pi r^2$, or $2\pi r(h + r)$ Curved Surface = $2\pi rh$
where r is the radius at base and h is the height.

(6) For a right circular cone
Volume = $\frac{1}{3}\pi r^2 h$, Total surface Area = $\pi r^2 + \pi rl$ or $\pi r(r+l)$ where $l = \sqrt{h^2 + r^2}$, Curved Surface Area = πrl
where h is the height, l is the slant-height and r is the radius at base.

(7) For a cube – Volume = x^3, Surface area = $6x^2$
where x is the side (edge) of the cube.

(8) For a cuboid volume = xyz, Surface area = 2(xy + yz + zx)

(9) For a triangle are = $\sqrt{S(S-a)(S-b)(S-c)}$ (*Hero's Formula*) and perimeter = a+b+c
where $S = \frac{a+b+c}{2}$ and a,b, c are the three sides of the triangle.

Example 7. Find the radius of closed right circular cylinder of volume 100 cubic centimeters which has the minimum total surface area.

Sol. Let r and h be the radius and height of the cylinder.
Now volume = $\pi r^2 h = 100$

$$h = \frac{100}{\pi r^2}$$

Let S be the total surface area of the cylinder

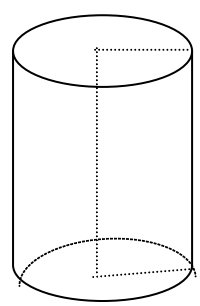


Fig. 3.3

$$\therefore \quad S = 2\pi rh + 2\pi r^2$$

$$\therefore \quad = 2\pi r\left(\frac{100}{\pi r^2}\right) + 2\pi r^2$$

$$S = \frac{200}{r} + 2\pi r^2$$

$$\frac{dS}{dr} = -\frac{200}{r^2} + 4\pi r$$

$$= -\frac{-200 + 4\pi r^3}{r^2}$$

For max, or min., $\frac{dS}{dr} = 0$

$$\Rightarrow \quad \frac{200 + 4\pi r^3}{r^2} = 0$$

or $4\pi r^3 - 200 = 0$, $\Rightarrow \pi r^3 = \frac{200}{4} = 50$

$$r = \left(\frac{50}{\pi}\right)^{1/3}$$

$$\frac{d^2S}{dr^2} = \frac{400}{r^3} + 4\pi$$

$$\therefore \quad \left(\frac{d^2S}{dr^2}\right)_{r=\left(\frac{50}{\pi}\right)^{1/3}} = \frac{400}{50/\pi} + 4\pi = 12\pi > 0$$

$\therefore$ S is minimum when $r = \left(\frac{50}{\pi}\right)^{1/3}$

When $r = \left(\frac{50}{\pi}\right)^{1/3}, h = \frac{100}{\pi r^2} = \frac{100}{\pi\left[\left(\frac{50}{\pi}\right)^{1/3}\right]^2}$

$$= \frac{100}{\pi^{1/3}(50)^{2/3}} = \frac{2\cdot(50)^{1/3}}{(\pi)^{1/3}} = 2\left(\frac{50}{\pi}\right)^{1/3} = 2r$$

Hence the total surface area is minimum when radius

$r = \left(\frac{50}{\pi}\right)^{1/3}$ cm and height = $2\left(\frac{50}{\pi}\right)^{1/3}$ cm.

Example 8. A cylinder is such that sum of its height and the circumference of its base is 10 meters. Find the greatest volume of the cylinder.

Sol. Let V and r be the volume and radius of base of the cylinder then circumference of base is $2\pi r$. If h is the height of the cylinder, then

$h + 2\pi r = 10, \quad \therefore h = 10 - 2\pi r$ …(1)

$\therefore \quad V = \pi r^2 h = \pi r^2 (10 - 2\pi r)$ [Using (1)]

$= 10\pi r^2 - 2\pi^2 r^3$

Differentiating w.r.t. r,

$$\frac{dV}{dr} = 10\pi(2r) - 2\pi^2(3r^2)$$

$$= 20\pi r - 6\pi^2 r^2$$

For max. or min. volume, $\frac{dV}{dr} = 0$

$\Rightarrow \quad 20\pi r - 6\pi^2 r^2 = 0$

$\therefore \quad 2\pi r(10 - 3\pi r) = 0$

$\Rightarrow \quad r = 0 \quad$ or $\quad 10 - 3\pi r = 0$, i.e., $r = \frac{10}{3\pi}$

Since $r \neq 0$, as it makes $V = 0$

$\therefore \quad r = \frac{10}{3\pi}$

Again, $\frac{d^2V}{dr^2} = \frac{d}{dr}\left(\frac{dV}{dr}\right) = \frac{d}{dr}(20\pi r - 6\pi^2 r^2)$

$= 20\pi - 12\pi^2 r$

Now $\left(\frac{d^2V}{dr^2}\right)_{r=\left(\frac{10}{3\pi}\right)} = 20\pi - 12\pi^2, \frac{10}{3\pi}$

$= 20\pi - 40\pi = -20\pi < 0$

$\therefore$ V is maximum when $r = \frac{10}{3\pi}$

and maximum volume, $V = \pi r^2 h$

$$= \pi\left(\frac{10}{3\pi}\right)^2\left[10 - 2\pi\left(\frac{10}{3\pi}\right)\right]$$

$$= \frac{100}{9\pi}\left(10 - \frac{20}{3}\right) = \frac{100}{9\pi} \times \frac{10}{3} = \frac{1000}{27\pi} m^2.$$

Example 9. The perimeter of a triangle is 8 cm. If one of the sides is 3cm, what are the other two sides for maximum area of the triangle ?

Sol. We know that S $\frac{a+b+c}{2}$

$\therefore$ 25 = a + b + c = 8

$\therefore$ s = 4.

Let a = 3

$\therefore$ b + c = 8 − 3 = 5

Now or $\Delta = \sqrt{s(s-a)(s-b)(s-c)}$

Then or $\Delta^2 = s(s-a)(s-b)(s-c)$

$= 4(4-3)(4-b)(4-c) = 4(4-b)(4-c)$

$= (4-5+c)(4-c)$ $(\therefore b + c = 5)$

$\therefore$ $\Delta^2 = 4(c-1)(4-c) = 4(5c-4-c+2)$

When Δ is maximum , Δ^2 is also maximum.

Now $\frac{d(\Delta^2)}{dc} = 4(5-2c).$

For max. or min. $\frac{d(\Delta^2)}{dc} = 0$

$\therefore$ $5-2c = 0$ or $c = \frac{5}{2}$

$\frac{d^2(\Delta^2)}{dc^2} = 4(0-2) = -8 = -$ ve quantity

$\therefore$ Area is max. when $c = \frac{5}{2}$.

Now b+c = 5, $\therefore$ $b = 5 - \frac{5}{2} = \frac{5}{2}$

Thus area is max. when $b = c = \frac{5}{2}$.

For maximum area of the triangle, it must be isosceles.

Example 10. Cost and revenue functions of a company are as given below :
Total cost $C = 100 + 0.015x^2$
Total revenue $R = 3x$ where x are the number of units produced. Find the production rate which will maximise the profit. Also find that no profit.

Sol. Profit (P) – total revenue – Total cost
$= 3x - (100 + 0.015x^2)$

$\therefore \quad \frac{dP}{dx} = 3 - 0.030x$

For max. or min values $\frac{dP}{dx} = 0$

or $-0.030x + 3 = 0$

$\therefore \quad x = \frac{3}{.03} = \frac{3\times 100}{3} = 100$ units.

Also $\frac{d^2y}{dx^2} = -0.03x$ which is negative

Hence profit is maximum at $x = 100$

and maximum profit $= 3\times 100 - 100 - .015(100)^2 = 50$ Rs.

Exercise 3.1

1. If $y = 3x^2+6x$ prove that $x^2\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + 2y = 0$
2. Find the local maximum and local minimum values of the following functions.
   (i) $2x^3-9x^2+12x+4$ (ii) x^2-3x+2
   (iii) x^3-3x (iv) $2x^2 - 3x^3$
   (v) $x^3(x-1)^2$ (vi) $x^5-5x^4+5x^3-1$
   (vii) $(x+3)^3 (x-4)^4$ (viii) $\frac{x}{(x-1)(x-4)}$ $(1 < x < 4)$
3. Find the maximum and minimum values of the following functions.
   (i) $f(x) = \frac{x+1}{\sqrt{x^2+1}}$. $0 \le x \le 2$ (ii) $f(x) = (x-1)^2+3$ $[-3 < x < 1]$
   (iii) $f(x) = x^4 - 62x^3 + 120x + 9$ (iv) $f(x) = 41 + 24x - 18x^2$
   (v) $f(x) = 3x^4-8x^3 + 12x^2 - 48x + 25$ on the interval (0, 3)
   (vi) $f(x) = \frac{1+x+x^2}{1-x+x^2}$ $-2 \le x \le 2$
4. A wire of length 4 cm is to form a rectangle. Find out the dimensions of the rectangle so that it has maximum area.
5. Show that of all the rectangles having the same area, square has the least perimeter.
6. Prove that the perimeter of a right angled triangle of given hypotenuse is maximum when it is an isosceles triangle.
7. An open box is to the made out of a piece of cardboard measuring 24cm × 24cm by cutting off equal squares from the corners and turning up the sides. Find the height of the box for maximum volume.

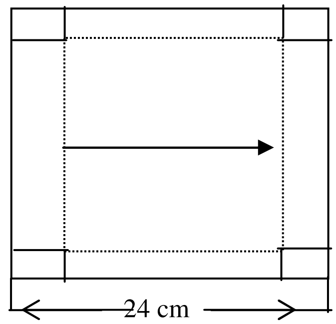


fig 3.4

8. Determine two positive numbers whose sum is 30 and whose product is maximum.
9. Find two numbers whose sum is 12 and sum of whose cubes is minimum.
10. A metal wire 36cm long is bent to form a rectangle. Find the dimensions when its area is maximum.
11. A close cylindrical can is to have a volume 1024π cm^3. Find the radius of the can so that the area of metal used will be as small as possible.
12. A right circular cylinder is to be made so that the sum of its radius and its height is 6 meters. Find the maximum volume of the cylinder.
13. Show that $x^3 - 3x^2 + 3x + 7$ has neither maximum nor minimum value
14. For a product total, revenue function is shown as

    $R = 4000000 - (x - 2000)^2$

    Where x is the number of units sold

    For what value of x :

    (i) Total revenue is maximum

    (ii) Total profit is maximum when total cost function is given by $C = 1500000 + 400x$ also find the max. Revenue & max profit

Answers

2. (a) 9, 8 (ii) Min $-\frac{1}{4}$ (iii) 2, −2 (iv) $\frac{32}{27}$,0 (v) $\frac{108}{3125}$ at $x = \frac{3}{5}$ & 0 at x = 1

   (vi) 0 at x = 1 and −28 at x = 3 (vii) 6912 at x = 0 &0 at x = 4

   (viii) −1 at x = 2 & $-\frac{1}{9}$ at x = −2

3. (i) 1 at x = 0 and 1.34 at x = 2 (ii) 19 at x = −3 and 3 at x = 1 (iii) 68 at x = 1 & −1647 at x = −6 (iv) 49 (v) 16 at x = 3 & −39 at x = 2 (vi) 3, $\frac{1}{3}$

4. Length = 1 cm, Breath = 1cm. The rectangle is a square

7. 4 8. 15,15 9. 6,6

10. It is a square whose each side is 9 cm 11. 8 cm

12. 32 π cubic metres 14. (i) 2000 (ii) 1980.

अध्याय .4

अनिश्चित समाकलन
(Indefinite Integral)

अब हम अवकलन ;कपििमितमदजपंजपवदद्ध के व्युत्क्रम संक्रिया ;तमअमतेम चतवबमेद्ध पर विचार करेंगे। अवकल गणित में, हम किसी दिये हुए फलन का अवकल गुणांक ज्ञात करते हैं जबकि यदि हमको किसी फलन का अवकल गुंणाक दिया हो तो समाकल गणित में हमको वह फलन ज्ञात करना होता है, इसी कारण समाकलन ;पदजमहतंजपवदद्ध प्रति अवकलज ;।दजप.कमतपअंजपअमद्ध कहलाता है। इस समाकलन को अनिश्चित समाकलन भी कहते हैं।

Now we will consider the inverse process of differentiation. In differentiation, we find the differential co-efficient of a given function while in integration if we are given the differential co-efficient of a function, we have to find the function. That is why integration is called anti-derivative i.e. in differentiation if $y = f(x)$ we find $\frac{dy}{dx}$. In integration, we are given $\frac{dy}{dx}$ and we have to find y. This integration is also called indefinite integral.

Definition of Integration

Integration is the inverse process of differentiation.

If $\frac{d}{dx}[\phi(x)] = f(x)$ then

$\phi(x)$ is called the integral or anti-derivative or primitive of f(x) with respect to x. Symbolically, it is written as

$$\int f(x)\,dx = \phi(x)$$

The symbol ∫ -----dx denotes integration w.r.t. x. Here dx conveys that x is a variable of integration. The given function whose integral is to be found, is known as integrand.

Example. $\because \frac{d}{dx}(x^2) = 2x$

$\therefore \int 2x\,dx = x^2$

Constant of integration

We know that $\frac{d}{dx}(x^3) = 3x^2$

Therefore integral of $3x^2$ may be x^3, $x^3 + 1$ or $x^3 + C$ where C is any arbitrary constant, Thus

$$\int 3x^2\,dx = x^3+C$$

Example. Find $\int 5x^6\,dx$

Solution. $\int 5x^6.\,dx = 5\int x^6\,dx = 5\times\frac{x^7}{7}+C = \frac{5}{7}x^7+C$

O;kid lq= (Standard Formulae)

1. $\int x^n dx = \frac{x^{n+1}}{n+1} + C, n \neq -1 \quad \left[\because \frac{d}{dx}\left(\frac{x^{n+1}}{n+1}\right) = x^n\right]$

2. $\int \frac{1}{x} dx = \log_e x + C \quad \left[\because \frac{d}{dx}(\log_e x) = \frac{1}{x}\right]$

3. $\int e^x dx = e^x + C \qquad \left[\because \frac{d}{dx}(e^x) = e^x\right]$

4. $\int a^x dx = \frac{a^x}{\log_e a} + C \qquad \left[\because \frac{d}{dx}\left(\frac{a^x}{\log_e a}\right) = a^x\right]$

5. $\int e^{ax+b} dx = \frac{e^{ax+b}}{a} + C \qquad \left[\because \frac{d}{dx}\left(e^{ax+b}\right) = ae^{ax+b}\right]$

6. $\int (ax+b)^n dx = \frac{(ax+b)^{n+1}}{a(n+1)} + C \left[\because \frac{d}{dx}\frac{(ax+b)^{n+1}}{a(n+1)} = (ax+b)^n\right]$ (if $n \neq -1$)

7. $\int \frac{dx}{ax+b} = \frac{1}{a} \log |ax+b| + C \qquad \left[\because \frac{d}{dx}\frac{\log|ax+b|}{a} = \frac{1}{ax+b}\right]$

nks lkekU; izes; (Two general Theorems)

Theorem 1. The integral of the product of a constant and a function is equal to the product of a constant, and integral of the function i.e.,

$\int kf(x)\, dx = k \int f(x)\, dx$, 'k' being a constant.

Proof. Let $\int f(x)\, dx = \phi(x)$, $\quad \therefore \frac{d}{dx}[\phi(x)] = f(x)$

Now $\frac{d}{dx}[k\phi(x)] = k. \frac{d}{dx}[\phi(x)]$

[$\because$ The derivative of the product of a constant and a function is equal to the product of the constant and the derivative of the function]

$= kf(x) \qquad | \because \frac{d}{dx}[\phi(x)] = f(x)$

$\therefore$ By definition,

$\int k. f(x)\, dx = k. \phi(x) = k. \int f(x)\, dx.$

Theorem 2. The integral of the sum or the difference of two functions is equal to the sum or difference of their integrals i.e.,

$\int [f_1(x) \pm f_2(x)]\, dx = \int f_1(x)\, dx \pm \int f_2(x)\, dx.$

Proof. Let $\int f_1(x)\, dx = \phi_1(x)$ and $\int f_2(x)\, dx = \phi_2(x)$

$\therefore \qquad \frac{d}{dx}[\phi_1(x)] = f_1(x)$ and $\frac{d}{dx}[\phi_2(x)] = f_2(x)$

Now $\frac{d}{dx}[\phi_1(x) \pm \phi_2(x)] = \frac{d}{dx}[\phi_1(x)] \pm \frac{d}{dx}[\phi_2(x)]$

$= f_1(x) \pm f_2(x)$

[$\because$ The derivative of the sum or difference of two functions is equal to the sum or difference of their derivatives].

$\therefore$ By definition of the integral of a function

$\int [f_1(x) \pm f_2(x)]\, dx = \phi_1(x) \pm \phi_2(x)$

$= \int f_1(x)\, dx \pm \int f_2(x)\, dx.$

Remark. We can extend this theorem to a finite number of functions and can have the following result.

$\int[f_1(x) \pm f_2(x) \pm \ldots\ldots \pm f_n(x)]\, dx$

$\int f_1(x)dx \pm \int f_2(x)\, dx \pm \ldots\ldots \pm \int f_n(x)\, dx.$

Example 1. Write down the integral of

(i) x^2 (ii) x^{-9} (iii) 1

(iv) $\sqrt{x}$. (v) $\frac{1}{x^2}$ (vi) $x^{-2/3.}$

Solution.

(i) $\int x^2\, dx = \frac{x^{2+1}}{2+1} + C = \frac{1}{3}x^3 + C$

(ii) $\int x^{-9}\, dx = \frac{x^{-9+1}}{-9+1} + C = \frac{x^{-8}}{-8} + C = -\frac{1}{8x^8} + C$

(iii) $\int 1.\, dx = \int x^0\, dx = \frac{x^{0+1}}{0+1} + C = x + C$

(iv) $\int x^{1/2}\, dx = \frac{x^{1/2+1}}{\frac{1}{2}+1} + C = \frac{x^{3/2}}{3/2} + C = \frac{2}{3}x^{3/2} + C$

(v) $\int \frac{1}{x^2} dx = \int x^{-2}\, dx = \frac{x^{-2+1}}{-2+1} + C = \frac{x^{-1}}{-1} + C = -\frac{1}{x} + C$

(vi) $\int x^{-2/3}\, dx = \frac{x^{-2/3+1}}{-\frac{2}{3}+1} = \frac{x^{1/3}}{\frac{1}{3}} + C = 3x^{1/3} + C.$

Example 2. Find the integrals of the following

(i) $\sqrt{x} - \frac{1}{\sqrt{x}}$ (ii) $\frac{(1+x)^2}{x^3}$ (iii) $\frac{x^4}{x^2+1}$

(iv) $x\sqrt{x+2}$ (v) $(1+x)\sqrt{1-x}$

Solution.

(i) $\int \sqrt{x} - \frac{1}{\sqrt{x}}\, dx = \int (x^{1/2} - x^{-1/2})\, dx$

$$= \frac{x^{3/2}}{\frac{3}{2}} - \frac{x^{1/2}}{\frac{1}{2}} = \frac{2x^{3/2}}{3} - 2x^{1/2} + C$$

(ii) $$\int \frac{(1+x)^2}{x^3} dx = \int\left(\frac{1+2x+x^2}{x^3}\right)dx = \int\left(\frac{1}{x^3} + \frac{2}{x^2} + \frac{1}{x}\right)dx$$

$$= \int x^{-3}\, dx + 2\int x^{-2}\, dx + \int \frac{1}{x}dx$$

$$= \frac{x^{-2}}{(-2)} + 2\frac{x^{-1}}{(-1)} + \log x + C$$

$$= -\frac{1}{2x^2} - \frac{2}{x} + \log x + C.$$

(iii) $$\int \frac{x^4}{x^2+1} dx = \int \frac{(x^4-1)+1}{x^2+1} dx$$

$$= \int \frac{x^4-1}{x^2+1} dx + \int \frac{1}{x^2+1} dx = \int (x^2-1) dx + \int \frac{1}{1+x^2} dx$$

$$= (x^3/3) - x + \tan^{-1} x + C \quad \left[\because \frac{d}{dx}(\tan^{-1} x) = \frac{1}{1+x^2}\right]$$

(iv) $$I = \int x\sqrt{x+2}\,dx$$

$$= \int [(x+2)-2]\sqrt{x+2}\,dx$$

$$= \int (x+2)\sqrt{x+2}\,dx - \int 2\sqrt{x+2}\,dx$$

$$= \int (x+2)^{3/2} dx - 2\int (x+2)^{1/2} dx$$

$$= \frac{(x+2)^{5/2}}{\frac{5}{2}} - 2\frac{(x+2)^{3/2}}{\frac{3}{2}} + C$$

$$= \frac{2}{5}(x+2)^{5/2} - \frac{4}{3}(x+2)^{3/2} + C.$$

(v) $$I = \int (1+x)\sqrt{1-x}\,dx$$

$$= \int [2-(1-x)]\sqrt{1-x}\,dx$$

$$= 2\int (1-x)^{1/2} dx - \int (1-x)(1-x)^{1/2} dx$$

$$= 2\int (1-x)^{1/2} dx - \int (1-x)^{3/2} dx$$

$$= 2\frac{(1-x)^{3/2}}{\frac{3}{2}(-1)} - \frac{(1-x)^{5/2}}{\frac{5}{2}(-1)} + C$$

$$= -\frac{4}{3}(1-x)^{3/2} + \frac{2}{5}(1-x)^{5/2} + C.$$

Example 3. Integrate a^{3x+3} dx, $a \neq -1$

Solution. $I = \int a^{3x+3} dx = \int a^{3x} . a^3 dx$

$$= a^3 \int a^{3x} dx$$

$$= a^3 . \int e^{3x \log a} dx$$

$\because$ $e^{\log f(x)} = f(x)$

$\therefore$ $e^{\log a^{3x}} = a^{3x}$

Also $e^{\log a^{3x}} = e^{3x \log a}$

$\therefore$ $a^{3x} = e^{3x \log a}$

$$= a^3 \int e^{(3 \log a)x} dx$$

$$= a^3 . \frac{e^{(3\log a)x}}{3\log a} + C \quad \left[\because \int e^{kx} dx = \frac{e^{kx}}{k}\right]$$

$$= a^3 . \frac{e^{3x \log a}}{3 \log a} + C = \frac{a^3 . a^{3x}}{3 \log a} + C \qquad [\because e^{3x \log a} = a^{3x}]$$

$$= \frac{a^{3x+3}}{3 \log a} + C .$$

प्रतिस्थापन द्वारा समाकलन (Integration by Substitution)

अनेक फलनों का उचित प्रतिस्थापन द्वारा व्यापक रूपों में से किसी एक रूप में बदलकर छोटा किया जा सकता है तथा अब इस प्रकार के फलनों को सुगमता से समाकलन किया जा सकता है।

प्रतिस्थापन की विधि (Method of substitution)

यदि हमें प्रतिस्थापन विधि से ∫श्;गद्धकग का मान ज्ञात करना है तथा यदि ग त्र ;जिद्ध, जहाँ ज एक नया चर है, तो यह उचित प्रतिस्थापन है जो दिये हुए फलन श्;गद्ध को थ्क्ष्;जिद्धद्ध में बदल देता है। श्;गद्ध के साथ ही हमको नयी चर राशी ज के पदों में कग को भी बदलना होता है।

By substitution, many functions can be converted into smaller functions which can be integrated easily.

When we apply method of substitution for finding the value of $\int f(x)\, dx$ and if $x = f(t)$ where t is a new variable then f(x) is converted into F[f(t)] and also dy/dx.

Now $\quad x = f(t)$

$$\therefore \frac{dx}{dt} = f'(t) \qquad \text{or } dx = f'(t)\, dt .$$

Two important forms of integrals :

(i) $$\int \frac{f'(x)}{f(x)} dx = \log|f(x)| + C$$

(ii) $$\int [f(x)]^n . f'(x)\, dx = \frac{[f(x)]^{n+1}}{n+1} \quad \text{when } n \neq -1 .$$

Example 4. Evaluate the following :

(i) $\int \frac{2x+9}{x^2+9x+10} dx$ (ii) $\int \frac{6x-8}{3x^2-8x+5} dx$

(iii) $\int 3x^2 . e^{x^3} dx$ (iv) $\int \frac{e^{1/x^2}}{x^3} dx$

(v) $\int \frac{\log x}{x} dx$ (vi) $\int \frac{dx}{x \log_e x}$

(vii) $\int \frac{x^3}{\sqrt{1+x^3}} dx$ (viii) $\int \frac{1}{x+\sqrt{x}} dx$

Solution. (i) $I = \int \frac{2x+9}{x^2+9x+10} dx$

Put $x^2+9x+10 = t$

$$\therefore \quad 2x+9 = \frac{dt}{dx} \text{ or } (2x+9)\, dx = dt$$

$$\therefore \quad I = \int \frac{dt}{t} = \log|t| + C = \log|x^2+9x+10| + C$$

(ii) $I = \int \frac{6x-8}{3x^2-8x+5}dx$

Put $3x^2 - 8x + 5 = t$

$\therefore$ $6x-8 = \frac{dt}{dx}$ or $(6x-8)\,dx = dt$

$= \int \frac{dt}{t} = \log|t| + C = \log|3x^2-8x+5| + C.$

(iii) Put $x^3 = t$

$\therefore$ $\frac{dt}{dx} = 3x^2$ or $3x^2.dx = dt$

$\therefore$ $\int 3x^2.\, e^{x^3}\, dx = \int e^{x^3}\, 3x^2\, dx = \int e^t\, dt = e^t + C = e^{x^3} + C$

(iv) Let $\frac{1}{x^2} = t$ or $x^{-2} = t$

$\therefore$ $-\frac{2}{x^3}\, dx = dt$ or $\frac{1}{x^3}\, dx = -\frac{1}{2}dt$

$\therefore$ $\int \frac{e^{1/x^2}}{x^3}\, dx = \int e^{1/x^2}\, \frac{1}{x^3}\, dx = \int e^t\, (-\frac{1}{2}dt)$

$= -\frac{1}{2}\int e^t\, dt = -\frac{1}{2}e^t + C = -\frac{1}{2}e^{1/x^2} + C$

(v) Let $\log x = t$. So $(1/x)dx = dt$

$\therefore$ $\int \frac{\log x}{x}dx = \int \log x.\frac{1}{x}dx = \int t\, dt = \frac{t^2}{2} + C = \frac{(\log x^2)}{2} + C$

(vi) Let $\log_e x = t$, so $(1/x)dx = dt$

$\therefore$ $\int \frac{dx}{x\log_e x} = \int \frac{1}{\log_e x}.\frac{1}{x}dx = \int \frac{1}{t}dt$

$= \log_e t + C = \log_e(\log_e x) + C$

(vii) Let $1+x^3 = t^2$ or $x^3 = t^2-1$

Differentiating we get $3x^2\, dx = 2t\, dt$ or $x^2\, dx = \left(\frac{2}{3}\right)t\, dt$

$\therefore$ $\int \frac{x^3}{\sqrt{(1+x^3)}}dx = \int \frac{x^3}{\sqrt{(1+x^3)}}.x^2dx = \int \frac{t^2-1}{t}.\frac{2}{3}tdt$

$= \left(\frac{2}{3}\right)\int (t^2-1)\, dt = \left(\frac{2}{3}\right)(\frac{1}{3}t^3 - t) + C$

$= (\frac{2}{9})\,(1+x^3)^{3/2} - (\frac{2}{3})(1+x^3)^{1/2} + C$

(viii) $\int \frac{1}{x+\sqrt{x}}dx = \int \frac{1}{\sqrt{x}(\sqrt{x}+1)}dx$

Let $\sqrt{x} = t$, so

$$\frac{1}{2\sqrt{x}}dx = dt \quad \text{or} \quad \frac{1}{\sqrt{x}}dx = 2dt$$

$$\therefore \quad \int \frac{1}{x+\sqrt{x}}dx = 2\int \frac{7}{\sqrt{x}(\sqrt{x}+1)} = \int \frac{1}{t+1}dx$$

$$= 2\log(t+1) + C = 2\log\left(\sqrt{x}+1\right) + C.$$

Example 5. Integrate the following :

(i) $x\sqrt{x+2}$ (ii) $\dfrac{2+3x}{3+2x}$ (iii) $\dfrac{(x+1)(x+\log x)^2}{x}$

(iv) $\dfrac{1}{e^x-1}$

Solution. (i) $I = \int x\ \sqrt{x+2}dx$

Putting $x+2 = t \Rightarrow x = t-2$

$\therefore \quad dx = dt$

$$I = \int (t-2)\, t^{1/2}\, dt = \int t^{3/2}\, dt - 2\int t^{1/2}\, dt$$

$$= \frac{t^{5/2}}{\frac{5}{2}} - 2\frac{t^{3/2}}{\frac{3}{2}} + C = \frac{2}{5}t^{5/2} - \frac{4}{3}t^{3/2} + C$$

$$= \frac{2}{5}(x+2)^{5/2} - \frac{4}{3}(x+2)^{3/2} + C.$$

(ii) $I = \int \dfrac{2+3x}{3-2x}dx$

Let $t = 3-2x \Rightarrow 2x = 3-t \Rightarrow x = \dfrac{3-t}{2}$

$\therefore \quad dt = -2dx \Rightarrow dx = -\dfrac{dt}{2}$

$$I = -\frac{1}{2}\int \frac{2+3\left(\frac{3-t}{2}\right)}{t}dt$$

$$= -\frac{1}{2}\int \frac{\left(2+\frac{9}{2}-\frac{3}{2}t\right)}{t}\frac{dt}{2}$$

$$= -\int \frac{dt}{t} - \frac{9}{4}\int \frac{dt}{t} + \frac{3}{4}\int dt$$

$$= -\log|t| - \frac{9}{4}\log|t| + \frac{4}{3}t + C$$

$$= -\log|3-2x| - \frac{9}{4}\log|3-2x| + \frac{3}{4}(3-2x) + C$$

$$= \frac{3}{4}(3-2x) - \log|3-2x| - \frac{9}{4}\log|3-2x| + C$$

(iii) $I = \int \dfrac{(x+1)(x+\log x)^2}{x}dx$

Put $x+\log x = t$, $\therefore \left(1+\frac{1}{x}\right)dx = dt$

or $\left(\frac{x+1}{x}\right)dx = dt$

$\therefore \quad I = \int(x+\log x)^2 \left(\frac{x+1}{x}\right)dx = \int t^2\, dt$

$= \frac{t^3}{3} + C = \frac{1}{3}(x+\log x)^3 + C$.

(iv) $\int \frac{1}{e^x - 1}dx = \int \frac{e^x}{e^x(e^x - 1)}dx$

Let $e^x = t$, then on differentiation $e^x.dx = dt$

$\therefore \int \frac{1}{e^x - 1}dx = \int \frac{dt}{t(t-1)} = \int\left(\frac{1}{t-1} - \frac{1}{t}\right)dt$

$= \log(t-1) - \log t + C$

$= \log\left(\frac{t-1}{t}\right) = \log\left(\frac{e^x - 1}{e^x}\right) + C$

Exercise 4.1.

Q.1. Evaluate (i) $\int(4x^3+3x^2 - 2x+5)dx$ (ii) $\int\left(\sqrt{x} - \frac{1}{2}x + \frac{2}{\sqrt{x}}\right)dx$

(iii) $\int\left(\frac{x^4+1}{x^2}\right)dx$ (iv) $\int\left(x - \frac{1}{x}\right)^3 dx$

(v) $\int\left(2^x + \frac{1}{2}e^{-x} + \frac{4}{x} - \frac{1}{\sqrt[3]{x}}\right)dx$ (vi) $\int \frac{x}{x-3}dx$

(vii) $\int (e^{a \log x} + e^{x \log a})\, dx$ (viii) $\int \frac{1}{\sqrt{5x+3} + \sqrt{5x+2}}dx$

(ix) $\int\left(e^{3x} - 2e^x + \frac{1}{x}\right)dx$ (x) $\int \frac{(x^3+1)(x-2)}{x^2 - x - 2}dx$

(xi) $\int \frac{(a^x + b^x)^2}{a^x b^x}\, dx$ (xii) $\int \frac{1}{\sqrt{x+1} + \sqrt{x-1}}dx$

Q.2. Evaluation (i) $\int \frac{3x+5}{(3x^2+10x+2)^{2/3}}dx$ (ii) $\int \frac{\sqrt{2+\log x}}{x}dx$

(iii) $\int \frac{e^x - e^{-x}}{e^x + e^{-x}}dx$ (iv) $\int \frac{dx}{x^2 - a^2}$

(v) $\int x(x^2+4)^5\, dx$ (vi) $\int \frac{8x^2}{(x^3+2)^3}dx$

(vii) $\int \frac{x^3}{(x^2+1)^3}dx$ (viii) $\int \frac{x+2}{\sqrt{x^2+4x+5}}dx$

(ix) $\int \frac{(x+1)(x+\log x)^3}{3x}dx$ (x) $\int \frac{e^{x-1}+x^{e-1}}{e^x+x^e}dx$

(xi) $\int \frac{1}{(1+e^x)(1+e^{-x})}dx$ (xii) $\int (x+1).\, 2^{x^2+2x}\ dx$

Answers

1. (i) $x^4+x^3-x^2+5x+C$ (ii) $\frac{2}{3}x^{3/2}-\frac{1}{4}x^2+4\sqrt{x}+C$ (iii) $\frac{x^3}{3}-\frac{1}{x}+C$

(iv) $\frac{x^4}{4}-\frac{3}{2}x^2+3\log x+\frac{1}{2x^2}+C$ (v) $\frac{2^x}{\log 2}-\frac{1}{2}e^{-x}+4\log x-\frac{3}{2}x^{2/3}+C$

(vi) $x+\log|x-3|+C$ (vii) $\frac{x^{a+1}}{a+1}+\frac{a^x}{\log a}+C$

(viii) $\frac{2}{15}[(5x+3)^{3/2}-(5x+2)^{3/2}]+C$ (ix) $\frac{e^{3x}}{3}-2e^x+\log|x|+C$

(x) $\frac{x^3}{3}-\frac{x^2}{2}+x+C$ (xi) $\frac{\left(\frac{a}{b}\right)^x}{\log\frac{a}{b}}+2x+\frac{\left(\frac{b}{a}\right)^x}{\log\frac{b}{a}}+C$

(xii) $\frac{1}{3}(x+1)^{3/2}-\frac{1}{3}(x-1)^{3/2}+C$

2. (i) $\frac{3}{2}(3x^2+10x+2)^{1/3}+C$ (ii) $\frac{2}{3}(2+\log x)^{3/2}+C$ (iii) $\log|e^x+e^{-x}|+C$

(iv) $\frac{1}{2a}\log\left|\frac{x-a}{x+a}\right|+C$ (v) $\frac{1}{12}(x^2+4)^6$ (vi) $\frac{4}{3(x^2+2)^2}$

(vii) $-\frac{1}{4}.\frac{2x^2+1}{(x^2+1)^2}$ (viii) $\frac{1}{15}(1+x^6)^{5/2}+C$ (ix) $\frac{1}{12}(x+\log x)^4+C$

(x) $\frac{1}{e}\log|e^x+x^e|+C$ (xi) $-\frac{1}{1+e^x}+C$ (xii) $\frac{2^{x^2+2x}}{2\log_2}+C$

खण्डशः समाकलन ;पदजमहतंजपवद इल चंतजेद्ध

दो फलनों के गुणनफल का समाकल, खण्डशः समाकलन ;पदजमहतंजपवद इल चंतजेद्ध विधि से किया जा सकता है।

दो फलनों के गुणनफल का समाकल ज्ञात करना (To find the integral of the product of two functions)

If u and v be two functions of x, then

$$\frac{d}{dx}(uv) = u.\frac{dv}{dx} + v.\frac{du}{dx}$$

$$\Rightarrow \quad u.\frac{dv}{dx} = \frac{d}{dx}(uv) - v\frac{du}{dx}$$

Integrating both sides w.r.t x, we get

$$\int u.\frac{dv}{dx}dx = uv - \int v\frac{du}{dx}dx \qquad \ldots(1)$$

Let $u = f_1(x)$ and $\frac{dv}{dx} = f_2(x)$

Since $\frac{dv}{dx} = f_2(x)$, $\therefore \int f_2(x)dx = v$

Hence (1) becomes

$$\int f_1(x).\, f_2(x)dx = f_1(x)\int f_2(x)dx - \int [f_1'(x)\int f_2(x)dx]\, dx\ .$$

In words, this rule of integration by parts can be stated as :

Integral of the product of two functions

= First function × Integral of the second
– Integral of [diff. coeff. of the first × Integral of the second)

Integral of the product of two functkions

In finding integrals by this method proper choice of Ist and 2nd function is essential. Although there is no fixed law for taking Ist and 2nd function and their choice is possible by practice, yet following rule is helpful in the choice of functions 1st and 2nd.

(i) If the two functions are of different types take that function as Ist which comes first in the word ILATE.

Where I, stands for Inverse circular function.
L, stands for Logarithmic function.
A, stands for Algebraic function.
T, stands for Trigonometrical function.
and E, stands for Exponential function.

(ii) If both the functions are trigonometrical take that function as 2nd whose integral is simpler.

(iii) If both the functions are algebraic take that function as 1st whose d.c. is simpler.

(iv) Unity may be taken as one of the functions.

(v) The formula of integration by parts can be applied more than once if necessary.

Example 6. Evaluate $\int x^n \log x\, dx$

Solution. Let $I = \int x^n \log x = \int (\log x)\ .x^n\text{-}\ dx$

So $I = (\log x).\ \frac{x^{n+1}}{n+1} - \int \frac{1}{x}\frac{x^{n+1}}{x+1}dx$

$$= \frac{x^{n+1}.(\log x)}{n+1} - \frac{1}{n+1}\int x^n dx$$

$$= \frac{x^{n+1}.(\log x)}{n+1} - \frac{1}{n+1}\frac{x^{n+1}}{n+1} + C = \frac{x^{n+1}.\log x}{n+1} - \frac{x^{n+1}}{(n+1)^2} + C$$

Example 7. Evaluate $\int x\, e^x\, dx$

Solution. Let $I = \int x\, e^x\, dx$

[Here x is algebraic function and e^x is exponential function and A occurs before T in ILATE, therefore, we take x as 1st and e^x as 2nd functions].

$$I = \int x\, e^x\, dx = x \int e^x\, dx - \int\left(\frac{d}{dx}(x).\int e^x dx\right)dx$$

$$= x\, e^x - \int 1.e^x\, dx = x\, e^x - e^x + C = e^x\,(x-1) + C.$$

Example 8. Evaluate $\int x^3\, e^{-x}\, dx$

$$I = \int x^3\, e^{-x}\, dx = x^3\,(-e^{-x}) - \int 3x^2\,(-e^{-x})dx$$
$$= -x^3\, e^{-x} + 3\int x^2\, e^{-x}\, dx$$
$$= -x^3\, e^{-x} + 3[x^2(-e^{-x}) - \int 2x.(-e^{-x})\, dx]$$
$$= -x^3e^{-x} - 3x^2e^{-x} + 6\int x\, e^{-x}\, dx$$
$$= -x^3e^{-x} - 3x^2e^{-x} + 6\,[x.(-e^{-x}) - \int 1.(-e^{-x})dx]$$
$$= -x^3e^{-x} - 3x^2\, e^{-x} - 6\, e^{-x} + 6x\, e^{-x} + C$$
$$= -e^{-x}(x^3 + 3x^2 - 6x + 6) + C$$

Example 9. Integrate $x^3.e^{x^2}$

Solution. $I = \int x^3\; e^{x^2}\, dx.$

Put $x^2 = t$, $\therefore$ $2x\, dx = dt$ or $x\, dx = \frac{dt}{2}$

$$I = \int x^3\; e^{x^2}\, dx = \int x^2\; e^{x^2}\, x\, dx$$
$$= \int t\, e^t\, \frac{dt}{2} = \frac{1}{2}\int t\, e^t\, dt$$
$$= \frac{1}{2}[te^t - \int (1.e^t)\, dt]$$
$$= \frac{1}{2}te^t - \frac{1}{2}e^t + C = \frac{1}{2}x^2e^{x^2} - \frac{1}{2}e^{x^2} + C.$$

Example 10. Evaluate $\int x^3\, e^{ax}\, dx$

Solution. Let $I = x^2\, e^{ax}\, dx$

$$= x^2\left(\frac{e^{ax}}{a}\right) - \int 2x.\frac{e^{ax}}{a}dx$$
$$= \frac{x^a e^{ax}}{a} - \frac{2}{a}\left[x.\left(\frac{e^{ax}}{a}\right) - \int 1.\frac{e^{ax}}{a}dx\right]$$
$$= \frac{x^2 e^{ax}}{a} - \frac{2}{a}\left[x.\frac{e^{ax}}{a} - \frac{1}{a}e^{ax}dx\right]$$
$$= e^{ax}\left(\frac{x^2}{a} - \frac{2x}{a^2} + \frac{2}{a^2}\right) + C.$$

Example 11. Evaluate $\int \log x\, dx$

Solution. Let $I = \int \log x = \int (\log x).\, 1\, dx$.

Integrating by parts, taking log x as the 1st function

$= \log x\,(x) - \int \frac{1}{x}.\, x\, dx \;=\; x \log x - \int 1.\, dx$

$= x \log x - x + C \;=\; x(\log x - 1) + C.$

Example 12. Evaluate $\int (\log x)^2 .x\, dx$

Let $I = \int (\log x)^2 .\, x\, dx$

$$= (\log x)^2 .\, \frac{x^2}{2} - \int (2 \log x).\frac{1}{x} \times \frac{x^2}{2} dx$$

$$= \frac{x^2}{2}(\log x)^2 - \int (\log x).x\, dx$$

$$= \frac{x^2}{2}(\log x)^2 - \left[(\log x.\frac{x^2}{2} - \int \frac{1}{x}.\frac{x^2}{2} dx \right]$$

$$= \frac{x^2}{2}(\log x)^2 - \frac{x^2}{2}\log x + \frac{1}{2}\int x\, dx$$

$$= \frac{x^2}{2}\left[(\log x)^2 - \log x + \frac{1}{2} \right] + C$$

Example 13. Evaluate $\int e^x(1+x) \log (x\, e^x)\, dx$

Solution. $I = \int e^x\,(1+x) \log (x\, e^x)\, dx.$

Put $x\, e^x = t$, $\therefore$ $e^x(1+x)\, dx = dt$

$\therefore$ $I = \int \log t\, .\, 1\, dt$

$= \log t.\, (t) - \int \frac{1}{t}.t\, dt$

$= t \log t - \int 1.\, Dt \; = t \log t - t + C$

$= t\,(\log t - 1) + C = (x\, e^x)\, [\log (x\, e^x) - \log e] + C$

$= (x\, e^x) \log \left(\frac{xe^x}{e} \right) + C$

Example 14. Evaluate $\int \frac{\log x}{(x+1)^2} dx$

$I = \int \log x.\, \frac{1}{(x+1)^2} dx$.

Now integrating by parts, taking log x as first function

$$I = \log x\, .\, \frac{-1}{1+x} - \int \frac{1}{x}.\frac{-1}{1+x} dx = -\frac{\log .x}{1+x} dx = -\frac{\log x}{1+x} + \int \frac{1}{x(1+x)} dx$$

$$= -\frac{\log x}{1+x} + \int \left(\frac{1}{x} + \frac{1}{1+x} \right) dx$$

$$= -\frac{\log x}{1+x} + \log|x| - \log|1+x| + C$$

$$= -\frac{\log x}{1+x} + \log\left|\frac{x}{1+x}\right| + C .$$

vkaf'kd fHkUuks ls lekdyu (Integration by partial fractions)

Rational Function. An expression of the form $\frac{f(x)}{\phi(x)}$ where f(x) and $\phi(x)$ are rational integral algebraic functions or polynomials.

$f(x) = a_0x^m + a_1x^{m-1} + \ldots + a_{m-1}x + a_m$

$\phi(x) = b_0x^n + b_1x^{n-1} + \ldots + b_{n-1}x + b_n$.

Where m, n are positive integers and $a_0, a_1, a_2, \ldots, a_m, b_0, b_1, b_2, \ldots, b_n$ are constants is called a rational function or rational fraction. It is assumed that f(x) and $\phi(x)$ have no common factor.

e.g., $\frac{x+1}{x^2+x^2-6x}, \frac{x-1}{(x+1)(x^2+1)}$ are rational functions.

Such fractions can always be integrated by splitting the given fraction into partial fractions.

Note on Partial Fractions

1. Proper rational algebraic fraction. A proper rational algebraic fraction is a rational algebraic fraction in which the degree of the numerator is less than that of the denominator.

2. The degree of the numerator f(x) must be less than the degree of denominator $\phi(x)$ and if the degree of the numerator of a rational algebraic fraction is equal to or greater than, that of the denominator, we can divide the numerator by the denominator until the degree of the remainder is less than that of the denominator.

Then

Given fraction = a polynomial + a proper rational algebraic fraction.

For example, consider a rational algebraic fraction.

$$\frac{x^2}{(x-1)(x-2)} = \frac{x^2}{x^2-3x+2}$$

Hence the degree of the numerator is 3 and the degree of the denominator is 2. We divide numerator by denominator.

$x^2-3x+2)x^2$ $(x+3$

x^2-3x^2+2x

$-\ +\ -$

$3x^2-2x$

$+3x^2-9x+6$

$-\ \ +\ \ -$

$7x-6$

$$\therefore \quad \frac{x^2}{(x-1)(x-2)} = x+3+\frac{7x-6}{(x-1)(x-2)}$$

Working rule. (i) The degree of the numerator (x) must be less than the degree of denominator $\phi(x)$ and if not so, then divide f(x) by $\phi(x)$ till the remainder of a lower degree than $\phi(x)$.

(ii) Now break the denominator $\phi(x)$ into linear and quadratic factors.

(iii) (a) Corresponding to non-repeated linear factor of $(x-\alpha)$ type in the denominator $\phi(x)$. Put a partial fraction of the form $\frac{A}{x-\alpha}$.

Therefore, the partial fraction of $\frac{x^2}{(x+2)(x-4)(x-5)}$ are of the form

$$\frac{A}{x+2}+\frac{B}{x-4}+\frac{C}{x-5}$$

(b) Corresponding to non-repeated quadratic factor $(ax^2 + bx + c)$ of $\phi(x)$, partial fraction will be of the form

$$\frac{Ax+b}{ax^2+bx+c}$$

For example, the partial fraction of

$$\frac{2x-3}{(x-1)(x-4)^2(x^2-5x+10)} = \frac{A}{x-1}+\frac{B}{x-4}+\frac{C}{(x-4)^2}+\frac{D}{x^2-5x+10}$$

(c) Corresponding to a repeated quadratic factor of the form $(ax^2+b+c)^m$ in $\phi(x)$, there corresponds m partial fractions of the form

$$\frac{A_1x+B_1}{(ax^2+bx+c)}+\frac{A_2x+B_2}{(ax^2+bx+c)^2}+......+\frac{A_mx+B_m}{(ax^2+bx+c)^m}$$

Therefore the partial fractions of

$$\frac{3x-5}{(x+5)(x^2+7x+8)^2}=\frac{A}{x+5}+\frac{Bx+C}{x^2+7x+8}+\frac{Dx+E}{(x^2+7x+8)^2}$$

Thus we see that when we resolve the denominator $\phi(x)$ into real factors, they can be of four types :

(a) Linear non-repeated.
(b) Linear repeated.
(c) Quadratic non-repeated.
(d) Quadratic repeated.

The proper fraction $\frac{f(x)}{\phi(x)}$ is equal to the sum of partial fractions as suggested above. After this, multiply both sides by $\phi(x)$. The relation, we get will be an identity. So the values of the constants of R.H.S. will be obtained by equating the coefficients of like powers of x, and then solving the equation so obtained. Sometimes we can get the values of constants by some short cut methods i.e., by giving certain values to x etc.

Example 15. Evaluate the following :

(i) $\int \frac{3x+2}{(x-2)(2x+3)}dx$ (ii) $\int \frac{3x-1}{(2x+1)(3x+2)(6x-1)}dx$

Solution. (i) Let $\frac{3x+2}{(x-2)(2x+3)}=\frac{A}{x-2}+\frac{B}{2x+3}$

Multiplying both sides by (x–2) (2x+3)

$3x+2 = A(2x+3) + B(x-2)$

Put $x = -\frac{3}{2}$, $3 \times -\frac{3}{2} + 2 = A(-3+3) + B(-\frac{3}{2} - 2)$

or $-\frac{5}{2} = -\frac{7}{2}B$ or $B = \frac{5}{2} \times \frac{2}{7} = \frac{5}{7}$

Put x= 2 $\quad 3 \times 2 + 2 = A(2 \times 2 + 3) + B(2-1)$

$8 = 7A$ or $A = \frac{8}{7}$

$$\therefore \frac{3x+2}{(x-2)(x-3)} = \frac{8}{7(x-2)} + \frac{5}{7(2x+3)}$$

$$\therefore \int \frac{3x+2}{(x-2)(2x+3)} dx = \frac{8}{7}\int \frac{1}{x-2} dx + \frac{5}{7}\int \frac{1}{2x+3} dx$$

$$= \frac{8}{7}\log|x-2| + \frac{5}{7}\log|2x+3| + c$$

(ii) Let $\frac{3x-1}{(2x+1)(3x+2)(6x-5)} = \frac{A}{2x+1} + \frac{B}{3x+2} + \frac{C}{6x-5}$

Multiplying both sides by (2x+1) (3x+2) (6x–5)

$(3x-1) = A(3x+2)(6x-5)+B(2x+1)(6x-5) + C(2x+1(3x+2)$

Put $x = -\frac{1}{2}, -\frac{3}{2} - 1 = A\left(-\frac{3}{2} + 2\right)(-3-5)$

or $-\frac{5}{2} = -4A \Rightarrow A = \frac{5}{8}$

Put $x = -\frac{2}{3}$, $-2-1 = B\left(-\frac{4}{3} + 1\right)(-4-5)$

or $-3 = 3B \Rightarrow B = -5$

Put $x = \frac{5}{6}$ $\quad \frac{5}{2} - 1 = C\left(\frac{5}{3} + 1\right)\left(\frac{5}{2} + 2\right)$

$\frac{3}{2} = 12C \Rightarrow C = \frac{1}{8}$

$$\therefore \qquad \frac{3x-1}{(2x+1)(3x+2)(6x-5)} = \frac{5}{8(2x+1)} - \frac{1}{3x+2} + \frac{1}{8(6x-5)}$$

$$\therefore \qquad I = \frac{5}{8}\int \frac{dx}{2x+1} - \int \frac{dx}{3x+2} + \frac{1}{8}\int \frac{dx}{6x-5}$$

$$= \frac{5}{8} \cdot \frac{\log|2x+1|}{2} - \frac{\log|3x+2|}{3} + \frac{1}{8} \cdot \frac{\log|6x-5|}{6}$$

$$= \frac{5}{16}\log|2x+1| - \frac{1}{3}\log|3x+2| + \frac{1}{48}\log|6x-5| + C \ .$$

Example 11. Evaluate

(i) $\frac{17x-2}{4x^2+7x-2}dx$ (ii) $\int \frac{dx}{x-x^3}dx$

Solution. (i) $\frac{17x-2}{4x^2+7x-2} = \frac{17x-2}{(x+2)(4x-1)} = \frac{A}{x+2} + \frac{B}{4x-1}$

$17x-2 = A(4x-1) + B(x+2)$

Put $x = \frac{1}{4}$

$\therefore \quad \frac{17}{4} - 2 = B\left(\frac{1}{4}+2\right)$

$\frac{9}{4} = B\left(\frac{9}{4}\right)$, or $B = 1$.

Put $x = -2$

$-34-2 = A(-8-1)$ or $A = 4$.

$\therefore \quad \frac{17x-2}{4x^2+7x-2} = \frac{4}{x+2} + \frac{1}{4x-1}$

$\therefore \quad \int \frac{17x-2}{4x^2+7x-2}\, dx = 4\int \frac{dx}{x+2} + \int \frac{dx}{4x-1}$

$= 4 \log |x+2| + \frac{1}{4} \log |4x-1| + c.$

(iii) $\int \frac{dx}{x-x^2} = \int \frac{dx}{x(1-x^2)} = \int \frac{dx}{x(1-x)(1+x)}$

Let $\frac{1}{x(1-x)(1+x)} = \frac{A}{x} + \frac{B}{1-x} + \frac{C}{1+x}$ …(i)

Multiplying both sides by $x(1-x)(1+x)$, we get

$1 = A(1-x)(1+x)+Bx(1+x)+Cx(1-x)$ …(ii)

Putting $x= 0$, 1 and -1 in (ii), we get

$1 = A, \therefore A = 1$ and $1 = 2B \quad \therefore \quad B = \frac{1}{2}$

and $1 = -2C \quad \therefore \quad C = \frac{-1}{2}$.

Putting these values of A, B and C in (i) , we get

$\frac{1}{x(1-x)(1+x)} = \frac{1}{x} + \frac{1}{2}\times\frac{1}{1-x} - \frac{1}{2}\times\frac{1}{1+x}$

$\therefore \quad \int \frac{dx}{x-x^2} = \int \left[\frac{1}{x} + \frac{1}{2(1-x)} - \frac{1}{2(1+x)}\right] dx$

$= \log |x| - \frac{1}{2}\log|1-x| - \frac{1}{2}\log|1+x| + c$

$= \frac{1}{2}[2\log|x| - \log|1-x| - \log|1+x| + c$

$$= \frac{1}{2}\left[\log\left|\frac{x^2}{(1-x)(1+x)}\right|\right] + c = \frac{1}{2}\log\left|\frac{x^2}{1-x^2}\right| + c.$$

Example 12. Evaluate (i) $\frac{dx}{1+3e^x+2e^{2x}}$ (ii) $\int \frac{dx}{x[6(\log x)^2+7\log x+2}$

(i) Put $e^x = t$, $\therefore$ $e^x\,dx = dt$

$$\therefore\ I = \int \frac{dt}{e^x(1+3t+2t^3)} = \int \frac{dt}{t(2t+1)(t+1)}$$

Now $$\frac{1}{t(2t+1)(t+1)} = \frac{1}{t} + \frac{1}{t+1} - \frac{4}{2t+1}$$

$$\therefore\ I = \int \frac{dt}{t} + \int \frac{dt}{t} - \int \frac{4}{2t+1}dt$$

$$= \log|t| + \log|t+1| - 4 \times \frac{1}{2} \log|2t+1| + c$$

$$= \log|e^x| + \log|e^x+1| - 2\log|2e^x+1| + c$$

$$= x + \log|e^x+1| - 2\log|1|2e^{2x}| + c.$$

(ii) $$\int \frac{dx}{x[6(\log x)^2 7\log x+2]}$$

Put $\log x = t$, then $\frac{1}{x}dx = dt$

$$I = \int \frac{dt}{6t^2+7t+2} = \int \frac{dt}{(2t+1)(3t+2)}$$

$$= \int \left(\frac{2}{2t+1} - \frac{3}{3t+2}\right)dt \qquad \text{[By Partial Fraction]}$$

$$= 2\int \frac{dt}{2t+1} - 3\int \frac{dt}{3t+2}$$

$$= 2 \times \frac{1}{2}\log|2t+1| - 3 \times \frac{1}{2}\log|3t+2| + c$$

$$= \log\frac{2t+1}{3t+2} + c = \log\left|\frac{2\log x+1}{3\log x+2}\right| + c.$$

Exercise 4.2

1. Evaluate : (i) $\int x^2 e^{3x}dx$ (ii) $\int x^n \log x\,dx$

(iii) $\int \frac{xe^x}{(x+1)^2}dx$ (iv) $\int(\log x)\,dx$

(v) $\int \sqrt{4x^2-9}\,dx$ (vi) $\int \frac{dx}{(x+1)\sqrt{x+2}}$

(vii) $\int \frac{dx}{(x+1)\sqrt{x^2+x+1}}$ (viii) $\int \frac{dx}{(x^2-1)\sqrt{x^2+1}}$

(ix) $\int x \log(1+x)\, dx$ (x) $\int x^3 a^{x^2} dx$

2. Evaluate : (i) $\int \frac{x}{(x-1)(x-2)}dx$ (ii) $\int \frac{2x}{(x^2+1)(x^2+3)}dx$

(iii) $\int \frac{3x+5}{x^2-x^3-x^2+1}dx$ (iv) $\int \frac{x^2+1}{(2x+1)(x+1)(x-1)}dx$

(v) $\int \frac{26x+6}{8-10x-3x^2}dx$ (vi) $\int \frac{2x^3-3x^2-9x+1}{2x^2-x-10}dx$

(vii) $\int \frac{dx}{(x+1)^2)(x^2+1)}$ (viii) $\int \frac{dx}{(e^x-1)^2}$

(ix) $\int \frac{x^2+x+1}{(x-3)^3}$ (x) $\int \frac{9x^2+bx+c}{(x-a)(x-b)(x-c)}dx$

Answers

1. (i) $\frac{x^2e^{3x}}{3}-\frac{2xe^{3x}}{9}+\frac{2}{27}e^{3x}$ (ii) $\log x. \frac{x^{n+1}}{n+1}-\frac{x^{n+1}}{(n+1)^2}$ (iii) $\frac{1}{x+1}.e^x$

(iv) $x(\log x)^2 - 2x \log x + 2x$ (v) $\frac{\sqrt{4x^2-9}}{2}-\frac{9}{4}\log|2x+\sqrt{4x^2-9}|+c$

(vi) $\log\left|\frac{\sqrt{x+2}-1}{\sqrt{x+2}+1}\right| + c$ (vii) $1-\log\left|\frac{1}{x+1}-\frac{1}{2}+\frac{\sqrt{x^2+x+1}}{x+1}\right|+c$

(viii) $-\frac{1}{2\sqrt{2}}\log\left|\frac{\sqrt{2x}+\sqrt{x^2+1}}{\sqrt{2x}-\sqrt{x^2+1}}\right| + c$ (ix) $\frac{1}{2}(x^2-1)\log(1+x)-\frac{1}{4}x^2+\frac{1}{2}x+c$

(x) $\frac{x^2a^{x^2}}{2\log a}-\frac{a^{x^2}}{2(\log a)^2}+c$

2. (i) $-\log|x-1| + 2\log|x-2| + c$ (ii) $\frac{1}{2}\log\left|\frac{x^2+1}{x^2+3}\right| + c$

(iii) $\frac{1}{2}\log\left|\frac{x+1}{x-1}\right|-\frac{4}{x-1}+c$ (iv) $-\frac{5}{6}\log|2x+1|+\frac{1}{3}\log|x-1|+\log|x+1|+c$

(v) $-\frac{5}{3}\log|3x-2|-7\log+x+4|+c$ (vi) $\frac{x^2}{2}-x+\log\left|\frac{x+2}{2x-5}\right|+c$

(vii) $\frac{1}{2}\log|x+1| - \frac{1}{2(x+1)} - \frac{1}{4}\log|x^2+1| + c$

(viii) $\log\left|\frac{e^x}{e^x-1}\right| - \frac{1}{e^x-1} + c$ (ix) $\log|x-3| - \frac{7}{x-3} - \frac{13}{2(x-3)^2} + c$

(x) $a^3 + ab + c \log|x-A| + \frac{ab^2+b^2+c}{(b-a)(b-a)}\log(x-b) + \frac{c(ac+b+1)}{(c-a)(c-b)} \log|x-c| + k.$

Chapter-5

निश्चित समाकलन तथा क्षेत्रफल
(Definite Integral and Area)

निश्चित समाकलन ;क्मपिदपजम प्दजमहतंसेद्ध

कभी–कभी समाकलन गणित के ज्यामिति तथा दूसरी शाखाओं में एक चर के दो दिये हुए मानों ;मानाद्ध ं और इ के लिए एक फलन ;िगद्ध के समाकल मानो में अन्तर ज्ञात करना आवश्यक हो जाता है। इस अन्तर को ;िगद्ध का निश्चित समाकल ;कमपिदपजम पदजमहतंसद्ध सीमाओं ं व इ के बीच या ं से इ तक कहा जाता है।

इस निश्चित समाकल को दर्शाया जाता है

$$\int_a^b f(x)dx$$

तथा इसको पढ़ा जाता है श्सीमाओं ं तथा इ के बीच ग के सापेक्ष ;िगद्ध का समाकालश।

चूंकि, हम जानते हैं कि यदि $\int f(x)dx = F(x)$, vr%

$$\int_a^b f(x)dx = [F(x)]_a^b = F(b) - F(a),$$

tgk¡ a rFkk b Øe'k% fuEu rFkk mPp lhek,a dgykrh gSaA

Sometimes, in geometry and other branches of integral calculus, it becomes necessary to find the differences in two values (say a and b) of a variable x for integral values of function f(x). This difference is called definite integral of f(x) within limits a and b or b and a.

This definite integral is shown as follows :

$$\int_a^b f(x)dx$$

and is read as integration of f(x) between limits a and b. As we know that if $\int f(x)dx = F(x)$

So $\int_a^b f(x)dx = [F(x)]_a^b = F(b) - F(a),$

where a and b are called lower and upper limits.

निश्चित समाकल के गुणधर्म (General Properties of Definite Integral)

Property 1. $\int_a^b f(x)dx = \int_a^b f(t)dt$

Property 2. $\int_a^b f(x)dx = -\int_b^a f(x)dx$

Property 3. $\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$,
where $a < c < b$.

Property 4. $\int_0^a f(x)dx = \int_0^b f(a-x)dx$.

Property 5. $\int_{-a}^a f(x)dx = 0$ if f(x) is an odd function of x

$= 2\int_0^a f(x)dx$ if f(x) is an even function of x

Note : (i) f(x) is called odd function
if $f(-x) = -f(x)$

(ii) f(x) is called even function
if $f(-x) = f(x)$

Property 6. $\int_0^{2a} f(x)dx = \int_0^a f(x)dx + \int_0^a f(2a-x)dx$

Example 1. Find the values of

(i) $\int_0^1 x^2 dx$ (ii) $\int_{-1}^2 (3x-1)(2x+1)dx$ (iii) $\int_{-1}^1 (x+1)dx$

(iv) $\int_2^3 \frac{dx}{x^2-1}$ (v) $\int_0^4 (t^2+1)dt$ (vi) $\int_0^4 (\sqrt{x} - -2x + x^2)dx$

(vii) $\int_a^b \frac{1}{x} dx$ (viii) $\int_0^2 \frac{e^x - e^{-x}}{5} dx$ (ix) $\int_0^1 e^{2x}(e^{2x}+3)dx$

(x) $\int_0^1 \frac{1-x}{1+x} dx$ (xi) $\int_0^1 \frac{dx}{\sqrt{x+1}+\sqrt{x}}$ (xii) $\int_0^1 \frac{dx}{[ax+b)(1-x)]^2}$

Solution. (i) $I = \int_0^1 x^2 dx = \left[\frac{x^3}{3}\right]_0^1 = \frac{1}{3}[x^3]_0^1 = \frac{1}{3}[1^3 - 0] = \frac{1}{3}$

(ii) $I = \int_{-1}^2 (3x-1)(2x+1)dx = \int_{-1}^2 (6x^2+x-1)dx$

$= 6\int_{-1}^2 x^2 dx + \int_{-1}^2 x dx - \int_{-1}^2 dx = 6\left[\frac{x^3}{3}\right]_{-1}^2 + \left[\frac{x^2}{2}\right]_{-1}^2 + [x]_{-1}^2$

$= 2[x^3]_{-1}^2 + \frac{1}{2}[x^2]_{-1}^2 - [x]_{-1}^2$

$= 2[2^3 - (-1)^3] + \frac{1}{2}[2^2 - (-1)^2] - [2-(-1)]$

$= 2(9) + \frac{1}{2}(3) - (3) = 16\frac{1}{2}$

(iii) $I = \int_{-1}^1 (x+1)dx = \left[\frac{x^2}{2} + x\right]_{-1}^1 = \left[\frac{1}{2}+1\right] - \left[\frac{(-1)^2}{2} - 1\right]$

$= \frac{3}{2} - \left(-\frac{1}{2}\right) = \frac{3}{2} + \frac{1}{2} = 2$

(iv) $I = \int_2^3 \frac{dx}{x^2-1} = \int_2^3 \frac{dx}{x^2-1^2}$

$= \left[\frac{1}{2\times1}\log\left|\frac{x-1}{x+1}\right|\right]_2^3 = \frac{1}{2}\left[\log\frac{2}{4} - \log\frac{1}{3}\right]$

$= \frac{1}{2}\left(\log\frac{1}{2} - \log\frac{1}{3}\right) = \frac{1}{2}\log\left(\frac{\frac{1}{2}}{\frac{1}{3}}\right)$

$= \frac{1}{2}\log\left(\frac{1}{2}\times\frac{3}{1}\right) = \frac{1}{2}\log\frac{3}{2}$.

(v) $I = \int_0^4 (t^2+1)dt = \left|\left(\frac{t^3}{3} + t\right)\right|_0^4$

$= \left(\frac{4^3}{3} + 4\right) - 0 = \frac{64}{3} + 4 = \frac{76}{3} = 25\frac{1}{3}$

(vi) $I = \int_0^4 (\sqrt{x} - 2x + x^2)dx = \int_0^4 (x^{1/2} - 2x + x^2)dx$

$= \left|\frac{2}{3}.x^{3/2} - x^2 + \frac{1}{3}x^3\right|_0^4 = \left(\frac{2}{3}.4^{3/2} - 4^2 + \frac{1}{3}.4^3\right) - 0$

$$= \frac{2}{3}\times 8 - 16 + \frac{64}{3} = \frac{32}{3}$$

(vii) $I = \int_a^b \frac{1}{x} dx = [\log |x|]_a^b$

$$= \log b - \log a = \log\left(\frac{b}{a}\right)$$

(viii) $I = \int_0^2 \frac{e^x - e^{-x}}{5} dx = \frac{1}{5}\int_0^2 (e^x - e^{-x})\, dx$

$$= \frac{1}{5}\left[e^x - \frac{e^{-x}}{-1}\right]_0^2$$

$$= \frac{1}{5}\left[e^x + e^{-x}\right]_0^2 = \frac{1}{5}\left[(e^2+e^{-2}) - (e^0+e^0)\right]$$

$$= \frac{1}{5}(e^2+e^{-2}-2) = \frac{1}{5}\left(e^2 + \frac{1}{e^2} - 2\right)$$

$$= \frac{1}{5}\left(e - \frac{1}{e}\right)^2$$

(ix) $I = \int_0^1 e^{2x}(e^{2x}+3)\, dx = \int_0^1 (e^{4x} + 3e^{2x})dx$

$$= \left|\frac{e^{4x}}{4} + 3.\frac{e^{2x}}{2}\right|_0^1 = \left(\frac{e^4}{4} + \frac{3}{2}e^2\right) - \left(\frac{e^0}{4} + \frac{3}{2}e^0\right)$$

$$= \frac{1}{4}e^4 + \frac{3}{2}e^2 - \left(\frac{1}{4} + \frac{3}{2}\right) = \frac{1}{4}e^4 + \frac{3}{2}e^2 - \frac{7}{4}$$

(x) $I = \int_0^1 \left(\frac{1-x}{1+x}\right)dx = \int_0^1 \frac{2-(1+x)}{1+x}dx$

$$= \int_0^1 \left(\frac{2}{1+x} - 1\right)dx = |2\log(1+x) - x|_0^1$$

$$= (2\log 2 - 1) - (2\log 1 - 0) = 2\log 2 - 1$$

[$\because$ $\log 1 = 0$]

(xi) $I = \int_0^1 \frac{dx}{\sqrt{x+1} + \sqrt{x}}$

Now $\int \frac{dx}{\sqrt{x+1}+\sqrt{x}} = \int \frac{\sqrt{x+1}-\sqrt{x}}{(\sqrt{x+1}+\sqrt{x})(\sqrt{x+1}-\sqrt{x})}dx$

$$= \int \frac{\sqrt{x+1}-\sqrt{x}}{x+1-x}dx = \int (\sqrt{x+1} - \sqrt{x})\, dx$$

$$= \frac{2}{3}(x+1)^{3/2} - \frac{2}{3}x^{3/2}$$

$\because$ $\int_0^1 \frac{dx}{\sqrt{x+1}+\sqrt{x}} = \frac{2}{3}|(x+1)^{3/2} - x^{3/2}|_0^1$

$$= \frac{2}{3}[2^{3/2} - 1^{3/2}) - (1^{3/2} - 0)]$$

$$= \frac{2}{3}[(2^3)^{1/2}-2] = \frac{2}{3}(2\sqrt{2}-2) = \frac{4}{3}(\sqrt{2}-1).$$

(xii) $I = \int_0^1 \frac{dx}{[ax + b(1-x)]^2} = \int_0^1 \frac{dx}{[(a-b)x + b]^2}$

$$= \int_0^1 [(a-b)x+b]^{-2} dx = \left[\frac{[(a-b)x+b]^{-1}}{-1\times(a-b)}\right]_0^1$$

$$= \frac{1}{(b-a)}\left[\frac{1}{(a-b)x+b}\right]_0^1 = \frac{1}{(b-a)}\left[\frac{1}{a-b+b} - \frac{1}{b}\right]$$

$$= \frac{1}{(b-a)}\left(\frac{1}{a} - \frac{1}{b}\right) = \frac{1}{(b-a)}\left(\frac{b-a}{ab}\right) = \frac{1}{ab}$$

Example 2. If $\int_0^a 3x^2 dx = 8$, find the value of a.

Solution. $\int_0^a 3x^2 dx = 3\int_0^a x^2 dx$

$$= 3\left[\frac{x^3}{3}\right]_0^a = [a^3-0] = a^3$$

Since $\int_0^a 3x^2 dx = 8$ (Given)

$\Rightarrow a^2 = 8$

$\therefore$ $a = (8)^{1/3} = 2.$

Example 3. Show that when f(x) is of the form

$a + b x + c x^2$

$$\int_0^1 f(x)\, dx = \frac{1}{6}\left[f(0) + 4f\left(\frac{1}{2}\right) + f(1)\right]$$

Solution. $f(x) = a + bx + cx^2$

$$f(0) = a,\ f\left(\frac{1}{2}\right) = a+b\times\frac{1}{2}+c\times\frac{1}{4} = a + \frac{1}{2}b+\frac{1}{4}c$$

$$f(1) = a+b+c$$

$$\text{R.H.S.} = \frac{1}{6}[f(0) + 4f\left(\frac{1}{2}\right)+f(1)]$$

$$= \frac{1}{6}[a+4\times(a+\frac{1}{2}b+\frac{1}{4}c) + a + b + c]$$

$$= \frac{1}{6}[6a + 3b + 2c] = a+\frac{b}{2}+\frac{c}{3}$$

$$\text{L.H.S.} = \int_0^1 f(x)\, dx = \int_0^1 (a+bx + cx^2)\, dx$$

$$= \left|ax + \frac{bx^2}{2}+\frac{cx^3}{3}\right|_0^1 = a + \frac{b}{2}+\frac{c}{3}$$

Hence L.H.S. = R.H.S.

Example 4. Evaluate the following definite integrals

(i) $\int_0^{1/2} \frac{x}{\sqrt{1-x^2}} dx$ (ii) $\int_1^2 3x\sqrt{5-x^2} dx$ (iii) $\int_1^2 x\sqrt[3]{x-4} dx$

(iv) $\int_a^b \frac{\log x}{x} dx$ (v) $\int_0^2 \frac{dx}{4+x-x^2}$ (vi) $\int_8^{15} \frac{dx}{(x-3)\sqrt{x+1}}$

Solution. (i) $I = \int_0^{1/2} \frac{x}{\sqrt{1-x^2}} dx$

Put $1-x^2 = t$, $\therefore -2x dx = dt$ or $x\,dx = -\frac{1}{2} dt$

when $x = 0, t = 1$

when $x = \frac{1}{2}, t = 1 - \frac{1}{4} = \frac{3}{4}$

$$\therefore \quad I = \int_0^{1/2} \frac{x}{\sqrt{1-x^2}} dx = \int_1^{3/4} \frac{-\frac{1}{2}dt}{\sqrt{t}} = -\frac{1}{2}\int_1^{3/4} t^{-1/2} dt$$

$$= -\frac{1}{2} \cdot \left[\frac{t^{1/2}}{\frac{1}{2}}\right]_1^{3/4} = -\left[\sqrt{\frac{3}{4}} - 1\right] = 1 - \frac{\sqrt{3}}{2}$$

(ii) $I = \int_1^2 3x\sqrt{5-x^2} dx$

Put $5-x^2 = t$ $\therefore -2x\,dx = dt$ or $x\,dx = -\frac{1}{2}dt$

when $x = 1, t = 5-1 = 4$, when $x = 2, t = 5-4 = 1$

$$\therefore \quad I = \int_1^2 3x\sqrt{5-x^2} dx = \int_4^1 -\frac{3}{2}\sqrt{t}dt = -\frac{3}{2}\int_4^1 t^{1/2}$$

$$= -\frac{3}{2}\left[\frac{t^{3/2}}{\frac{3}{2}}\right]_4^1 = -\frac{3}{2} \times \frac{2}{3}\left[t^{3/2}\right]_4^1 = [1 - 4^{3/2}] .$$

$$= -(1-8) = 7$$

(iii) $I = \int_4^8 x\sqrt[3]{x-4} dx = \int_4^8 x(x-4)^{1/3} dx$

Put $x-4 = t$, $\therefore dx = dt$

when $x = 4$, $t = 0$ and when $x = 8, t = 4$

$$\therefore \quad I = \int_0^4 (t+4)t^{1/3}dt = \int_0^4 t^{4/3}dt + 4\int_0^4 t^{1/3} dt$$

$$= \frac{3}{7}\left|t^{7/3}\right|_0^4 + 4 \times \frac{3}{4}\left|t^{4/3}\right|_0^4 = \frac{3}{7}[(4)^{7/3} - 0] + 3[(4)^{4/3} - 0]$$

$$= \frac{3}{7}(4)^{7/3} + 3(4)^{4/3} = (4)^{4/3}\left[\frac{3}{7} \times 4 + 3\right] = \left(\frac{33}{7}\right) \times 4 \times (4)^{1/3}$$

$$= \frac{132}{7}(4)^{1/3} .$$

(iv) $I = \int_a^b \frac{\log x}{x} dx$

Put $\log x = t$, $\therefore \frac{1}{x}dx = dt$

when $x = a, \; t = \log a$

when $x = b, \; t = \log b$

$$\therefore \quad I = \int_{\log a}^{\log b} t\,dt = \left|\frac{t^2}{2}\right|_{\log a}^{\log b} = \frac{1}{2}[(\log b)^2 - (\log a)^2]$$

$$= \frac{1}{2}(\log b + \log a)(\log b - \log a) = \frac{1}{2}\log(ab)\log\left(\frac{b}{a}\right)$$

(v) $I = \int_0^2 \frac{dx}{x+4-x^2} = \int_0^9 \frac{dx}{4-(x^2-x)}$

$$= \int_0^2 \frac{dx}{4-(x-\frac{1}{2})^2+\frac{1}{4}}$$

$$= \int_0^2 \frac{dx}{17/4-\left(x-\frac{1}{2}\right)^2} = \int_0^2 \frac{dx}{\left(\frac{\sqrt{17}}{2}\right)^2-\left(x-\frac{1}{2}\right)^2}$$

$$= \frac{1}{2\sqrt{\frac{17}{2}}}\left[\log\left(\frac{x-\frac{1}{2}+\sqrt{\frac{17}{2}}}{-x+\frac{1}{2}+\sqrt{\frac{17}{2}}}\right)\right]_0^2$$

$$= \frac{1}{\sqrt{17}}\left[\log\left(\frac{2-\frac{1}{2}+\sqrt{\frac{17}{2}}}{-2+\frac{1}{2}+\sqrt{\frac{17}{2}}}\right)-\log\left(\frac{-\frac{1}{2}+\sqrt{\frac{17}{2}}}{\frac{1}{2}+\sqrt{\frac{17}{2}}}\right)\right]$$

$$= \frac{1}{\sqrt{17}}\left\{\log\left(\frac{\sqrt{17}+3}{\sqrt{17}-3}\right)-\log\left(\frac{\sqrt{17}-1}{\sqrt{17}+1}\right)\right\}$$

$$= \frac{1}{\sqrt{17}}\left\{\log\left(\frac{\sqrt{17}+3}{\sqrt{17}-3}\right)\div\left(\frac{\sqrt{17}-1}{\sqrt{17}+1}\right)\right\}$$

$$= \frac{1}{\sqrt{17}}\log\left(\frac{\sqrt{17}+3}{\sqrt{17}-3}\right)\times\left(\frac{\sqrt{17}+1}{\sqrt{17}-1}\right)$$

$$= \frac{1}{\sqrt{17}}\log\left(\frac{17+3+4\sqrt{17}}{17+3-4\sqrt{17}}\right) = \frac{1}{\sqrt{17}}\log\left(\frac{20+4\sqrt{17}}{20-4\sqrt{17}}\right)$$

$$= \frac{1}{\sqrt{17}}\log\frac{4(5+\sqrt{17})}{4(5-\sqrt{17})} = \frac{1}{\sqrt{17}}\log\left(\frac{5+\sqrt{17}}{5-\sqrt{17}}\times\frac{5+\sqrt{17}}{5+\sqrt{17}}\right)$$

$$= \frac{1}{\sqrt{17}}\log\left(\frac{25+5\sqrt{17}+5\sqrt{17}+17}{25-17}\right)$$

$$= \frac{1}{\sqrt{17}}\log\left(\frac{42+10\sqrt{17}}{8}\right)$$

$$= \frac{1}{\sqrt{17}}\log\left(\frac{21+5\sqrt{17}}{4}\right)$$

(vi) $I = \int_8^{15}\frac{dx}{(x-3)\sqrt{x+1}}$ Put $x + 1 = t^2$, then $x = t^2 - 1$, $\therefore$ $dx = 2t\,dt$

when $x = 8$, $t^2 = 9$, $\therefore$ $t = 3$
when $x = 15$, $t^2 = 16$, $\therefore$ $t = 4$

$$I = 2\int_3^4\frac{dt}{t^2-2^2} = 2.\frac{1}{2(2)}\left[\log\left|\frac{t-2}{t+2}\right|_3^4\right]$$

$$= \frac{1}{2}\left(\log\left|\frac{4-2}{4+2}\right| - \log\left|\frac{3-2}{3+2}\right|\right)$$

$$= \frac{1}{2}\left(\log\frac{2}{6} - \log\frac{1}{5}\right) = 2\left(\log\frac{1}{3} - \log\frac{1}{5}\right)$$

$$= 2\log\left(\frac{1/3}{1/5}\right) = 2\log\frac{5}{3}$$

Example 5. Evaluate the following

(i) $\int_0^1 x^2 e^{2x}\,dx$ (ii) $\int_0^1 (x-2)(2x+3)\,e^x\,dx$

(iii) $\int_2^4 \frac{x^2+x}{\sqrt{2x+1}}dx$ (iv) $\int_1^e \frac{e^x}{x}(1+x\log x)\,dx$

Solution. (i) **I =** $\int_0^1 x^2 e^{2x}\,dx$

$$= \left|x^2\left(\frac{e^{2x}}{2}\right)\right|_0^1 - \int_0^1 (2x).\left(\frac{1}{2}e^{2x}\right)dx$$

$$= \frac{1}{2}\,|\,x^2e^{2x}\,|_0^1 - \int_0^1 xe^{2x}dx$$

$$= \frac{1}{2}(e^2-0) - \left[\left|\frac{xe^{2x}}{2}\right|_0^1 - \int_0^1 1.\left(\frac{e^{2x}}{2}\right)dx\right]$$

$$= \frac{1}{2}e^2 - \left[\frac{1}{2}e^2 - \frac{1}{2}\int_0^1 e^{2x}dx\right] = \frac{1}{2}\int_0^1 e^{2x}dx$$

$$= \frac{1}{2}\left|\frac{1}{2}e^{2x}\right|_0^1 = \frac{1}{4}(e^2-1).$$

(ii) $I = \int_0^1 (x-2)(2x+3)\,e^x\,dx$

$= \int_0^1 (2x^2 - x - 6)\, e^x\, dx.$

Integrating by parts, we get

$= \left[(2x^2 - x - 6)e^x\right]_0^1 - \int_0^1 (4x-1)\, e^x\, dx$

$= (2-1-6)\, e - (-6) - \int_0^1 (4x-1)\, e^x\, dx$

$= -5e + 6 - \left[\left|(4x-1)e^x\right|_0^1 - \int_0^1 4.e^x\, dx\right]$

$= -5e + 6 - \left[(4-1)e - (-1) - 4\left|e^x\right|_0^1\right]$

$= -5e + 6 - [3e + 1 - 4(e-1)]$

$= -5e + 6 - 3e - 1 + 4(e-1) = -5e + 6 - 3e - 1 + 4e - 4$

$= 1 - 4e.$

(iii) $I = \int_2^4 \frac{x^2 + x}{\sqrt{2x+1}}\, dx$

Integrating by parts taking x^2+x as first function and $\frac{1}{\sqrt{2x+1}}$ as the 2nd function.

$$I = \left|(x^2 + x)\int \frac{dx}{\sqrt{2x+1}}\right|_2^4 - \int_2^4 \left\{(2x+1).\int \frac{dx}{\sqrt{2x+1}}\right\} dx$$

Now $\int \frac{dx}{\sqrt{2x+1}} = \frac{(2x+1)^{-\frac{1}{2}+1}}{2.\frac{1}{2}} = \sqrt{2x+1}$

$\therefore\ I = \left|(x^2 + x).\sqrt{2x+1}\right|_2^4 - \int_2^4 (2x+1)\sqrt{2x+1}\, dx$

$= (60 - 6\sqrt{5}) - \int_2^4 (2x+1)^{3/2}\, dx$

$= (60 - 6\sqrt{5}) - \left|\frac{(2x+1)^{5/2}}{2.\frac{5}{2}}\right|_2^4$

$= 60 - 6\sqrt{5} - \frac{1}{5}(9^{5/2} - 5^{5/2}) = 60 - 6\sqrt{5} - \frac{243}{5} + 5\sqrt{5}$

$= \frac{57}{5} - \sqrt{5}.$

(iv) $\int \frac{e^x}{x}(1 + x\log x)\, dx = \int e^x\left(\frac{1}{x} + \log x\right) dx$

$= \int e^x\, [f'(x) + f(x)]\, dx$ where $f(x) = \log x$

$= e^x f(x) = e^x \log x$

$\therefore\ \int_1^e \frac{e^x}{x}(1 + x\log x)\, dx = \left[e^x \log x\right]_1^e = e^x \log^e - e\log 1$

$= e^x \quad \left[\because \begin{array}{l} \log e = 1 \\ \log 1 = 0 \end{array}\right]$

Exercise 5.1

Q. 1. Evaluate the following:

(i) $\int_2^4 (3x-2)^2\,dx$ (ii) $\int_6^{10} \frac{dx}{x+2}$ (iii) $\int_3^{11} \sqrt{2x+3}\,dx$

(iv) $\int_0^2 \frac{\sqrt{x}\,dx}{\sqrt{x}+\sqrt{2-x}}$ (v) $\int_0^2 \frac{dx}{(x+1)\sqrt{x^2-1}}$ (vi) $\int_0^1 \frac{3x^3-4x^2+1}{\sqrt{x}}\,dx$

Q. 2. Evaluate the following :

(i) $\int_0^1 \frac{x^5}{1+x^6}\,dx$ (ii) $\int_1^2 x\sqrt{3x-2}\,dx$ (iii) $\int_2^4 \frac{6x^2-1}{\sqrt{2x^3-x-2}}\,dx$

(iv) $\int_1^2 \frac{(\log x)^2}{x}\,dx$

Q. 3. Evaluate the following :

(i) $\int_0^1 x\,e^x\,dx$ (ii) $\int_0^1 x\log\left(1+\frac{x}{2}\right)dx$ (iii) $\int_0^1 x^2\,e^x\,dx$

(iv) $\int_a^b \frac{\log x}{x^2}\,dx$

Definite Integral as area under the curve

Let f(x) be finite and continuous in $a \le x \le b$. Then area of the region bounded by x–axis, y = f(x) and the ordinates at x = a and x = b is equal to $\int_a^b f(x)\,dx$.

Proof. Let AB be the curve y = f(x) and P(x, y) be any point on the curve such that $a \le x \le b$. Let DA and CB be the ordinates x = a and x = b

Take point Q(x+δx, y + δy) near to the point P(x, y). Draw PS and QT parallel to x-axis.

Clearly PS = δx and QS = δy.

Let S represent the area bounded by the curve y = f(x), x-axis and the ordinates AD (x = a) and the variable ordinate PM.

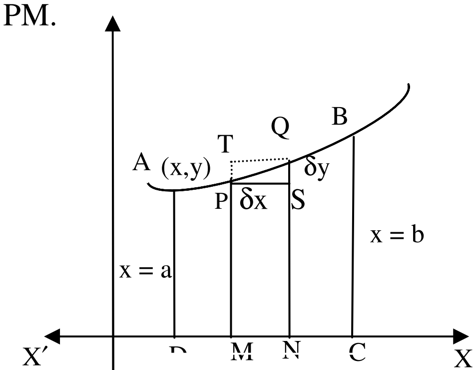


Fig. 1

∴ If δx is increment in x, then δS is increment in S.

It is clear from figure that δS is the area that lies between the rect. PMNS and rect. TQNM.

Also area of rect. PMNS = y. δx and area of rect. TQNM = (y + δy) δx

∴ $y\delta x < \delta S < (y+\delta y)\,\delta x$

or $y < \frac{\delta S}{\delta x} < (y+\delta y)$

when $Q \to P$, $\delta x \to 0$, $\delta y \to 0$

and $\lim_{\delta x \to 0} \frac{\delta S}{\delta x} \to \frac{dS}{dx}$, we get

$$\frac{dS}{dx} = y = f(x)$$

$$\therefore \quad \int_a^b f(x)\,dx = \int_a^b \frac{ds}{dx}\,.\,dx = \int_a^b dS = |S|_a^b$$

$$= (S)_{x=b} - (S)_{x=a}$$

But it is clear from the figure, when x = a, S = 0, because then PM and AD coincide and then x = b, S = area ABCD = reqd. area.

$\therefore \int_a^b f(x)\,dx$ = Area ABCD.

Thus the area bounded by the curve y = f(x), the x axis and the ordinates x = a and x = b is

$\int_a^b f(x)\,dx$

Remarks. In the figure given, we assumed that f(x ≥ 0) for all x in $a \le x \le b$. However, if (i) $f(x) \le 0$ for all x in $a \le x \le b$, then area bounded by x-axis,

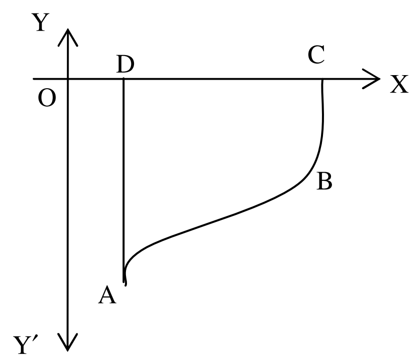
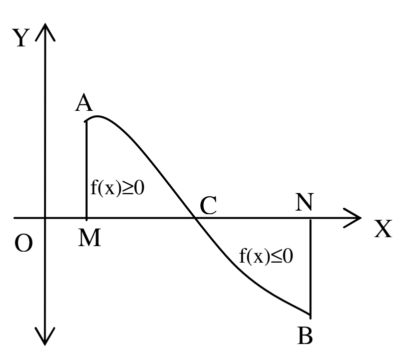


Fig. 2

the curve y = f(x) and the ordinate x = a to x = b is given by

$$= -\int_a^b f(x)\,dx.$$

(ii) If $f(x) \ge 0$ for $a \le x \le c$ and $f(x) \le 0$ for $c \le x \le b$, then area bounded by x = f(x), x-axis and the ordinates x = a, x = b, is



Y′

Fig. 3

$$= \int_a^c f(x)\,dx + \int_c^b - f(x)\,dx$$

$$= \int_a^c f(x)\,dx - \int_c^b f(x)\,dx$$

(iii) The area of the region bounded by $y_1 = f_1(x)$ and $y_2 = f_2(x)$ and the ordinates x = a and x = b is given by

$$= \int_a^b f_2(x)\,dx - \int_a^b f_1(x)\,dx$$

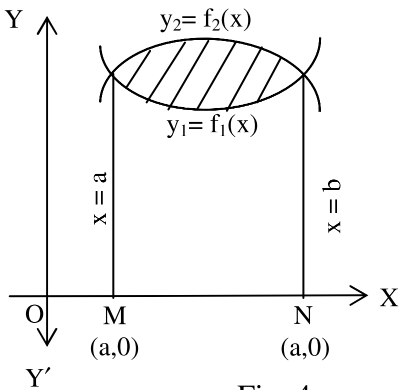


Fig. 4

where $f_2(x)$ is y_2 of upper curve and $f_1(x)$ is y_1 of lower curve i.e.,

Required area = $\int_a^b [f_2(x) - f_1(x)]\,dx = \int_a^b (y_2 - y_1)\,dx$

Example 5. (a) Calculate the area under the curve $y = 2\sqrt{x}$ included between the lines x = 0 and x = 1.

(b) Find the area under the curve $y = \sqrt{3x+4}$ between x = 0 and x = 4.

Solution. (a) $y = 2\sqrt{x}$ $\Rightarrow y^2 = 4x$

$y = 2\sqrt{x}$ is the upper part of the parabola $y^2 = 4x$. We have to find the area of the shaded region OAB.

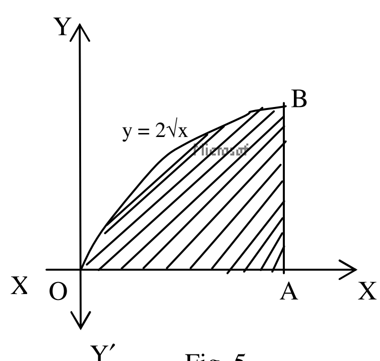


Fig. 5

Required area $= \int_0^1 y\,dx$

$$= \int_0^1 2\sqrt{x}\,dx = 2\left|\frac{x^{3/2}}{3/2}\right|_0^1$$

$$= \frac{4}{3}[(1)^{3/2} - 0] = \frac{4}{3}(1-0)$$

$= \frac{4}{3}$ sq. units.

(b) $y = \sqrt{3x+4}$, $\therefore$ $y^2 = 3x+4$. $y = \sqrt{3x+4}$ is the upper part of the parabola $y^2 = 3x+4$. We have to find the area of the shaded region.

Required area OABC

$= \int_0^4 y\,dx = \int_0^4 \sqrt{3x+4}\,dx$

$= \left| \frac{(3x+4)^{3/2}}{3.\frac{3}{2}} \right|_0^4 = \frac{2}{9} \left| (3x+4)^{3/2} \right|_0^4$

$= \frac{2}{9}[64-8] = \frac{2}{9} \times 56 = \frac{112}{9}$

$= 12\frac{4}{9}$ sq. units.

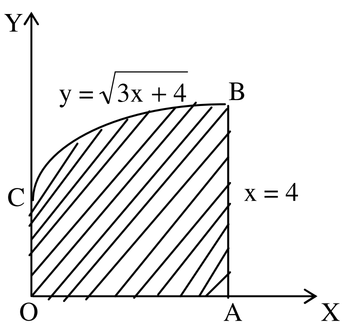


Fig. 6.

Example 6. Find the area bounded by $x = \log_e x$, $y = 0$ and $x = 2$.

Solution. Required area ABC

$= \int_1^2 y\,dx = \int_1^2 \log x\,dx$

$= \left| x \log x - x \right|_1^2$

$= 2 \log 2 - 2 - (0-1)$

$= 2 \log 2 - 1 = \log 2^2 - 1$

$= \log 4 - 1$.

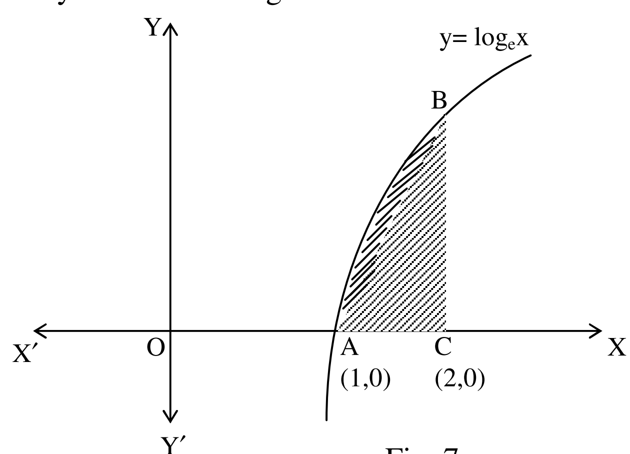


Fig. 7

Example 7. Find the area included between two curves $y^2 = 4ax$ and $x^2 = 4ay$.

Solution. As shown in the figure, we have to find the area OAPBO.

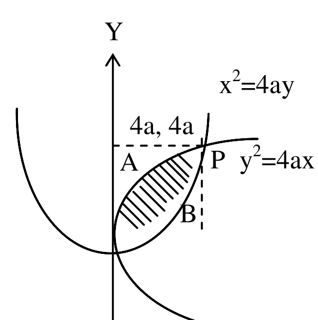
Solving the given two equations simultaneously, we have

$x^4 = 16a^2y^2 = 16a^2(4ax)$

or $x^4 = 64a^3x$

$\Rightarrow x^4 - 64a^3x = 0$,

or $x(x^3 - 64a^3) = 0$



$\Rightarrow \quad x = 0,\ x^3 = 64a^3$

$\Rightarrow x^3 = (4a)^3 \Rightarrow x = 4a$

$\therefore \qquad x = 0$ at O

and $\qquad x = 4a$ at B.

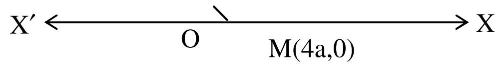


Fig. 8

Now Area OAPBO = Area OAPMO − Area OBPMO

$$= \int_0^{4a} y_1 dx - \int_0^{4a} y_2 dx - \int_0^{4a} 2a^{1/2}x^{1/2}dx - \int_0^{4a}\frac{x^2}{4a}dx$$

$$= \quad 2a^{1/2}\int_0^{4a} x^{1/2}\ dx - \frac{1}{4a}\int_0^{4a} x^2 dx$$

$$= \quad 2a^{1/2}\times\frac{2}{3}\left|x^{3/2}\right|_0^{4a} - \frac{1}{4a}\times\frac{1}{3}\left|x^3\right|_0^{4a}$$

$$= \quad \frac{4}{3}a^{1/2}[(4a)^{3/2}-0] - \frac{1}{12a}[(4a)^3-0]$$

$$= \quad \frac{4}{3}a^{1/2}\times 8a^{3/2} - \frac{1}{12a}\times 64a^3$$

$$= \quad \frac{32}{3}a^2 - \frac{16}{3}a^2 = \frac{32a^2-16a^2}{3} = \frac{16}{3}a^2 \text{ sq. units.}$$

Example 8. Find the area cut-off from the parabola $4y = 3x^2$ by the straight line $2y = 3x+12$.

Solution. Let the points of intersection of the parabola and the line be A and B as shown in the figure. Draw AM and BN ⊥ s to x-axis.

Now putting $y = \frac{3}{4}x^2$ in $2y = 3x+12$

we set $\quad \frac{3}{2}x^2 = 3x+12$

or $\quad 3x^2 - 6x + 24 = 0$

or $\quad x^2 - 2x - 8 = 0$

$\Rightarrow (x-4)(x+2) = 0$

or $\quad x = 4,\ x = -2$

$\therefore \quad y = \frac{3}{4}\times 16 = 12$,

$y = \frac{3}{4}\times 4 = 3$.

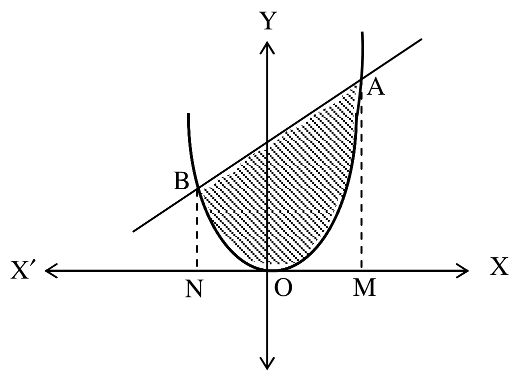


Fig. 9

The co-ordinates of the point A are (4, 12) and co-ordinates of B are (−2, 3).

Now Required ares AOB

= Area of trapezium BNMA − [Area BNO + Area OMA]

But area of trapezium

$= \frac{1}{2}(\text{sum of} \parallel \text{sides}) \times$ Height

$$= \frac{1}{2} \times (12+3) \times 6 = 15 \times 3 = 45$$

Area BNO + Area OMA $= \int_{-2}^{4} y\,dx$

But $4y = 3x^2$, $\therefore$ $y = \frac{3}{4}x^2$

$$\therefore \text{ Area } = \frac{3}{4}\int_{-2}^{4} x^2 dx = \frac{3}{4}\left|\frac{x^3}{3}\right|_{-2}^{4} = \frac{3}{12}[(4)^3 - (-2)^3] = \frac{3}{12}(64+8) = \frac{3}{12} \times 72 = 18$$

Hence required area = 45 − 18 = 27 sq. units.

Example 9. Find the area bounded by the parabola $y^2 = 2x$ and the ordinates $x = 1$ and $x = 4$.

Solution. The equation of the parabola is $y^2 = 2x$ which is of the form $y^2 = 4ax$. The parabola is symmetrical about x-axis and opens towards right.

In the first quadrant $y \geq 0$. Required Area = PMM′P′

$$= 2 \text{ area ABMP}$$

$$= 2\int_{1}^{4} y\,dx = 2\int_{1}^{4} \sqrt{2}\, x^{1/2} dx$$

$$= 2\sqrt{2}\left|\frac{x^{3/2}}{3/2}\right|_{1}^{4}$$

$$= 2\sqrt{2} \times \frac{2}{3}[4^{3/2} - 1]$$

$$= \frac{4\sqrt{2}}{3}(8-1)$$

$$= \frac{4\sqrt{2}}{3} \times 7 = \frac{28\sqrt{2}}{3} \text{ sq. units.}$$

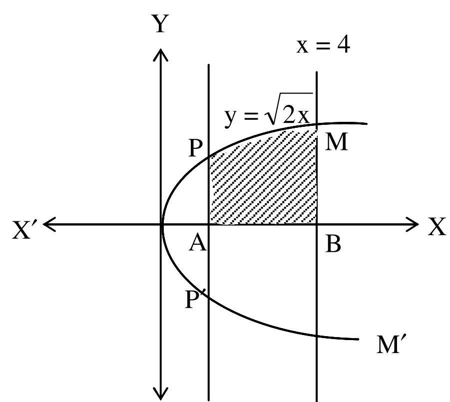


Fig. 10

Example 10. Make a rough sketch of the graph of the function $y = \frac{4}{x^2}$, $(1 \leq x \leq 3)$, and find the area enclosed between the curve, the x-axis and the liens $x = 1$ and $x = 3$.

Solution. Given equation of the curve is

$y = \frac{4}{x^2}$, $(1 \leq x \leq 3)$ $\because$ $\frac{4}{x^2} > 0$,

$\therefore$ $y > 0$

i.e., the curve lies above the x-axis.

Table of Values

x	1	2	3
$y = \frac{4}{x^2}$	4	1	$\frac{4}{9} = 0.44$

Required area $= \int_{1}^{3} y\,dx$

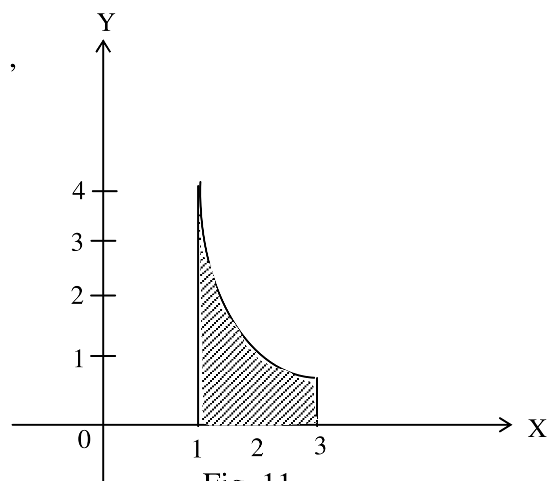


Fig. 11

Required area = $\int_1^3 \frac{4}{x^2}dx = 4\left|-\frac{1}{x}\right|_1^3 = 4\left(-\frac{1}{3}+1\right) = 4\left(\frac{2}{3}\right) = \frac{8}{3}$ sq. units.

Example 11. Find the area of the region
$\{(x,y) : x^2 \le y \le x\}$.

Solution. Let us first sketch the region whose area is to be found out.

The required area is the area included between the curves

$x^2 = y$ and $y = x$.

Solving these two equations simultaneously, we have

$x^2 = x \Rightarrow x^2 - x = 0$

$\Rightarrow x(x-1) = 0$

$\Rightarrow x = 0 , x = 1$

when $x = 0 , y = 0$

when $x = 1 , y = 1$.

$\therefore$ These two curves intersect each other at two points O(0, 0) and A(1.1).

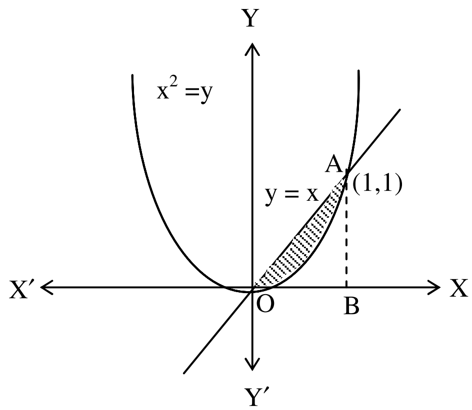


Fig. 12

Required area

$= \int_0^1 x dx - \int_0^1 x^2 dx$

$= \left|\frac{x^2}{2}\right|_0^1 - \left|\frac{x^3}{3}\right|_0^1 = \left(\frac{1}{2}-0\right)-\left(\frac{1}{3}-0\right) = \frac{1}{2}-\frac{1}{3}$

$= \frac{3-2}{6} = \frac{1}{6}$ sq. units.

Example 12. Find the area of the region
$\{(x, y) : x^2 \le y \le |x|\}$.

Solution. Let us first sketch the region whose area is to be found out.

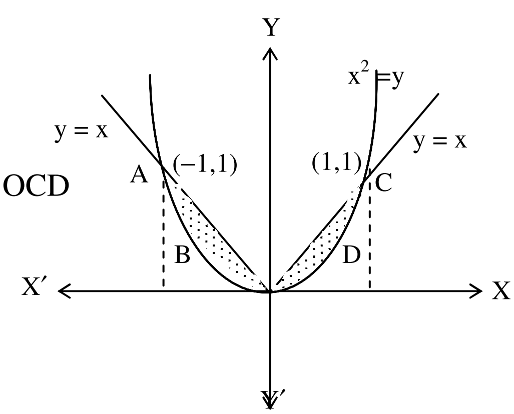
The required area is the area included between the curves

$x^2 = y$ and $y = |x|$.

The graph of $x^2 = y$ is a parabola with vertex (0, 0) and axis y-axis as shown in figure.

The graph of $y = |x|$ is the union of lines $y = x, x \ge 0$ and $y = -x, x \le 0$.

The required region is the shaded region.



$\therefore$ The required area = Area OAB+Area OCD

= 2 Area OCD

$= 2\int_0^1 x dx - 2\int_0^1 x^2 dx$

$= 2\left|\frac{x^2}{2}\right|_0^1 - 2\left|\frac{x^3}{3}\right|_0^1$

$$= 2\left(\frac{1}{2} - 0)\right) - 2\left(\frac{1}{3} - 0\right)$$

Fig. 13

$$= 2\left(\frac{1}{2} - \frac{1}{3}\right) = 2\left(\frac{3-2}{6}\right)$$

$$= 2 \times \frac{1}{6} = \frac{1}{3} \text{ sq. units.}$$

Example 13. Using integration find the area of the triangular region whose sides have the equation

$$y = 2x+1 \qquad \ldots(1)$$

$$y = 3x+1 \qquad \ldots(2)$$

and $$x = 4 \qquad \ldots(3)$$

Solution. Solving (1) and (3), we get $x = 4$, $y = 2\times4+1 = 9$.

$\therefore$ (4, 9) is the point of intersection of lines (1) and (3).

Solving (1) and (2), we get $x = 0$, $y = 1$.

$\therefore$ (0, 1) is the point of intersection of lines (1) and (2).

Solving (2) and (3), we get $x = 4$, $y = 3 \times 4 + 1 = 13$.

(4, 13) is the point of intersection of lines (2) and (3)

Required area ABC

$$= \int_0^4 (3x+1)dx - \int_0^4 (2x+1)dx$$

$$= \left| \frac{3x^2}{2} + x \right|_0^4 - \left| \frac{2x^2}{2} + x \right|_0^4$$

$$= \left[\frac{3}{2}(4)^2 + 4\right] - [(4)^2 + 4]$$

$$= (24+4) - (16+4) = 28 - 20$$

$$= 8 \text{ sq. units.}$$

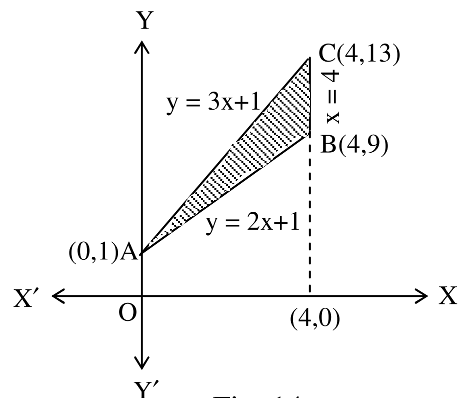


Fig. 14

Example 14. Using integration, find the area of the region bounded by the triangle whose vertices are (−1, 1), (0, 5) and (3,2).

Solution. Let A(−1, 1), B(0, 5) and C(3, 2) are three given vertices of a triangle, as shown in the figure. Equation of AB

$$y-1 = \frac{5-1}{0+1}(x+1)$$

$$y-1 = 4(x+1)$$

$$\therefore \quad y = 4x+5 \qquad \ldots(i)$$

Equation of BC

$$y-5 = \frac{2-5}{3-0}(x-0)$$

$$3y - 15 = -3x$$

$$3y = 15-3x$$

i.e., $y = 5-x \qquad \ldots(ii)$

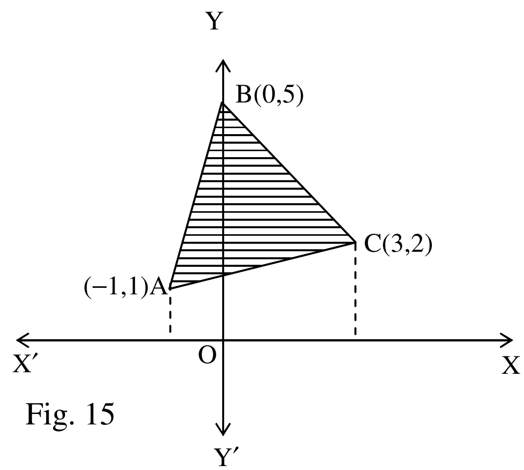


Fig. 15

Equation of AC

$$y-1 = \frac{2-1}{3+1}(x+1)$$

or $4y-4 = 1.(x+1)$

$4y-4 = x+1 \qquad \Rightarrow 4y = x+5$

i.e. $y = \frac{x}{4}+\frac{5}{4}$ …(iii)

Area of Δ ABC

$$= \int_{-1}^{0}(4x+5)dx + \int_{0}^{3}(5-x)dx - \int_{-1}^{3}\left(\frac{1}{4}x+\frac{5}{4}\right)dx$$

$$= \left|\frac{4x^2}{2}+5x\right|_{-1}^{0} + \left|5x-\frac{x^2}{2}\right|_{0}^{3} - \left|\frac{x^2}{8}+\frac{5}{4}x\right|_{-1}^{3}$$

$$= \left|2x^2+5x\right|_{-1}^{0} + \left(15-\frac{9}{2}-0-0\right) - \left(\frac{9}{8}+\frac{15}{4}-\frac{1}{8}+\frac{5}{4}\right)$$

$$= (0+0-2+5) + \frac{21}{2} - \left(\frac{9+30-1+10}{8}\right)$$

$$= 3+\frac{21}{2}-\frac{48}{8} = 3+\frac{21}{2}-6 = \frac{21}{2}-3 = \frac{15}{2}$$

Exercise 5.2

1. Find the area of the region included between the parabola $y = \frac{3}{4}x^2$ and the line $3x-2y+12 = 0$.
2. Find the area bounded by the curve $y = x^2$ and the line $y = x$.
3. Make a rough sketch of the graph of the function $y = 9-x^2$, $0 \le x \le 3$ and determine the area enclosed between the curve and the axis.
4. Using integration, find the area of the region bounded by the triangle whose vertices are (1, 0), (2, 2) and (3.1).
5. Find the area of the region bounded by

   $y = -1$
   $y = 2$
   $x = y^2$
   $x = 0$
6. Find the area between the parabola $y^2 = x$ and the line $x = 4$
7. Find the area bounded by the curve $y = x^2-4$ and the lines $y = 0$ and $y = 5$.
8. Find the area of the region enclosed between the curve $y = x^2+1$ and the line $y = 2x+1$.
9. Find the area bounded by the curve $x = at^2$, $y = 2at$ between the ordinates corresponding to $t = 1$ and $t = 2$.
10. Find the area of the region enclosed by the parabola $y^2 = 4ax$ and chord $y = mx$.

Answers

Exercise 5.1

1. (i) 104 (ii) $\log \frac{3}{2}$ (iii) $\frac{98}{3}$ (iv) 1 (v) $\frac{1}{\sqrt{3}}$ (vi) $-\frac{52}{15}$

2. (i) $\frac{1}{6}\log 2$ (ii) $\frac{326}{135}$ (iii) $2(\sqrt{122}-\sqrt{12})$ (iv) $\frac{1}{3}(\log_e 2)^3$

3. (i) 1 (ii) $\frac{3}{4}+\frac{2}{3}\log\frac{2}{3}$ (iii) e^{-2} (iv) $\frac{\log a+1}{\log a}-\frac{\log b+1}{\log b}$

Exercise 5.2

1. 27 2. $\frac{1}{6}$ 3. 18 4. $\frac{3}{2}$

5. $\frac{15}{4}$ 6. $\frac{32}{3}$ 7. $\frac{76}{3}$ 8. $\frac{4}{3}$

9. $\frac{56a^2}{3}$ 10. $\frac{8a^2}{3m^2}$

सीखने की वक्र (Learning Curve)

सीखने की वक्र एक ऐसी तकनीक है जिसकी सहायता से हम किसी भी उत्पाद के उत्पादन विधि में समय तथा खर्च का अनुमान कर सकते हैं। समय के साथ, उत्पादन की विधि परिपक्व होती जाती है व एक स्थिर स्थिति पर पहुंचती है। ऐसा इसलिए होता है क्योंकि समय के साथ ज्ञान में वृद्धि के कारण किसी भी उत्पाद की एक इकाई बनाने के लिए लिया गया समय कम होता जाता है व अंत में स्थिर हो जाता है।

Learning curve is a technique with the help of which we can estimate the cost and time of production process of a product. With passage of time, the production process becomes increasingly mature and reaches a steady state. It so happens because with gain in experience with time, time taken to produce one unit of a product steadily decreases and in the last attains a stable value.

The general form of the learning curve is given by

$$y = f(x) = ax^{-b}$$

where y is the average time taken to produce one unit, and x is the number of units produced, a and b are the constants.

a is defined as the time taken for producing the first unit (x = I) and b is calculated by using the formula

$$b = -\frac{\log(\text{learning rate})}{\log 2}$$

If the learning curve is known, then total time (labour hours) required to produce units numbered from a to b is given by

$$L = \int_a^b f(x)dx = \int_a^b A.x^a dx \qquad \text{(another form of learning curve)}$$

Example 1. The first batch of 10 dolls is produced in 30 hours. Determine the time taken to produce next 10 dolls and again next 20 dolls, assuming a 60% learning rate. Estimate the time taken to produce first unit.

New Time taken to produce one batch = Previous time taken to produce one batch × learning rate

No. of dolls	Total time (hours)	Total increase in time	Average time (hrs/doll)
0	0	-	-
10	30	30	3
20	$20\left(\frac{30\times 60}{100}\right)=36$	6	1.8
40	$20\left(\frac{36\times 60}{100}\right)=43.2$	7.2	1.08

Now

$$\beta = -\frac{\log(0.6)}{\log 2} = -0.7369$$

when x = 10 , y = 3 , then 3 = $a.10^{-0.7369}$

Solving the equation, we get

A = 16.38 hours.

Example 2. Because of learning experience, there is a reduction in labour requirement in a firm. After producing 36 units, the firm has the learning curve $f(x) = 1000\,x^{-1/2}$. Find the labour hours required to produce the next 28 units.

Solution L = $\int_{36}^{64} 1000x^{-0.5}\,dx$

$$= 1000\,\left[2x^{1/2}\right]_{36}^{64} = 2000[x^{1/2}]_{36}^{64}$$

= 2000 [8-6] = 4000 hours

Example 3. A firm's learning curve after producing 100 units is given by $f(x) = 2400\,x^{-0.5}$ which is the rate of labour hours required to produce the x^{th} unit. Find the hours needed to produce an additional 800 units.

Solution. Labour hours required $L = \int_{100}^{900} f(x)dx$

$$= \int_{100}^{900} 2400\,x^{-0.5}dx = 2400\left[\frac{x^{1/2}}{1/2}\right]_{100}^{900}$$

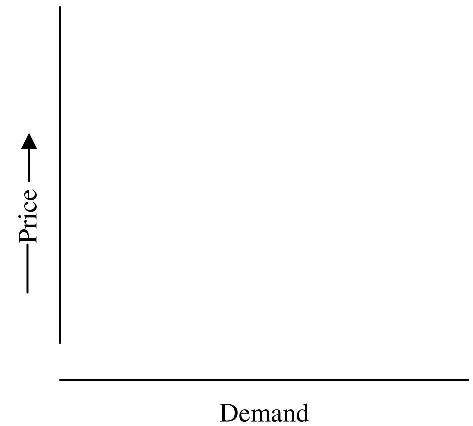
$$= 2400 \times 2\,\left[x^{1/2}\right]_{100}^{900} = 4800[30-10]$$

= 96000 hours.

MiHkksDrk rFkk mRikn cpr (Consumer and Producer Surplus) किसी भी वस्तु के लिए एक उपभोक्ता जो कीमत देना चाहता है तथा वास्तविक कीमत जो वह देता है, इन दोंनों के अन्तर को उपभोकता बचत कहते हैं एक वस्तु से प्राप्त संतोष का दर्जा एक वैयक्रिक मामला है।

Consumer surplus is the difference between the price that a consumer is willing to pay and the actual price he pays for a commodity. The degree of satisfaction derived from a commodity is a subjective matter.

If DD_1 is the market demand curve then demand x_0 corresponds to the price p_0. The consumer surplus is given by DD_1p_0.



$$DD_1p_0 = \text{Area } DD_1x_00 - \phi_0D_1\, x_0O$$
$$= \int_0^{x_0} f(x)dx - p_0x_0$$

where f(x) is the demand function.
It is assumed that the area is defined at x = 0 and that the satisfaction is measurable in terms of price for all consumers. In other words, we assume that utility function is same for all consumers and marginal utility of money is constant.

Example. Find the consumer surplus if the demand function is $p = 25-2x$ and the surplus function is $4p = 10+x$.

Solution. First find the equilibrium price p_0 and equilibrium demand, x_0 by solving the above two equations simultaneously.

$$p = 25-2x$$
or $$4p = 100-8x$$
$$4p = 10 + x$$
$$- \quad - \quad -$$

$$0 = 90-9x$$
or $9x = 90$
or $x = 10$

So $x_0 = 10$ units.
Substitute the value in first equation

$$p_0 = 25 - 2x_0 = 25 - 20$$
$$= 5$$

Now consumer surplus = $\int_0^{x_0} f(x)dx - p_0x_0$

$$= \int_0^{10} (25-2x)dx - 5\times10$$
$$= \left[25x - x^2\right]_0^{10} - 50$$
$$= [250-100] - 50 = 100$$

mRiknd cpr (Producer Surplus) fdlh Hkh oLrq ds fy, ,d mRiknd tks dher pkgrk gS rFkk okLrfod dher tks mls feyrh gS] muds vUrj dks mRiknd cpr dgrs gSaA

Producer surplus is the difference in the prices a producer expects to get and the price which he actually gets for a commodity.

If SS_1 is the market supply curve and if x_0 is the supply at the market price p_0, the producer surplus is the area PS.

$$PS = \text{Area } SS_1P_0 = p_0x_0 - \int_0^{x} g(x)dx$$

where g(x) is the supply function.

Example. Find the producer surplus for the supply function

$$p^2 - x = 9 \quad \text{when } x_0 = 7$$

Solution. We are given $p^2 - x = 9$

or $p_0^2 - x_0 = 9$

Also given $x_0 = 7$

$\therefore\ p_0^2 - 7 = 9$

or $p_0^2 = 16$

or $p_0 = 4$

$$\therefore \quad PS = p_0 x_0 - \int_0^{x_0} g(x)dx$$

$$= 4 \times 7 - \int_0^7 (x+9)^{1/2} dx$$

$$= 28 - \left[\frac{2}{3}(x+9)^{3/2}\right]_0^7$$

$$= 28 - \frac{2}{3}\left[(16)^{3/2} - (9)^{3/2}\right]$$

$$= 28 - \frac{2}{3}[64 - 27]$$

$$= 28 - \frac{74}{3}$$

$$= \frac{10}{3}$$

Chapter-6

आव्यूह
MATRICES

लगभग एक शताब्दी पूर्व निर्देशाक ज्यामिति में ज्यामितीय आकृतियों को सरल बनाने के लिए आव्यूहों ;उंजतपबमेद्ध की खोज हुई। वास्तव में ष्आव्यूहष् नवीन गणित का एक मुख्य आधार है तथा विज्ञान की प्रत्येक शाखाओं में इसका प्रयोग दिन–प्रतिदिन महत्त्वपूर्ण होता जा रहा है। यह समाज शास्त्र, जनसंख्या सम्बन्धी आकलन, अर्थशास्त्र, सांख्यिकी, इंजीनीयरिंग, आदि के क्षेत्र में प्रयोग की जाती है।

6.1. परिभाषा (Definition)

किसी भी उद संख्याओं ;वास्तविक अथवा सम्मिश्र संख्याओंद्ध के एक निकाय को, जो उ पंक्तियों ;तवेद्ध और द स्तम्भों ;बवसनउदेद्ध में आयताकार सारणी ;तमबजंदहनसंत ंततंलद्ध में व्यवस्थित ;ंततंदहमद्ध हो उ×द क्रम या कोटि ;वतकमतद्ध का उ×द आव्यूह कहते हैं या आव्यूह भी कहते हैं, अर्थात

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & \dots & a_{1j} & \dots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \dots & a_{2j} & \dots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \dots & a_{3j} & \dots & a_{n} \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ a_{i1} & a_{i2} & a_{i3} & \dots & a_{ij} & \dots & a_{in} \\ \dots & \dots & \dots & \dots & \dots. & \dots & \dots \\ a_{m_1} & a_{m_2} & a_{m_3} & \dots & a_{m_j} & \dots & a_{mn} \end{bmatrix}$$

एक उ × द आव्यूह है, जहॉं ंपर प्रतीक किसी भी संख्या को दर्शाता है जो प वी पंक्ति तथा र वें स्तम्भ में स्थित है। संख्याएं ं11एइ्णएंउद आव्यूह के अवयव ;मसमउमदजद्ध कहलाते हैं।

उदाहरणार्थ मान लें कि एक कक्षा में 40 लड़के तथा 35 लड़कियॉ है, दूसरी कक्षा में 30 लड़के तथा 40 लड़कियॉ हैं तथा तीसरी कक्षा में 24 लड़के तथा 16 लड़कियॉ हैं। इन आंकड़ों को हम आव्यूह की तरह निम्नलिखित ढंग से प्रस्तुत कर सकते हैं:

	लड़के	लड़कियॉ	
izFke d{kk	40	35	→ प्रथम पंक्ति
nwljh d{kk	30	30	→ द्वितीय पंक्ति
rhljh d{kk	24	16	→ तृतीय पंक्ति
	↓ izFke LraHk	↓ f}rh; LraHk i	

नोट 1. एक आव्यूह को निम्न प्रतीयों से दर्शाते हैं:

[ ] , ( ), ‖ ‖

या आव्यूह को साधारणतया बड़े लैटिन अक्षरों ;स्जपद समजजमतेद्ध ।ए ठएब्ए आदि से दर्शाते हैं जबकि इसके अवयवों को छोटे लैटिन अक्षरों तथा इन अक्षरों से पर आव्यूह में स्थिति को बताते हुए दर्शाते हैंय जैसे ं11ए ं12ए ं13 आदि।

नोट 2. प्रत्येक उ × द संख्या, उ×द आव्यूह बनाती है जो आव्यूह के अवयव कहलाते हैं। आव्यूह के अवयव संदिश या अदिश ;अमबजवत वत ेबंसंतद्ध राशियां हो सकती हैं।

6.2. आव्यूहों के मुख्य प्रकार ";चमबपंस जलचमे व िउंजतपबमेद्ध

आयतीत आव्यूह (Square matrix). जब उ ≠ दए अर्थात जब पंक्तियों की संख्या तथा स्तभों की संख्या समान नहीं हो तो ऐसी आव्यूहों को आयतीय आव्यूह कहते हैं, जैसे

$$\begin{bmatrix} -3 & 2 & 4 \\ 1 & -4 & 6 \end{bmatrix} 2 \times 3 \text{ कोटि का आव्यूह है।}$$

वर्ग आव्यूह ";नंतम उंजतपगद्धण यदि उ त्र दए अर्थात जब पंक्तियों की संख्या, स्तम्भों की संख्या के समान हो तो ऐसी आव्यूहों को वर्ग आव्यूह कहते हैं, जैसे

$$\begin{bmatrix} 5 & 2 & -3 \\ 0 & 4 & 2 \\ -3 & 6 & 0 \end{bmatrix}$$, 3×3 कोटि का आव्यूह है।

पंक्ति आव्यूह (Row matrix) यदि किसी आव्यूह में केवल एक पंक्ति हो ;अर्थात उ त्र1द्ध तो ऐसी आव्यूह पंक्ति कहलाती हैं, जैसे

$[1 \quad -3 \quad 2]$ 1×3 कोटि का आव्यूह हैं।

स्तम्भ आव्यूह (Column matrix). यदि किसी आव्यूह में केवल एक स्तम्भ हो तो ऐसी आव्यूह स्तम्भ आव्यूह कहलाती है, जैसे

$$\begin{bmatrix} 10 \\ -5 \\ 8 \end{bmatrix}$$ 3×1 कोटि का आव्यूह है।

शून्य आव्यूह (Null or Zero matrix). यदि किसी आव्यूह के प्रत्येक अवयव शून्य हों तो शून्य आव्यूह कहलाती है तथा इसे ट से प्रदर्शित करते हैं, जैसे

$$O = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

मात्रक आव्यूह (Unit matrix). एक वर्ग आव्यूह निम्न प्रकार हों

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

जिसके मुख्य विकर्ण ;समंकपदह कपंहवदंसद्ध के सभी अवयव 1 के बराबर हों तथा शेष अवयव शून्य हों तो ऐसी आव्यूह, मात्रक आव्यूह अथवा इकाई आव्यूह कहलाती है तथा इसे प् से दर्शाते हैं।

उप–आव्यूह (Sub-matrixद्धण दी हुई आव्यूह से बनायी गयी ऐसी आव्यूह जिसकी कितनी भी पंक्तियॉ तथा स्तम्भों को छोड़ दिया गया हो, उप–आव्यूह कहलाती हैं जैसे

आव्यूह $\begin{bmatrix} 2 & 1 & 5 \\ 0 & 3 & 2 \\ 3 & 5 & 1 \end{bmatrix}$ का एक उप–आव्यूह $\begin{bmatrix} 3 & 2 \\ 5 & 1 \end{bmatrix}$ है।

समान आव्यूह (Equal matrix). दो आव्यूह । त्र ख्ंपरद्व तथा ठ त्र ख्इपरद्व समान होती हैं यदि प तथा र के सभी मानों के लिए ंपर त्र इपर अर्थात । आव्यूह का प्रत्येक अवयव, ठ आव्यूह के संगन अवयव के समान हो.
उदाहरण के लिए यदि

$$A = \begin{bmatrix} 2 & -3 \\ 4 & 5 \end{bmatrix} \text{rFkk} \quad B = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$ यदि ं त्र 2ए इ त्र –3ए ब त्र 4 तथा क त्र 5

उपरोक्त परिभाषा से स्पष्ट होता है कि

(i) यदि । कोई आव्यूह है तब A = A (LorqY;rk reflexitivity)

(ii) यदि A = B rc B = A (सममितता] Symmetric)

आव्यूहों पर बुनियादी कारवाई ;Basic operations on Matrices)

1ण आव्यूह का अदिश राशि से गुणन ;डनसजपचसपबंजपवद वि उंजतपग इल ेबंसंतद्धण यदि । त्र ख्ंपरद्वए उ × द क्रम का आव्यूह हैं तथा λ एक अदिश राशि ;ेबंसंत नुंदजपजलद्ध है तो । का λ से गुणन वह आव्यूह होता है जिसका प्रत्येक अवयव । के अवयवों का λ गुणा होता है तथा इसे λ। से निरूपित किया जाता है।

या, यदि $A = [a_{ij}]$ rc $\lambda A = [\lambda a_{ij}]$, जहॉ λ कोई अदिश संख्या है।

उदाहरण (a) If $A = \begin{bmatrix} 1 & 3 & 5 \\ 2 & 4 & 6 \\ 7 & 9 & 11 \end{bmatrix}$, then $3A = \begin{bmatrix} 3 & 9 & 15 \\ 6 & 12 & 18 \\ 21 & 27 & 33 \end{bmatrix}$

(b) If $A = \begin{bmatrix} 3 & 6 & 9 \\ 12 & 18 & 15 \\ 27 & 21 & 6 \end{bmatrix}$, then $\frac{1}{3}A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 6 & 5 \\ 9 & 7 & 2 \end{bmatrix}$

एक आव्यूह के अदिश राशि से गुणन की विशेषतायें : (Properties of Multiplication of a matrix by a scalar)

(i) यदि । तथा ठ दो उ×द कोटि के आव्यूह हैं तो ा;।ठद्ध त्र ा।. ाठ अर्थात आव्यूहों को अदिश गुणनफल आव्यूहों के योग पर वितरित है।

If A and B are two matrices each of the type m × n, then k(A+B) = kA+kB i.e., the scalar multiplication of matrices distributes over the addition of matrices.

Proof. Let $A = [a_{ij}]_{m\times n}$ and $B = [b_{ij}]_{m\times n}$

then $k(A+B) = k[a_{ij}]_{m\times n} + [b_{ij}]_{m\times n}$

$= k[a_{ij} + b_{ij}]_{m\times n}$ [By def. of addition of two matrices]

$= [k(a_{ij} + b_{ij})]_{m\times n}$ [By def. of scalar multiplication]

$= [ka_{ij} + kb_{ij}]_{m\times n}$ [By distributive law of numbers]

$= [ka_{ij}]_{m\times n} + [kb_{ij}]_{m\times n}$

$= k[a_{ij}]_{m\times n} + k[b_{ij}]_{m\times n} = kA + kB.$

(ii) यदि ा तथा स दो अदिश राशियॉ है तथा । उ×द कोटि का एक आव्यूह है, तो

$(k+l)\,A = kA + lA$

If k and l are two scalars and A is any m × n matrix, then

$(k+l)A = kA + lA$.

Proof. Let $A = [a_{ij}]_{m\times n}$

Then $(k+l)\,A = (k+l)\,[a_{ij}]_{m\times n}$

$= [(k+l)a_{ij}]_{m\times n} = [ka_{ij} + la_{ij}]_{m\times n}$

$= [ka_{ij}]_{m\times n} + [la_{ij}]_{m\times n} = k[a_{ij}]_{m\times n} + l[a_{ij}]_{m\times n}$

$= kA + lA$.

(iii) यदि ा तथा स दो आदिश राशियॉ हैं तथा ।ए उ×द कोटि का एक आव्यूह है तो

$k(lA) = (kl)\,A$

If k and l are two scalars and A is any m × n matrix, then

$K(lA) = (kl)A$.

Proof. Let $A = [a_{ij}]_{m\times n}$. Then

$k(lA) = k(l[a_{ij}]_{m\times n}) = k(la_{ij})_{m\times n}$

$= [(kl)\,a_{ij}]_{m\times n} = (kl)\,[a_{ij}]_{m\times n}\ (kl)A$.

[∵ Multiplication of numbers is associative]

(iv) यदि । एक उ × द कोटि का आव्यूह है तथा ा एक अदिश राशि है तो

$-k(A) = -(kA) = k(-A)$

If A be any m ×n matrix and k be any scalar, then

$(-k)\,A = -(kA) = k(-A)$ -‾‾‾-

Proof. Let $A = [a_{ij}]_{m\times n}$. Then

$(-k)A = [(-k)\,a_{ij}]_{m\times n} = [-(ka_{ij})]_{m\times n}$

$= [ka_{ij}]_{m\times n} = -\,(kA)$

Also $(-k)\,A = [(-k)\,a_{ij}]_{m\times n} = [k(-a_{ij})]_{m\times n}$

$= k[-a_{ij}]_{m\times n} = k(-A)$

(v) (a) $1.\,A = A$ (b) $(-1)\,A = -A$

Proof. $1.\,A = 1\,[a_{ij}] = [1.a_{ij}] = [a_{ij}] = A$

$(-1)\ A = -1\ [a_{ij}] = [(-1)a_{ij}] = [-a_{ij}] = -A$

2. (a) आव्यूहों का योग (Addition of matrix). ,क ही क्रम उ×द के दो आव्यूह $A = [a_{ij}]$ तथा $B = [b_{ij}]$ की योग का आव्यूह उनके संगत तत्वों का योग करने पर प्राप्त होती है तथा इसे A+B से प्रदर्शित किया जाता है, अर्थात

$$A+B = [a_{ij}+b_{ij}]$$

उदाहरण. यदि $A = \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{bmatrix}$ तथा, $B = \begin{bmatrix} a_3 & b_3 & c_3 \\ a_4 & b_4 & c_4 \end{bmatrix}$

rc $A+B = \begin{bmatrix} a_1+a_3 & b_1+b_3 & c_1+c_3 \\ a_2+a_4 & b_2+b_4 & c_2+c_4 \end{bmatrix}$

6.3. आव्यूहों के योग के गुण (Properties of Matrix addition)

(i) क्रम विनिमेय नियम (Commutative law).

अर्थात, $[a_{ij}] + [b_{ij}] = [b_{ij}] + [a_{ij}]$

Proof. Let $A = [a_{ij}]_{m\times n}$ and $B = [b_{ij}]_{m\times n}$

Then $A+B = [a_{ij}]_{m\times n} + [b_{ij}]_{m\times n}$

$= [a_{ij} + b_{ij}]_{m\times n}$ [By definition of addition of two matrices]

$= [b_{ij} + a_{ij}]_{m\times n}$ [Since a_{ij} and b_{ij} are numbers and addition of numbers is commutative]

$= [b_{ij}]_{m\times n} + [a_{ij}]_{m\times n}$ [By definition of addition of two matrices]

$= B + A.$

(ii) आव्यूहों का योग साहचर्य है (Matrix Addition is Associative.)

If A, B, C be three matrices each of the type m×n, then

$$(A+B) + C = A + (B+C).$$

Proof. Let $A = [a_{ij}]_{m\times n}$, $B = [b_{ij}]_{m\times n}$, $C = [c_{ij}]_{m\times n}$

Then $(A+B) + C = ([a_{ij}]_{m\times n} + [b_{ij}]_{m\times n}) + [c_{ij}]_{m\times n}$

$= [a_{ij} + b_{ij}]_{m\times n} + [c_{ij}]_{m\times n}$ [By definition of A +B]

$= [(a_{ij} + b_{ij}) + c_{ij}]_{m\times n}$ [By definition of addition of matrices]

$= [a_{ij} + (b_{ij} + c_{ij})]_{m\times n}$ [Since a_{ij}, b_{ij}, c_{ij} are numbers and addition of numbers is associative]

$= [a_{ij}]_{m\times n} + [b_{ij} + c_{ij}]_{m\times n}$ [By definition of addition of two matrices]

$= [a_{ij}]_{m\times n} + ([b_{ij}]_{m\times n}, + [c_{ij}]_{m\times n}) = A + (B + C).$

(iii) योज्य पहचान (Existence of Additive Identity).

If O be the m×n matrix each of whose elements is zero, then

$$A + O = A = O + A \text{ for } m\times n \text{ matrix } A.$$

Proof. Let $A = [a_{ij}]_{m\times n}$

Then $A + O = [a_{ij} + 0]_{m\times n} = [a_{ij}]_{m\times n} = A$

Also $O + A = [0 + a_{ij}]_{m\times n} = [a_{ij}]_{m\times n} = A$

Thus the null matrix O of the type m×n acts as the identity element for addition in the set of all m×n matrices.

(iv) योज्य व्युत्क्रम (Existence of additive inverse). ;fn vkO;wg A ds fy, ,d vkO;wg −A bl izdkj gks fd A + (−A) = O rks −A आव्यूह । का का योज्य व्युत्क्रम कहलाता है।

Let $A = [a_{ij}]_{m\times n}$. Then the negative of the matrix A is defined as the matrix $[-a_{ij}]_{m\times n}$ and is denoted by −A.

The matrix −A is the additive inverse of the matrix A. Obviously, −A + A = O = A +(−A). Here O is the null matrix of the type m×n. It is the identity element for matrix addition.

2 (b) आव्यूहों का व्यवकलन (Subtraction of matrices). ;fn A= $[a_{ij}]$ rFkk B = C rc A = C (संक्रामक, transitive) तथा B = $[a_{ij}]$ समान क्रम के दो आव्यूह हैं तो उनका अंतर A− B, वह आव्यूह होता हैं जिसका प्रत्येक अवयव, । और ठ के संगत अवयवों के अन्तर के बराबर है, अर्थात $A - B = [a_{ij} - b_{ij}]$

उदाहरण. यदि $A = \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{bmatrix}$ and $B = \begin{bmatrix} a_3 & b_3 & c_3 \\ a_4 & b_4 & c_4 \end{bmatrix}$

तो $A - B = \begin{bmatrix} a_1 - a_3 & b_1 - b_3 & c_1 - c_3 \\ a_2 - a_4 & b_2 - b_4 & c_2 - c_4 \end{bmatrix}$

Example 1. If $A = \begin{bmatrix} 1 & 5 & 6 \\ -6 & 7 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & -5 & 7 \\ 8 & -7 & 7 \end{bmatrix}$

Then find the value A + B and A−B

Sol. $A + B = \begin{bmatrix} 1 & 5 & 6 \\ -6 & 7 & 0 \end{bmatrix} + \begin{bmatrix} 1 & -5 & 7 \\ 8 & -7 & 7 \end{bmatrix}$

$$= \begin{bmatrix} 1+1 & 5-5 & 6+7 \\ -6+8 & 7-7 & 0+7 \end{bmatrix} = \begin{bmatrix} 2 & 0 & 13 \\ 2 & 0 & 7 \end{bmatrix}$$

and $A - B = \begin{bmatrix} 1 & 5 & 6 \\ -6 & 7 & 0 \end{bmatrix} - \begin{bmatrix} 1 & -5 & 7 \\ 8 & -7 & 7 \end{bmatrix}$

$$= \begin{bmatrix} 1-1 & 5-(-5) & 6-7 \\ -6-8 & 7-(-7) & 0-7 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 10 & -1 \\ -14 & 14 & -7 \end{bmatrix}$$

Example 2. If $A = \begin{bmatrix} 2 & 3 & 1 \\ 0 & -1 & 5 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 2 & -6 \\ 0 & -1 & 3 \end{bmatrix}$ then find the values of 3A −4B

Sol. $3A - 4B = 3\begin{bmatrix} 2 & 3 & 1 \\ 0 & -1 & 5 \end{bmatrix} - 4\begin{bmatrix} 1 & 2 & -6 \\ 0 & -1 & 3 \end{bmatrix}$

$$= \begin{bmatrix} 6 & 9 & 3 \\ 0 & -3 & 15 \end{bmatrix} - \begin{bmatrix} 4 & 8 & -24 \\ 0 & -4 & 12 \end{bmatrix}$$

$$= \begin{bmatrix} 6-4 & 9-8 & 3-(-24) \\ 0-0 & -3-(-4) & 15-12 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & 1 & 27 \\ 0 & 1 & 3 \end{bmatrix}$$

Example 3. Solve the following equations for A +B

$$2A - B = \begin{bmatrix} 3 & -3 & 0 \\ 3 & 3 & 2 \end{bmatrix}, \; 2B + A = \begin{bmatrix} 4 & 1 & 5 \\ -1 & 4 & -4 \end{bmatrix}$$

Sol. $2A - B = \begin{bmatrix} 3 & -3 & 0 \\ 3 & 3 & 2 \end{bmatrix}$

Multiplying both sides by 2

$$4A - 2B = 2\begin{bmatrix} 3 & -3 & 0 \\ 3 & 3 & 2 \end{bmatrix} = \begin{bmatrix} 6 & -6 & 0 \\ 6 & 6 & 4 \end{bmatrix} \quad \ldots(i)$$

Also it is given $2B + A = \begin{bmatrix} 4 & 1 & 5 \\ -1 & 4 & -4 \end{bmatrix}$ …(ii)

Adding (i) and (ii)

$$5A = \begin{bmatrix} 6 & -6 & 0 \\ 6 & 6 & 4 \end{bmatrix} + \begin{bmatrix} 4 & 1 & 5 \\ -1 & 4 & -4 \end{bmatrix}$$

$$= \begin{bmatrix} 6+4 & -6+1 & 0+5 \\ 6-1 & 6+4 & 4-4 \end{bmatrix} = \begin{bmatrix} 10 & -5 & 5 \\ 5 & 10 & 0 \end{bmatrix}$$

or $A = \frac{1}{5}\begin{bmatrix} 10 & -5 & 5 \\ 5 & 10 & 0 \end{bmatrix} = \begin{bmatrix} 2 & -1 & 1 \\ 1 & 2 & 0 \end{bmatrix}$

Again from (ii)

$$B = \begin{bmatrix} 1 & 1 & 2 \\ -1 & 1 & -2 \end{bmatrix}$$

Example. 4. If $= \begin{bmatrix} x+y & y-z \\ z-2x & y-x \end{bmatrix} = \begin{bmatrix} 3 & -1 \\ 1 & 1 \end{bmatrix}$ find x, y, z

Sol. Using equality of matrices

$x + y = 3$ …(i)

$z - 2x = 1$ …(ii)

$y - z = -1$ …(iii)

$y - x = 1$ …(iv)

Adding (i) and (iv)

$x + y = 3$

$- \quad x + y = 1$

$2y = 4$

$y = 2$

From (i) $x+2 = 3$ or $x = 3-2 = 1$

From (ii) $z-2x = 1$ or $z - 2 = 1$ or $z = 3$

$\therefore$ $x = 1, y = 2, z = 3$

Example 5. (a) $A = \begin{bmatrix} 2 & -1 \\ 4 & 2 \end{bmatrix}$, $B = \begin{bmatrix} 4 & 3 \\ -2 & 1 \end{bmatrix}$ and $C = \begin{bmatrix} -2 & -3 \\ -1 & -2 \end{bmatrix}$

Find each of the following :

(i) $A + B + C$ (ii) $2B + 3C$.

(b) If $A = \begin{bmatrix} 1 & 0 \\ 3 & 4 \end{bmatrix}$, $B = \begin{bmatrix} 0 & 3 \\ 2 & 5 \end{bmatrix}$, $C = \begin{bmatrix} 3 & 5 \\ 6 & 4 \end{bmatrix}$

show that $(A+B) + C = A+(B+C)$.

(c) Given $A = \begin{bmatrix} 1 & 2 & -3 \\ 5 & 0 & 2 \\ 1 & -1 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 3 & -1 & 2 \\ 4 & 2 & 5 \\ 2 & 0 & 3 \end{bmatrix}$

Find the matrix C, such that A + 2C = B .

Sol.

(a) (i) $A + B + C = \begin{bmatrix} 2 & -1 \\ 4 & 2 \end{bmatrix} + \begin{bmatrix} 4 & 3 \\ -2 & 1 \end{bmatrix} + \begin{bmatrix} -2 & -3 \\ -1 & -2 \end{bmatrix}$

$= \begin{bmatrix} 2+4-2 & -1+3-3 \\ 4-2-1 & 2+1-2 \end{bmatrix} = \begin{bmatrix} 4 & -1 \\ 1 & 1 \end{bmatrix}$

(ii) $2B + 2C = 2\begin{bmatrix} 4 & 3 \\ -2 & 1 \end{bmatrix} + 3\begin{bmatrix} -2 & -3 \\ -1 & -2 \end{bmatrix}$

$= \begin{bmatrix} 8 & 6 \\ -4 & 2 \end{bmatrix} + \begin{bmatrix} -6 & -9 \\ -3 & -6 \end{bmatrix}$

$= \begin{bmatrix} 8-6 & 6-9 \\ -4-3 & 2-6 \end{bmatrix} = \begin{bmatrix} 2 & -3 \\ -7 & -4 \end{bmatrix}$

(b) $A + B = \begin{bmatrix} 1 & 0 \\ 3 & 4 \end{bmatrix} + \begin{bmatrix} 0 & 3 \\ 2 & 5 \end{bmatrix} = \begin{bmatrix} 1+0 & 0+3 \\ 3+2 & 4+5 \end{bmatrix} = \begin{bmatrix} 1 & 3 \\ 5 & 9 \end{bmatrix}$

$\therefore \quad (A + B) + C = \begin{bmatrix} 1 & 3 \\ 5 & 9 \end{bmatrix} + \begin{bmatrix} 3 & 5 \\ 6 & 4 \end{bmatrix}$

$= \begin{bmatrix} 1+3 & 3+5 \\ 5+6 & 9+4 \end{bmatrix} = \begin{bmatrix} 4 & 8 \\ 11 & 13 \end{bmatrix}$

Again $B+C = \begin{bmatrix} 0 & 3 \\ 2 & 5 \end{bmatrix} + \begin{bmatrix} 3 & 5 \\ 6 & 4 \end{bmatrix} = \begin{bmatrix} 0+3 & 3+5 \\ 2+6 & 5+4 \end{bmatrix} = \begin{bmatrix} 3 & 8 \\ 8 & 9 \end{bmatrix}$

$\therefore \quad A+(B+C) = \begin{bmatrix} 1 & 0 \\ 3 & 4 \end{bmatrix} + \begin{bmatrix} 3 & 8 \\ 8 & 9 \end{bmatrix} = \begin{bmatrix} 1+3 & 0+8 \\ 3+8 & 4+9 \end{bmatrix}\begin{bmatrix} 4 & 8 \\ 11 & 13 \end{bmatrix}$

$\therefore \quad (A+B) + C = A + (B+C)$

(c) Given that A + 2C = B, or 2C = B − A

$\therefore \quad 2C = \begin{bmatrix} 3 & -1 & 2 \\ 4 & 2 & 5 \\ 2 & 0 & 3 \end{bmatrix} - \begin{bmatrix} 1 & 2 & -3 \\ 5 & 0 & 2 \\ 1 & -1 & 1 \end{bmatrix}$

$= \begin{bmatrix} 3-1 & -1-2 & 2-(-3) \\ 4-5 & 2-0 & 5-2 \\ 2-1 & 0-(-1) & 3-1 \end{bmatrix} = \begin{bmatrix} 2 & -3 & 5 \\ -1 & 2 & 3 \\ 1 & 1 & 2 \end{bmatrix}$

$C = \frac{1}{2}\begin{bmatrix} 2 & -3 & 5 \\ -1 & 2 & 3 \\ 1 & 1 & 2 \end{bmatrix} = \begin{bmatrix} 1 & -3/2 & 5/2 \\ -1/2 & 1 & 3/2 \\ 1/2 & 1/2 & 1 \end{bmatrix}$

Example 6. (a) If a matrix has 12 elements, what are the possible orders it can have? What if it has 7 elements.

Sol. (a) The possible orders of a matrix having 12 elements are 2×6, 6×2, 4×3, 3×4, 12×1, 1×12.

When a matrix has 7 elements, the possible orders are 7×1 and 1×7 .

Example 7. Is it possible to define the matrix A+B, when

(i) A has 3 rows and B has 2 rows.

(ii) A has 2 columns and B has 4 columns.

(iii) A has 3 rows and B has 2 columns.

(iv) Both A and B are square matrices of the same order.

Sol. (i) No, because A+B is defined only if A and B are of the same order.

(ii) No. As above.

(iii) Yes, only when A has 2 columns and B has 3 rows for in that case both will be of the same order.

(iv) Yes. Always.

Example 8. Construct a 3×4 matrix whose elements are

(i) $a_{ij} = i+j$ (ii) $a_{ij} = i-j$

(iii) $a_{ij} = i.j$ (iv) $a_{ij} = \frac{i}{j}$

Sol. a_{ij} denotes the element of a matrix which lies in the ith row and jth column.

(i) $a_{ij} = i+j$

$a_{11} = 1+1 = 2,\ a_{12} = 1+2 = 3,\ a_{13} = 1+3 = 4,\ a_{14} = 1+4 = 5$

$a_{21} = 2+1 = 3,\ a_{22} = 2+2 = 4,\ a_{23} = 2+3 = 5,\ a_{24} = 2+4 = 6$

$a_{31} = 3+1 = 4,\ a_{32} = 3+2 = 5,\ a_{33} = 3+3 = 6,\ a_{34} = 3+4 = 7$

$\therefore$ The required matrix is $\begin{bmatrix} 2 & 3 & 4 & 5 \\ 3 & 5 & 5 & 6 \\ 4 & 4 & 6 & 7 \end{bmatrix}_{3\times 4}$

(ii) $a_{ij} = i-j$

$a_{11} = 1-1 = 0,\ a_{12} = 1-2 = -1,\ a_{13} = 1-3 = -2,\ a_{14} = 1-4 = -3$

$a_{21} = 2-1 = 1,\ a_{22} = 2-2 = 0,\ a_{33} = 2-3 = -1,\ a_{24} = 2-4 = -2$

$a_{31} = 3-1 = 2,\ a_{32} = 3-2 = 1,\ a_{33} = 3-3 = 0,\ a_{34} = 3-4 = -1$

$\therefore$ The required matrix is $\begin{bmatrix} 0 & -1 & -2 & -3 \\ 1 & 0 & -1 & -2 \\ 2 & 1 & 0 & -1 \end{bmatrix}$

Exercise 6.1

1. (a) If a matrix has 10 elements what are the possible dimensions (order) it can have

(b) Construct a 2×3 matrix whose elements a_{ij} are given by

(i) $a_{ij} = i+j$ (ii) $a_{ij} = i-j$

(iii) $a_{ij} = ij$ (iv) $a_{ij} = \frac{i}{j}$

(c) Construct a matrix 3×4 whose $a_{ij} = i+j$.

(d) What is the type of the matrix given below :

$$A = \begin{bmatrix} 2 & 3 & 0 & 1 \\ -4 & 2 & 1 & 0 \\ 7 & 1 & -3 & 1 \end{bmatrix} ?$$

Write the elements a_{11}, a_{12}, a_{34} from this matrix.

(e) Are the following matrices equal ?

$A = [2 \quad 3 \quad 5]_{1\times 3},\ B = \begin{bmatrix} 6 & 2 \\ 6 & 1 \end{bmatrix}_{2\times 2}$

2. It is possible for the following pair of matrices to be equal and, if so, for what values of 'a' does equality occur.

$$A = \begin{bmatrix} 5 & a^3 \\ a^2 & 1 \end{bmatrix}, B = \begin{bmatrix} 5 & -27 \\ 9 & 1 \end{bmatrix}$$

3. Find the additive inverse of the matrix

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 6 & 6 & 8 \\ -9 & -4 & 2 \end{bmatrix}_{3\times3}$$

4. Does the sum $\begin{bmatrix} 1 & 5 & 3 \\ 4 & 2 & 6 \\ 7 & 8 & 9 \end{bmatrix} + \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$ make sense ? If so, find the sum and if not, point out the reason.

5. Find the value of a, b, c, d from the matrix equation

$$\begin{bmatrix} a+3 & 2b-8 \\ c+1 & 4d-6 \end{bmatrix} = \begin{bmatrix} 0 & -6 \\ -3 & 2d \end{bmatrix}$$

6. Solve the matrix equation

$$2\begin{bmatrix} x & y \\ z & t \end{bmatrix} - 4\begin{bmatrix} 1 & 3 \\ 5 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 2 \\ 4 & 3 \end{bmatrix}$$

7. Find a matrix B, if A + B − 4I = O, where

$$A = \begin{bmatrix} 2 & 3 & 5 \\ 1 & 0 & 2 \\ 3 & 4 & 5 \end{bmatrix}$$

8. Prove that

(A +B)+C = A + (B +C), when

$$A = \begin{bmatrix} 0 & -1 & 2 \\ 3 & 4 & -5 \end{bmatrix}, B = \begin{bmatrix} -2 & 0 & 3 \\ 4 & -5 & 6 \end{bmatrix}$$

$$C = \begin{bmatrix} 4 & 7 & -2 \\ 0 & -5 & 1 \end{bmatrix}$$

9. Choose the correct alternative

(i) If $2\begin{bmatrix} x & y \\ z & p \end{bmatrix} - 9\begin{bmatrix} -2 & 3 \\ 1 & 0 \end{bmatrix} = 18I$, then

(a) $x = 18, z = \frac{9}{2}$ (b) $x = 0, z = -\frac{9}{2}$

(c) $x = 0, z = \frac{9}{2}$ (d) None of these.

10. If $A = \begin{bmatrix} 1 & 2 & 3 \\ -1 & 0 & 2 \\ 1 & -3 & 1 \end{bmatrix}, B = \begin{bmatrix} 4 & 5 & 6 \\ -1 & 0 & 1 \\ 2 & 1 & 2 \end{bmatrix}$

$$C = \begin{bmatrix} -1 & -2 & 1 \\ -1 & -2 & 3 \\ -1 & -2 & 2 \end{bmatrix}$$

Find (i) A – 2B + 3C,

11. If $A = \begin{bmatrix} 5 & 3 & 2 \\ 4 & 1 & 0 \\ 5 & 2 & 3 \end{bmatrix}$ and $B = \begin{bmatrix} -2 & 4 & -6 \\ 3 & -3 & 9 \\ 4 & 2 & -5 \end{bmatrix}$ then find

(i) A + B (ii) A –B (iii) 2A +5B (iv) 3B–2A.

12. Solve the following equations for A and B

$$2A+3B = \begin{bmatrix} 0 & 1 & 2 \\ 2 & 3 & 4 \\ 4 & 5 & 6 \end{bmatrix} \text{ and } 3A - 4B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

3. अव्यूहों का गुणनफल (Multiplication of Matrices)

यदि A तथा B दो आव्यूह इस प्रकार हैं कि A में स्तम्भों की संख्या, B में पंक्तियों की संख्या के बराबर हैं, अर्थात यदि $A = [a_{ij}]$ तथा $[b_{jk}]$, तो A तथा B का गुणन $AB = [c_{ik}]$ द्वारा प्रदर्शित किया जाता है।

जहॉ $c_{ik} = \sum_{j=1} a_{ij}\, b_{jk}$.

Let $A = [a_{ij}]_{m\times n}$ and $B = [b_{jk}]_{n\times p}$ be two matrices such that the number of columns in A is equal to the number of rows in B. Then $m \times p$ matrix $C = [c_{ik}]_{m\times p}$ such that

$c_{jk} = \sum_{j=1} a_{ij}\, b_{jk}$ [Note that the summation is with respect to the repeated suffix]

is called the product of the matrices A and B in that order and we write C = AB.

mnkgj.k ds fy,

$$A = \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix}_{3\times 3}, \qquad B = \begin{bmatrix} x_1 & x_2 \\ y_1 & y_2 \\ z_1 & z_2 \end{bmatrix}_{3\times 2} \quad \text{rc}$$

$$AB = \begin{bmatrix} a_1x_1 + b_1y_1 + c_1z_1 & a_1x_2 + b_1y_2 + c_1z_2 \\ a_2x_1 + b_2y_1 + c_2z_1 & a_2x_2 + b_2y_2 + c_2z_2 \\ a_3x_1 + b_3y_1 + c_3z_1 & a_3x_2 + b_3y_2 + c_3z_2 \end{bmatrix}_{3\times 2} \quad \ldots(1)$$

व्याख्या (Explanations). vkO;wg (1) का प्रथम अवयव, आव्यूह A की पंक्ति में प्रत्येक अवयव को आव्यूह B के स्तम्भ में प्रत्येक अवयव के संगत गुणनफलों के योग से प्राप्त किया जाता है।

यदि । क्रम $m\times n$ का आव्यूह तथा ठ क्रम $n\times k$ का आव्यूह है, तो इनका गुणनफल AB एक $m \times k$ क्रम का आव्यूह होगा।

आव्यूह के उत्तर–गुणन तथा पूर्व–गुणन

(Post multiplication and pre-multiplication of matrices)

AB आव्यूह, आव्यूह B में आव्यूह A के गुणा करने से प्राप्त होती है, अर्थात AB आव्यूह, आव्यूह B में आव्यूह A द्वारा उत्तर गुणन से प्राप्त होती है जबकि आव्यूह BA] आव्यूह ठ से आव्यूह A की पूर्व गुणन से प्राप्त होती है।

गुणन AB में, आव्यूह A को पूर्व–गुणनखंड़ (pre-factor) तथा आव्यूह B को उत्तर– गुणनखंड (post-factor) कहते हैं।

उपरोक्त, दोनों परिस्थितियों में गुणन AB तथा BA समान, असमान तथा अस्तित्वहीन कुछ भी हो सकते हैं,

अर्थात, व्यापक रूप में हम कह सकते हैं $AB \neq BA$.
उपरोक्त को पुनः निम्न प्रकार से देखा जा सकता हैः

स्थिति प्. यदि । क्रम उ×द की तथा ठ क्रम द × ा की दो आव्यूह हों तो इनका गुणनफल AB तो अस्तित्व रखता है किन्तु BA अस्तित्वहीन है, क्योंकि हम जानते हैं कि AB का गुणनफल तभी सम्भव है जबकि A के स्तम्भों की संख्या की B की पंक्तियों की संख्या के बराबर है।

स्थिति प्प्. यदि । क्रम उ×द तथा B क्रम द × उ की दो आव्यूह हैं, तब AB तथा BA दोनों का अस्तित्व होता है। AB का क्रम उ×उ तथा BA का क्रम द×द होता है ।
अतरू $AB \neq BA$ जबकि AB तथा BA दोनों का अस्तित्व हैं A

Matrix AB is obtained by multiplying pre factor A with post. factor B while matrix BA is obtained by multiplying A with pre factor B

In product AB, matrix A is called prefactor and matrix B is called post-factor.

In both the above situations, AB and BA can be equal, unequal or non defined.

Note I. If A is of order m×n and B is of order n×k e then product AB exists but not BA.

Note II. If A is of order m×n and B is of order n×m then both AB and BA are defined. Order of AB will be m×m while that of BA will be n×n

Example 9. If $A = \begin{bmatrix} 2 & 1 & 0 \\ 3 & 2 & 1 \\ 1 & 0 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 0 & 1 & 2 \\ 3 & 1 & 0 & 5 \end{bmatrix}$

Then find AB. Does BA exist ?

Sol. Here the matrix A is of the order 3×3 and the matrix B is of the order 3×4. Since the number of columns of A is equal to the number of rows of B, therefore AB is defined i.e., the product AB exists and it will be a matrix of the order 3×4.

$$AB = \begin{bmatrix} 2 & 1 & 0 \\ 3 & 2 & 1 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 0 & 1 & 2 \\ 3 & 1 & 0 & 5 \end{bmatrix}$$

$$= \begin{bmatrix} 2\times1+1\times2+0\times3 & 2\times2+1\times0+0\times1 & 2\times3+1\times1+0\times0 & 2\times4+1\times2+0\times5 \\ 3\times1+2\times2+1\times3 & 3\times2+2\times0+1\times1 & 3\times3+2\times1+1\times0 & 3\times4+2\times2+1\times5 \\ 1\times1+0\times2+1\times3 & 1\times2+0\times0+1\times1 & 1\times3+0\times1+1\times0 & 1\times4+0\times2+1\times5 \end{bmatrix}$$

$$= \begin{bmatrix} 4 & 4 & 7 & 10 \\ 10 & 7 & 11 & 21 \\ 4 & 3 & 3 & 9 \end{bmatrix}_{3\times4}$$

BA does not exist because number of columns of B are 4 and number of rows of A and 3. Hence they can’t be multiplied.

आव्यूहों के गुणन की विशेषताऐं (Properties of Multiplication of Matrices)

1. आव्यूहों का गुणन साहचार्य होता है । (Multiplication of matrices is associative).

Let $A = [a_{ij}]$, $B = [b_{jk}]$ and $C = [c_{kr}]$ be three matrices of order $m \times n$, $n \times p$ rFkk $p \times l$ respectively, then

$(AB).\,C = A.\,(BC)$.

Proof. Let AB = $[d_{ik}]$ tgkW $d_{ik} = \sum_{j=1}^{n} a_{ij}\ b_{jk}$ …(i)

$\therefore$ (AB). C = $[d_{ik}] \times [c_{kr}] = [e_{ir}]$,

where $e_{ir} = \sum_{k=1}^{p} d_{ik} c_{kr} = \sum_{k=1}^{p}\left(\sum_{j=1}^{n} a_{ij} b_{jk}\right) c_{kr}$ [Using (i)]

or (ir)th element of (AB).C = $\sum_{k=1}^{p}\ \sum_{j=1}^{n} a_{ij} b_{jk} c_{kr}$ …(ii)

and let BC = $[g_{jr}]$, $g_{jr} = \sum_{k=1}^{p} b_{jk} c_{kr}$ …(iii)

$\therefore$ A. (BC) = $[a_{ij}] \times [g_{jr}] = [h_{ir}]$,

where $h_{ir} = \sum_{j=1}^{n} a_{ij} g_{jr}$

$= \sum_{j=1}^{n} a_{ij}\left(\sum_{k=1}^{p} b_{jk}\ c_{kr}\right)$ [(iii) से,

or (ir)th element of A. (BC) = $\sum_{k=1}^{p}\ \sum_{j=1}^{n} a_{ij}\ b_{jk}\ c_{kr}$ …(iv)

From (iii) and (iv) we see that (ir)th element of matrices

$\therefore$ (AB).C and (A. (BC) are same and they are of same order.

vr: (AB). C = A (BC).

2. vkO;wg ;ksx ds lkis{k] vkO;wgksa dk xq.kuQy forj.k fu;e dk ikyu djrk gS (Multiplication of matrices is distributive with respect to matrix addition).

Let A = $[a_{ij}]$, B = $[b_{jk}]$ and C = $[c_{ik}]$ be three matrices of orders, m × n, n × p and n × p respectively. Therefore,

A.(B+C) = AB + AC .

Proof. A.(B+C) = $[a_{jk}] \times \{[b_{ik}] + [c_{jk}]\}$

= $[a_{ij}]\ [b_{jk} + c_{jk}] = [d_{jk}]$ (Assumption)

tgkWa $d_{jk} = \sum_{j=1}^{n} a_{ij}\ (b_{jk} + c_{jk})$

or (jk)th element of A.(B+C) = $\sum_{j=1}^{n} a_{ij}\ b_{jk} + \sum_{j=1}^{n} a_{ij}\ c_{jk}$ …(i)

Again AB = $[a_{ij}]\ [\ b_{jk}] = [e_{ik}]$ (assumed)

where $e_{ik} = \sum_{j=1}^{n} a_{ij}\ b_{jk}$, or (ik)th element of AB = $\sum_{j=1}^{n} a_{ij}\ b_{jk}$. …(ii)

We can also show that

(ik)th element of AC = $\sum_{j=1}^{n} a_{ij}\ c_{jk}$ …(iii)

From (ii) and (iii) (ik)th element of AB + AC = $\sum_{j=1}^{n} a_{ij}\ b_{jk} = \sum_{j=1}^{n} a_{ij}\ c_{jk}$ …(iv)

Hence from (i) and (iv) we get

A.(B+C) = AB + AC

3. आव्यूहों का गुणनफल सदैव क्रम विनिमेय नियम का पालन नहीं करता है

(The multiplication of matrices is not always commulative)

(a) जब AB का अस्तित्व है तो यह आवश्यक नहीं है कि BA का भी अस्तित्व हो। उदाहरण के लिए यदि आव्यूह A का क्रम 6×7 है तथा आव्यूह B का क्रम 7×8 हो तब AB का तो अस्तित्व है किन्तु BA का अस्तित्व नहीं है।

(b) जब दोनों AB तथा BA का अस्तित्व हो तो यह आवश्यक नहीं है कि दोनों आव्यूहों का क्रम समान हो।

(c) जब दोनों AB तथा BA तथा का अस्तित्व हो तथा दोनों का क्रम भी समान हो तो यह आवश्यक नहीं है कि दोनों परस्पर समान भी हों।

(a) whenever AB exists, it is not always necessary that BA should also exist. For example if A be a 6×7 matrix while B be 7×8 matrix, then AB exists while BA does not exist.

(b) Whenever AB and BA both exist, it is always not necessary that they should be matrices of the same type. For example if A be a 4×3 matrix while B be a 3×4 matrix then AB exists and it is a 4×4 matrix. In this case BA also exists and it is a 3×3 matrix. Since the matrices AB and BA are not of the same order, therefore we have AB ≠BA.

(c) Whenever AB and BA both exist and are matrices of the same type, it is not necessary that AB = BA. For example, if

$$A = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \text{ and } B = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \text{ then}$$

$$AB = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1.0+0.1 & 1.1+0.0 \\ 0.0-1.1 & 0.1-1.0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

and

$$BA = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} 0.1+1.0 & 0.0-1.1 \\ 1.1-0.0 & 1.0-0.1 \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

Thus AB ≠ BA.

(d) It however does not imply that AB is never equal to BA

For example if $A = \begin{bmatrix} 1 & 2 & 1 \\ 3 & 4 & 2 \\ 1 & 3 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} 10 & -4 & -1 \\ -11 & 5 & 0 \\ 9 & -5 & 1 \end{bmatrix}$

Then

$$AB = \begin{bmatrix} 1 & 2 & 1 \\ 3 & 4 & 2 \\ 1 & 3 & 2 \end{bmatrix} \begin{bmatrix} 10 & -4 & -1 \\ -11 & 5 & 0 \\ 9 & -5 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1(10)+2(-11)+1(9) & 1(-4)+2(5)+1(-5) & 1(-1)+2(0)+1(1) \\ 3(10)+4(-11)+2(9) & 3(-4)+4(5)+2(-5) & 3(-1)+4(0)+2(1) \\ 1(10)+3(-11)+2(9) & 1(-4)+3(5)+2(-5) & 1(-1)+3(0)+2(1) \end{bmatrix}$$

$$= \begin{bmatrix} 10-22+9 & -4+10-5 & -1+0+1 \\ 30-44+18 & -12+20-10 & -3+0+2 \\ 10-33+18 & -4+15-10 & -1+0+2 \end{bmatrix} = \begin{bmatrix} -3 & 1 & 0 \\ 4 & -2 & -1 \\ -5 & 1 & 1 \end{bmatrix}$$

Also

$$BA = \begin{bmatrix} 10 & -4 & -1 \\ -11 & 5 & 0 \\ 9 & -5 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 1 \\ 3 & 4 & 2 \\ 1 & 3 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 10(1)-4(3)-1(1) & 10(2)-4(4)-1(3) & 10(1)-4(2)-1(2) \\ -11(1)+5(3)+0(1) & -11(2)+5(4)+0(3) & -11(1)+5(2)+0(2) \\ 9(1)-5(3)+1(1) & 9(2)-5(4)+1(3) & 9(1)-5(2)+1(2) \end{bmatrix}$$

$$= \begin{bmatrix} 10-12-1 & 20-16-3 & 10-8-2 \\ -11+15+0 & -22+20+0 & -11+10+0 \\ 9-15+1 & 18-20+3 & 9-10+2 \end{bmatrix} = \begin{bmatrix} -3 & 1 & 0 \\ 4 & -2 & -1 \\ -5 & 1 & 1 \end{bmatrix}$$

Hence AB = BA

Example 10. Find $A^2-4A-5I$, where

$$A = \begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix} \text{ and } I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Sol. $$A^2 = \begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix} \times \begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1.1+2.2+2.2 & 1.2+2.1+2.2 & 1.2+2.2+2.1 \\ 2.1+1.2+2.2 & 2.2+1.1+2.2 & 2.2+1.2+2.1 \\ 2.1+2.2+1.2 & 2.2+2.1+1.2 & 2.2+2.2+1.1 \end{bmatrix}$$

$$= \begin{bmatrix} 1+4+4 & 2+2+4 & 2+4+2 \\ 2+2+4 & 4+1+4 & 4+2+2 \\ 2+4+2 & 4+2+2 & 4+4+1 \end{bmatrix} = \begin{bmatrix} 9 & 8 & 8 \\ 8 & 9 & 8 \\ 8 & 8 & 9 \end{bmatrix}$$

$\therefore A^2 - 4A - 5I$

$$= \begin{bmatrix} 9 & 8 & 8 \\ 8 & 9 & 8 \\ 8 & 8 & 9 \end{bmatrix} - 4\begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix} - 5\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 9 & 8 & 8 \\ 8 & 9 & 8 \\ 8 & 8 & 9 \end{bmatrix} + \begin{bmatrix} -4 & -8 & -8 \\ -8 & -4 & -8 \\ -8 & -8 & -4 \end{bmatrix} + \begin{bmatrix} -5 & 0 & 0 \\ 0 & -5 & 0 \\ 0 & 0 & -5 \end{bmatrix}$$

$$= \begin{bmatrix} 9-4-5 & 8-8+0 & 8-8+0 \\ 8-8+0 & 9-4-5 & 8-8+0 \\ 8-8+0 & 8-8+0 & 9-4-5 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = O,$$

जहॉ c चून्य आव्यूह है।

Example 11. If $A = \begin{bmatrix} 1 & 2 \\ -2 & 3 \end{bmatrix}$, $B = \begin{bmatrix} 2 & 1 \\ 2 & 3 \end{bmatrix}$ and $C\begin{bmatrix} -3 & 1 \\ 2 & 0 \end{bmatrix}$ verify that (AB) C = A (BC) and A(B+C) = AB + BC.

Sol. (i) $AB = \begin{bmatrix} 1 & 2 \\ -2 & 3 \end{bmatrix}\begin{bmatrix} 2 & 1 \\ 2 & 3 \end{bmatrix}$

$$= \begin{bmatrix} 2+4 & 1+6 \\ -4+6 & -2+9 \end{bmatrix} = \begin{bmatrix} 6 & 7 \\ 2 & 7 \end{bmatrix}$$

$\therefore$ $$(AB)\,C = \begin{bmatrix} 6 & 7 \\ 2 & 7 \end{bmatrix} \begin{bmatrix} -3 & 1 \\ 2 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} -18+14 & 6+0 \\ -6+14 & 2+0 \end{bmatrix} = \begin{bmatrix} -4 & 6 \\ 8 & 2 \end{bmatrix}$$

$$BC = \begin{bmatrix} 2 & 1 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} -3 & 1 \\ 2 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} -6+2 & 2+0 \\ -6+6 & 2+0 \end{bmatrix} = \begin{bmatrix} -4 & 2 \\ 0 & 2 \end{bmatrix}$$

$\therefore$ $$A(BC) = \begin{bmatrix} 1 & 2 \\ -2 & 3 \end{bmatrix} \begin{bmatrix} -4 & 2 \\ 0 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} -4+0 & 2+4 \\ 8+0 & -4+6 \end{bmatrix} \begin{bmatrix} -4 & 6 \\ 8 & 2 \end{bmatrix}$$

Hence (AB)C = A(BC)

(ii) $$B + C = \begin{bmatrix} 2 & 1 \\ 2 & 3 \end{bmatrix} + \begin{bmatrix} -3 & 1 \\ 2 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 2-3 & 1+1 \\ 2+2 & 3+0 \end{bmatrix} = \begin{bmatrix} -1 & 2 \\ 4 & 3 \end{bmatrix}$$

$\therefore$ $$A(B+C) = \begin{bmatrix} 1 & 2 \\ -2 & 3 \end{bmatrix} \begin{bmatrix} -1 & 2 \\ 4 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} -1+8 & 2+6 \\ 2+12 & -4+9 \end{bmatrix} = \begin{bmatrix} 7 & 8 \\ 14 & 5 \end{bmatrix}$$

$$AC = \begin{bmatrix} 1 & 2 \\ -2 & 3 \end{bmatrix} \begin{bmatrix} -3 & 1 \\ 2 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} -3+4 & 1+0 \\ 6+6 & -2+0 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 12 & -2 \end{bmatrix}$$

$\therefore$ $$AB + AC = \begin{bmatrix} 6 & 7 \\ 2 & 7 \end{bmatrix} + \begin{bmatrix} 1 & 1 \\ 12 & -2 \end{bmatrix}$$

$$= \begin{bmatrix} 6+1 & 7+1 \\ 2+12 & 7-2 \end{bmatrix} = \begin{bmatrix} 7 & 8 \\ 14 & 5 \end{bmatrix}$$

Hence A (B+C) = AB +AC

Example 12. If $A = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$, show that

$$(A+B)\,(A-B) \neq A^2 - B^2$$

(d) $$A+B = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} + \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 0+0 & 1-1 \\ 1+1 & 1+0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 2 & 1 \end{bmatrix}$$

$$A-B = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} - \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 0-0 & 1-(-1) \\ 1-1 & 1-0 \end{bmatrix} = \begin{bmatrix} 0 & 2 \\ 0 & 1 \end{bmatrix}$$

$$\therefore\ (A+B)\,(A-B) = \begin{bmatrix} 0 & 0 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 0 & 2 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 5 \end{bmatrix}$$

$$A^2 = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 0+1 & 0+1 \\ 0+1 & 1+1 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}$$

$$B^2 = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 0-1 & 0-0 \\ 0+0 & -1+0 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$$

$$\therefore\quad A^2 - B^2 = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} - \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} 1+1 & 1-0 \\ 1-0 & 2+1 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 1 & 3 \end{bmatrix} \neq (A+B)\,(A-B)$$

Notes (1) यदि । तथा ठ दो द क्रम के वर्ग आव्यूह हैं तो

If A and B are two nth order square matrices then

(i) $(A+B)^2 = A^2 + AB + BA + B^2$

(ii) $(A-B)^2 = A^2 - AB - BA + B^2$

(iii) $(A+B)\,(A-B) = A^2 - AB + BA - B^2$

(2) If A and B commute i.e. AB = BA then

$(A+B)^2 = A^2 + 2AB + B^2$

$(A-B)^2 = A^2 - 2AB + B^2$

$(A+B)\,(A-B) = A^2 - B^2$

Example 13. $A = \begin{bmatrix} 1 & 2 \\ -2 & 1 \end{bmatrix}$ $B = \begin{bmatrix} 2 & 1 \\ 2 & 4 \end{bmatrix}$

$$(A+B)^2 = \begin{bmatrix} 1+2 & 2+1 \\ -2+2 & 1+4 \end{bmatrix}^2 = \begin{bmatrix} 3 & 3 \\ 0 & 5 \end{bmatrix} \begin{bmatrix} 3 & 3 \\ 0 & 5 \end{bmatrix} = \begin{bmatrix} 9 & 24 \\ 0 & 25 \end{bmatrix}$$

$$A^2 = \begin{bmatrix} 1 & 2 \\ -2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ -2 & 1 \end{bmatrix} = \begin{bmatrix} -3 & 4 \\ -4 & -3 \end{bmatrix}$$

$$B^2 = \begin{bmatrix} 2 & 1 \\ 2 & 4 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 2 & 4 \end{bmatrix} = \begin{bmatrix} 6 & 6 \\ 12 & 18 \end{bmatrix}$$

$$AB = \begin{bmatrix} 1 & 2 \\ -2 & 1 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 2 & 4 \end{bmatrix} = \begin{bmatrix} 6 & 9 \\ -2 & 2 \end{bmatrix}$$

$$BA = \begin{bmatrix} 2 & 1 \\ 2 & 4 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ -2 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 5 \\ -6 & 8 \end{bmatrix}$$

Now (i) To prove $(A+B)^2 = A^2 + AB + BA + B^2$

$$(A+B)^2 = \begin{bmatrix} 9 & 24 \\ 0 & 25 \end{bmatrix}$$

$$A^2 + AB + BA + B^2 = \begin{bmatrix} -3 & 4 \\ -4 & -3 \end{bmatrix} + \begin{bmatrix} 6 & 9 \\ -2 & 2 \end{bmatrix} + \begin{bmatrix} 0 & 5 \\ -6 & 8 \end{bmatrix} + \begin{bmatrix} 6 & 6 \\ 12 & 18 \end{bmatrix}$$

$$= \begin{bmatrix} -3+6+0+6 & 4+9+5+6 \\ -4-2-6+12 & -3+2+8+18 \end{bmatrix} = \begin{bmatrix} 9 & 24 \\ 0 & 25 \end{bmatrix} = (A+B)^2$$

Hence proved.

(ii) $(A-B)^2 = A^2 - AB - BA + B^2$

$$(A-B) = \begin{bmatrix} 1 & 2 \\ -2 & 1 \end{bmatrix} - \begin{bmatrix} 2 & 1 \\ 2 & 4 \end{bmatrix} = \begin{bmatrix} 1-2 & 2-1 \\ -2-2 & 1-4 \end{bmatrix} = \begin{bmatrix} -1 & 1 \\ -4 & -3 \end{bmatrix}$$

$$\text{Now } (A-B)^2 = \begin{bmatrix} -1 & 1 \\ -4 & -3 \end{bmatrix} \begin{bmatrix} -1 & 1 \\ -4 & -3 \end{bmatrix} = \begin{bmatrix} -3 & -4 \\ 16 & 5 \end{bmatrix}$$

$$A^2 - AB - BA + B^2 = \begin{bmatrix} -3 & 4 \\ -4 & -3 \end{bmatrix} - \begin{bmatrix} 6 & 9 \\ -2 & 2 \end{bmatrix} - \begin{bmatrix} 0 & 5 \\ -6 & 8 \end{bmatrix} + \begin{bmatrix} 6 & 6 \\ 12 & 18 \end{bmatrix}$$

$$= \begin{bmatrix} -3-6-0+6 & 4-9-5+6 \\ -4+2+6+12 & -3-2-8+18 \end{bmatrix} = \begin{bmatrix} -3 & -4 \\ 16 & 5 \end{bmatrix} = (A-B)^2$$

Hence proved.

(iii) $(A+B)(A-B) = A^2 - AB + BA - B^2$

$$(A+B) = \begin{bmatrix} 3 & 3 \\ 0 & 5 \end{bmatrix}, (A-B) = \begin{bmatrix} -1 & 1 \\ -4 & -3 \end{bmatrix}$$

$$\therefore \quad (A+B)(A-B) = \begin{bmatrix} 3 & 3 \\ 0 & 5 \end{bmatrix} \begin{bmatrix} -1 & 1 \\ -4 & -3 \end{bmatrix} = \begin{bmatrix} -15 & -6 \\ -20 & -15 \end{bmatrix}$$

$$A^2 - AB + BA - B^2 = \begin{bmatrix} -3 & 4 \\ -4 & -3 \end{bmatrix} - \begin{bmatrix} 6 & 9 \\ -2 & 2 \end{bmatrix} + \begin{bmatrix} 0 & 5 \\ -6 & 8 \end{bmatrix} - \begin{bmatrix} 6 & 6 \\ 12 & 18 \end{bmatrix}$$

$$= \begin{bmatrix} -3-6+0-6 & 4-9+5-6 \\ -4+2-6-12 & -3-2+8-18 \end{bmatrix} = \begin{bmatrix} -15 & -6 \\ -20 & -15 \end{bmatrix}$$

$$= (A+B)(A-B)$$

Hence proved.

Example 14. भारत के एक क्येिड्ढ राज्य में, एक फाइनेन्स कम्पनी के n∂rj प्रत्येक जिले, प्रत्येक कस्बे, प्रत्येक गाँव में हैं। मान लीजिए,राज्य में कुल 5 जिले] 30 कस्बे तथा 200 गाँव हैं। प्रत्येक n∂rj में एक बड़ा बाबू ; Head-clerk), एक खंजाची (Cashier)]

एक बाबू (Clerk) तथा एक चपरासी (Peon) है। जिला ऑफिस में, उपरोक्त के अतिरिक्त एक n∂rj अधीक्षक (Office Superintendent)] दो बाबू, एक टाइपिस्ट तथा एक चपरासी है। कस्बे के आफिस में गाँव के ऑफिस के अतिरिक्त एक बाबू तथा एक चपरासी है। सभी कर्मचारियों के मासिक वेतन निम्न प्रकार से है :

n∂rj अधीक्षक को रु॰ 500, बड़े बाबू को रु॰ 200, खंजाची को रु॰ 175, बाबू तथा टाइपिस्ट प्रत्येक को प्रत्येक को रु॰ 150, तथा चपरासी को रु॰ 100,

आव्यूह विधि से ज्ञात किजिए :

(i) सभी दफ्तरों के प्रत्येक प्रकार के पदों की संख्या

(ii) प्रत्येक दफ्तर के मासिक वेतन का बिल

(iii) सभी दफ्तरों के मासिक वेतन का कुल बिल

In a State in India, a finance company has its offices in every district, town and village. Suppose there are 5 districts, 30 towns and 200 villages. Each office has 1 head-clerk, 1 cashier, 1 clerk and 1 peon. In addition, each district office has 1 office superintendent, 2 clerks, 1 typist and 1 peon. Each town office has, in addition to village office staff, 1 clerk and 1 peon. Basic salary of all the employees is as follows:

Office superintendent Rs. 500, Head clerk Rs. 200, Cashier Rs. 175, Clerk and typist Rs. 150 each and peon Rs. 100. Using the matrix notation, find :

(i) Total number of posts of each kind in all the offices;

(ii) Monthly basic salary bill for each office and

(iii) Total of salary bills of all the offices.

Sol. Let matrix A represent the number of offices in district, towns and villages

$\therefore \quad A = [5 \quad 30 \quad 200]$

Let matrix B represent the number of posts in the three offices.

	O.S.	HC	Cr.	Cl.	T.	P.	where
District office	1	1	1	3	1	2	OS → Office Superintendent
Town office	0	1	1	2	0	2	HC → Head Clerk
Village office	0	1	1	1	0	1	Cr → Cashier
							Cl → Clerk
							T → Typist
							P → Peon

So $$B = \begin{bmatrix} 1 & 1 & 1 & 3 & 1 & 2 \\ 0 & 1 & 1 & 2 & 0 & 2 \\ 0 & 1 & 1 & 1 & 0 & 1 \end{bmatrix}$$

Further let matrix C represent basic salary of each kind post

So

$$C = \begin{bmatrix} 500 \\ 200 \\ 175 \\ 150 \\ 150 \\ 100 \end{bmatrix} \begin{matrix} -OS \\ -HC \\ -Cr \\ -Cl \\ T \\ P \end{matrix}$$

Now answer to part (i)

Total number of posts of each kind in all the offices

$$= A \times B = [5 \quad 30 \quad 200] \begin{bmatrix} 1 & 1 & 1 & 3 & 1 & 2 \\ 0 & 1 & 1 & 2 & 0 & 2 \\ 0 & 1 & 1 & 1 & 0 & 1 \end{bmatrix}$$

$$= [5 \quad 235 \quad 235 \quad 275 \quad 5 \quad 270]$$

Part (ii) Monthly basic salary bill for each office

$$= B \times C = \begin{bmatrix} 1 & 1 & 1 & 3 & 1 & 2 \\ 0 & 1 & 1 & 2 & 0 & 2 \\ 0 & 1 & 1 & 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 500 \\ 200 \\ 175 \\ 150 \\ 150 \\ 100 \end{bmatrix} = \begin{bmatrix} 500+200+175+450+150+200 \\ 0+200+175+300+0+200 \\ 0+200+175+150+0+100 \end{bmatrix}$$

$$= \begin{bmatrix} 1675 \\ 875 \\ 625 \end{bmatrix}$$

Part (iii) Total monthly salary bills of all offices

$$= A(BC) = [5 \quad 30 \quad 200] \begin{bmatrix} 1675 \\ 875 \\ 625 \end{bmatrix}$$

$$= [8375 + 26250 + 125000] = [159625] .$$

Example 15. If A = $\begin{bmatrix} 2 & 3 \\ 3 & 10 \end{bmatrix}$ and I, the identity matrix of order 2 show that

(2I – A) (10I – A) = 9I ,

Sol. $2I-A = 2\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 2 & 3 \\ 3 & 10 \end{bmatrix}$

$= \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} - \begin{bmatrix} 2 & 3 \\ 3 & 10 \end{bmatrix}$

$= \begin{bmatrix} 2-2 & 0-3 \\ 0-3 & 2-10 \end{bmatrix} = \begin{bmatrix} 0 & -3 \\ -3 & -8 \end{bmatrix}$

$10I-A = 10\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 2 & 3 \\ 3 & 10 \end{bmatrix}$

$= \begin{bmatrix} 10 & 0 \\ 0 & 10 \end{bmatrix} - \begin{bmatrix} 2 & 3 \\ 3 & 10 \end{bmatrix} = \begin{bmatrix} 10-2 & 0-3 \\ 0-3 & 10-10 \end{bmatrix}$

$= \begin{bmatrix} 8 & -3 \\ -3 & 0 \end{bmatrix}$

$\therefore$ $(2I-A)(10I-A) = \begin{bmatrix} 0 & -3 \\ -3 & -8 \end{bmatrix}\begin{bmatrix} 8 & -3 \\ -3 & 0 \end{bmatrix}$

$= \begin{bmatrix} 0+9 & 0+0 \\ -24+24 & 9+0 \end{bmatrix} = \begin{bmatrix} 9 & 0 \\ 0 & 9 \end{bmatrix}$

$= 9\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = 9I$.

Hence proved.

Example 16. If A = $\begin{bmatrix} 3 & -5 \\ -4 & 2 \end{bmatrix}$, find $A^2-5A-14I$.

Sol. $A^2 = A.A = \begin{bmatrix} 3 & -5 \\ -4 & 2 \end{bmatrix}\begin{bmatrix} 3 & -5 \\ -4 & 2 \end{bmatrix}$

$= \begin{bmatrix} 9+20 & -15-10 \\ -12-8 & 20+4 \end{bmatrix} = \begin{bmatrix} 29 & -25 \\ -20 & 24 \end{bmatrix}$

Now $A^2-5A-14I$

$= \begin{bmatrix} 29 & -25 \\ -20 & 24 \end{bmatrix} - 5\begin{bmatrix} 3 & -5 \\ -4 & 2 \end{bmatrix} - 14\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$= \begin{bmatrix} 29 & -25 \\ -20 & 24 \end{bmatrix} - \begin{bmatrix} 15 & -25 \\ -20 & 10 \end{bmatrix} - \begin{bmatrix} 14 & 0 \\ 0 & 14 \end{bmatrix}$

$= \begin{bmatrix} 29-15-14 & -25+25+0 \\ -20+20-0 & 24-10-14 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = O$.

Example 17. If the matrix A = $\begin{bmatrix} 5 & 3 \\ 12 & 7 \end{bmatrix}$, then verify that $A^2-12A-I = O$, where I is a unit matrix of order 2.

Sol. $A^2 = A.A. = \begin{bmatrix} 5 & 3 \\ 12 & 7 \end{bmatrix}\begin{bmatrix} 5 & 3 \\ 12 & 7 \end{bmatrix}$

$$= \begin{bmatrix} 25+36 & 15+21 \\ 60+84 & 36+49 \end{bmatrix} = \begin{bmatrix} 61 & 36 \\ 144 & 85 \end{bmatrix}$$

Now $A^2 - 12A - I$

$$= \begin{bmatrix} 61 & 36 \\ 144 & 85 \end{bmatrix} - 12 \begin{bmatrix} 5 & 3 \\ 12 & 7 \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 61-60-1 & 36-36-0 \\ 144-144-0 & 85-84-1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = O$$

Exercise 6.2

1. Find the matrix AB and BA, whichever possible, if

(i) $A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$, $B = \begin{bmatrix} 0 & 1 & 2 \\ 2 & 1 & 0 \end{bmatrix}$

(ii) $A = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 2 & 3 \\ 2 & 3 & 4 \end{bmatrix}$, $B = \begin{bmatrix} 1 & -2 \\ -1 & 0 \\ 1 & 1 \end{bmatrix}$

(iii) $A = \begin{bmatrix} 2 & 1 & 2 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}$, $B = \begin{bmatrix} 2 & -1 & 0 \\ 0 & 4 & -1 \\ -2 & 1 & 0 \\ 1 & -3 & 2 \end{bmatrix}$

2. If $A = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$ show that AB is a null matrix

3. If $A = \begin{bmatrix} 1 & 4 & 0 \\ 2 & 5 & 0 \\ 3 & 6 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 3 & 2 & 1 \\ 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$ and $C = \begin{bmatrix} 3 & 2 & 1 \\ 1 & 2 & 3 \\ 7 & 8 & 9 \end{bmatrix}$

then find AB − AC

4. ;fn $A = \begin{bmatrix} 1 & -1 & 1 \\ -3 & 2 & -1 \\ -2 & 1 & 0 \end{bmatrix}$ rFkk $B = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 1 & 2 & 3 \end{bmatrix}$, rks nÓkZb;s fd

$AB \neq BA$.

5. If $A = \begin{bmatrix} 1 & 2 & -1 \\ 2 & 0 & 3 \\ 0 & 1 & 2 \end{bmatrix}$, $B = \begin{bmatrix} 3 & -1 & 1 \\ 0 & 0 & 2 \\ 4 & -3 & 2 \end{bmatrix}$

verify that

$(A+B)(A-B) = A^2 - AB + BA - B^2$.

6. If $A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}$

where $i = \sqrt{-1}$

verify that $(A+B)^2 = A^2 + B^2$.

7. If $A=\begin{bmatrix}1 & -1\\0 & 2\end{bmatrix}$ and $B = \begin{bmatrix}1 & 0\\1 & 2\end{bmatrix}$

verify that

(i) $(A+B)^2 \neq A^2 + 2AB + B^2$

(ii) $(A+B)(A-B) \neq A^2 - B^2$.

8. Show that $A = \begin{bmatrix}1 & 0 & -2\\2 & 2 & 4\\0 & 0 & 2\end{bmatrix}$ satisfies the equation

$A^2 - 3A + 2I = O$.

9. A fruit seller has in stock 20 dozen mangoes, 16 dozen apples and 32 dozen bananas. Suppose the selling prices are Rs. 0.35, Rs. 0.75 and Rs. 0.80 per mango, apple and banana respectively. Find the total amount the fruit seller will get by selling his whole stock.

10. A firm has in stock 12 dozen blankets, 10 dozen coats and 5 dozen gowns. The selling prices are Rs. 200, Rs. 160 and Rs. 100 each respectively. Find the total amount the firm will receive from selling all the items.

11. In a development plan of a city, a contractor has taken a contract to construct certain houses for which he needs building materials like stones, sand etc. There are three firms, A, B, C that can supply him these material. At one time these firms A, B, C supplied him 40, 35 and 25 truck loads of stones and 10, 5 and 8 truck loads sand respectively. If the cost of one truck load of stone and sand is Rs. 1,200 and Rs. 500 respectively then find the total amount paid by the contractor to each of these firms, A, B, C respectively.

12. A man buys 8 dozens of mangoes, 10 dozens of apples and 4 dozens of bananas. Mangoes cost Rs 18 per dozen, apples Rs. 9 per dozen and bananas Rs. 6 per dozen. Represent the quantities bought by a row matrix and the prices by a column matrix and hence obtain the total cost.

13. A store has in stock 20 dozen shirts, 15 dozen trousers and 25 dozen pairs of socks. If the selling prices are Rs. 50 per shirt, Rs. 90 per trouser and Rs. 12 per pair of socks, then find the total amount the store owner will get after selling all the items in the stock.

14. A trust fund has Rs. 50,000 that is to be invested into two types of bonds. The first bond pays 5% interest per year and the second bond pays 6% interest per year. Using matrix multiplication determine how to divide by Rs. 50,000 among the two types of bonds so as to obtain an annual total interest of Rs. 2780.

15. The following matrix gives the proportionate mix of constituents used for three fertilizers.

		A	B	C	D
	I	0.5	0	0.5	0
Fertilizer	II	0.2	0.3	0	0.5
	III	0.2	0.2	0.1	0.5

(i) If sales are 1000 tins (of 1kg) per week, 20% being fertilizer I,30% being fertilizer II and 50% fertilizer III, how much of each constituent is used.

(ii) If the cost of each constituent is Rs. 0.50, 0.60, 0.75 and Rs. 1 per 100 gms. respectively, how much does a one kg. tin of each fertilizer cost.

(iii) What is the total cost per week.

Answers

Exercise 6.1

1. (a) 1×10, 10×1, 2×5, 5×2.

(b) (i) $\begin{bmatrix} 2 & 3 & 4 \\ 3 & 4 & 5 \end{bmatrix}$ (ii) $\begin{bmatrix} 0 & -1 & -2 \\ 1 & 0 & -1 \end{bmatrix}$

(iii) $\begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \end{bmatrix}$ (iv) $\begin{bmatrix} 1 & \frac{1}{2} & -\frac{1}{3} \\ 2 & 1 & \frac{2}{3} \end{bmatrix}$

(c) $\begin{bmatrix} 2 & 3 & 4 & 5 \\ 3 & 4 & 5 & 6 \\ 4 & 5 & 6 & 7 \end{bmatrix}$

(d) 3×4, $a_{11} = 2$; $a_{12} = 3$; $a_{24} = 0$

(e) $A \neq B$, because they are not of the same type.

2. $a = -3$.

3. $-A = \begin{bmatrix} -1 & 2 & -3 \\ -6 & -7 & -8 \\ 9 & -4 & -2 \end{bmatrix}_{3\times3}$

4. Since the two matrices are of different types, therefore their sum cannot be obtained.

5. $a = -3$, $b = 1$, $c = -4$, $d = 3$.

6. $x = 2$, $y = 11$, $z = 20$, $t = 7.5$

7. $B = \begin{bmatrix} 2 & -3 & -5 \\ -1 & 4 & 2 \\ -3 & -4 & -4 \end{bmatrix}$

10. (i) $\begin{bmatrix} -10 & -4 & -6 \\ -2 & 6 & 9 \\ -6 & -11 & 1 \end{bmatrix}$ (ii) $\begin{bmatrix} -2 & -1 & 0 \\ -1 & 0 & -7 \\ 0 & 3 & -4 \end{bmatrix}$

11. (i) $\begin{bmatrix} 3 & 7 & -4 \\ 6 & -2 & 9 \\ 9 & 4 & -2 \end{bmatrix}$ (ii) $\begin{bmatrix} 7 & -1 & 8 \\ 2 & 4 & -9 \\ 1 & 0 & 8 \end{bmatrix}$ (iii) $\begin{bmatrix} 0 & 20 & -26 \\ 18 & -13 & 45 \\ 30 & 14 & -19 \end{bmatrix}$ (iv) $\begin{bmatrix} -16 & 6 & 14 \\ -2 & -11 & 27 \\ 2 & 2 & -21 \end{bmatrix}$

12. $A = \frac{1}{17}\begin{bmatrix} 3 & 4 & 8 \\ 8 & 15 & 16 \\ 16 & 20 & 27 \end{bmatrix}$, $B = \frac{1}{17}\begin{bmatrix} -2 & 3 & 6 \\ 6 & 7 & 12 \\ 12 & 15 & 16 \end{bmatrix}$

Exercise 6.2

1. (i) $AB = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 1 & 2 \\ 2 & 1 & 0 \end{bmatrix}$, $BA = \begin{bmatrix} 1 & 2 \\ 1 & 2 \end{bmatrix}$

(ii) $AB = \begin{bmatrix} 3 & -2 \\ 5 & -5 \\ 7 & -8 \end{bmatrix}$, BA is not defined

(iii) AB = $\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}$, BA is not defined.

3. Null matrix 9. Rs 535.20 10. Rs. 54000
11. Rs 53000 to A, Rs 44500 to B and Rs 34000 to C 12. Rs. 258
13. Rs 31800
14. Rs. 22000 in first type of bond and Rs. 28000 in 2^{nd} type of bond

Chapter-7

सारणिक
(Determinants)

सारणिक एक वर्ग आव्यूह का विशेष प्रकार है। हर वर्ग आव्यूह । त्र ख्ंपर, के साथ एक सारणिक जुड़ा हुआ है जिसे हम द्य।द्य से बताते हैं। सारणिक एक वर्ग आव्यूह की केवल एक अदिश राशि है ॑;बंसंत ृनंदजपजलद्ध है अर्थात सारणिक एक मूल्य से जुड़ा है जबकि आव्यूह संख्याओं का एक निकाय है जिसका कोई मूल्य नहीं होता।

सामान्यतयाः द$^{जी}$ क्रम के एक सारणिक को हम निम्नलिखित तरीके से प्रस्तुत करते हैंः

$$|A| = \begin{vmatrix} a_{11} & a_{12} & a_{13}\dots & a_{1n} \\ a_{21} & a_{22} & a_{23}\dots & a_{2n} \\ a_{31} & a_{32} & a_{33}\dots & a_{3n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{n1} & a_{n2} & a_{n3}\dots & a_{nn} \end{vmatrix}$$

प्रथम क्रम का सारणिक (Determinant of first order)

$A = |a_{11}|$

द्वितीय क्रम का सारणिक (Determinant of second order)

$$|A| = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}$$

तृतीय क्रम का सारणिक (Determinant of third order)

$$|A| = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

किसी भी क्रम के सारणिक का मुल्य निकालने के लिए हमें उसे 2×2 के सारणिक में परिवर्तित करना पडेगा, जिसका मान हम सीधे ढंग से निकाल सकते हैं ! उदाहरण के लिए

$$A = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11} \times a_{22} - a_{12} \times a_{21}$$

इसका मुल्य निकालने के लिए हम आर पार के गुणफल ;बतवेउनसजपचसपबंजपवदद्ध का प्रयोग करते है ! तीसरे क्रम के सारणिक का मुल्य हम इस प्रकार ज्ञात करते है !

$$A = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12} \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13} \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

$$= a_{11}(a_{22} \times a_{33} - a_{23} \times a_{32}) - a_{12}(a_{21} \times a_{33} - a_{23} \times a_{31}) + a_{13}(a_{21} \times a_{32} - a_{22} \times a_{31})$$

सारणिक को Δ से भी बताते हैं।

Example 1. Find the value of

(i) $\begin{vmatrix} 2 & 4 \\ 3 & 6 \end{vmatrix}$ (ii) $\begin{vmatrix} 4 & 2 \\ -2 & 5 \end{vmatrix}$ (iii) $\begin{vmatrix} a+1 & a-2 \\ a+2 & a-1 \end{vmatrix}$ (iv) $\begin{vmatrix} \sqrt{5} & \sqrt{48} \\ \sqrt{3} & \sqrt{45} \end{vmatrix}$

Solution.

(i) $\begin{vmatrix} 2 & 4 \\ 3 & 6 \end{vmatrix} = 2 \times 6 - 3 \times 4 = 12-12 = 0$

(ii) $\begin{vmatrix} 4 & 2 \\ -2 & 5 \end{vmatrix} = 4\times 5 - 2(-2) = 20+4 = 24$

(iii) $\begin{vmatrix} a+1 & a-2 \\ a+2 & a-1 \end{vmatrix} = (a+1)(a-1) - (a-2)(a+2)$

$= a^2-1 - a^2+4 = 3$

(iv) $\begin{vmatrix} \sqrt{5} & \sqrt{48} \\ \sqrt{3} & \sqrt{45} \end{vmatrix} = \sqrt{5}\times\sqrt{45} - \sqrt{3}\times\sqrt{48}$

$= \sqrt{225} - \sqrt{144} = 15-12 = 3$.

Example 2. Solve for x

(i) $\begin{vmatrix} 2x+3 & x-3 \\ 2x+1 & x+2 \end{vmatrix} = 0$, (ii) $\begin{vmatrix} x-3 & x+1 \\ x+2 & x-1 \end{vmatrix} = 0$

Solution. (i) $\begin{vmatrix} 2x+3 & x-3 \\ 2x+1 & x+2 \end{vmatrix} = 0$

or $(2x+3)(x+2) - (2x+1)(x-3) = 0$

or $2x^2+7x+6 - 2x^2+5x+3 = 0$

$12x+9 = 0$

or $x = -\frac{9}{12} = -\frac{3}{4}$

(ii) $\begin{vmatrix} x-3 & x+1 \\ x+3 & x-1 \end{vmatrix} = 0$

or $(x-3)(x-1) - (x+2)(x+1) = 0$

or $x^2-4x+3-x^2-3x-2 = 0$

or $-7x+1 = 0$

or $x = -\frac{1}{7}$

Example 3. Evaluate $\begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{vmatrix}$

Solution. $\Delta = \begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{vmatrix} = 1 \begin{vmatrix} 5 & 6 \\ 8 & 9 \end{vmatrix} - 3 \begin{vmatrix} 4 & 6 \\ 7 & 9 \end{vmatrix} + \begin{vmatrix} 4 & 5 \\ 7 & 8 \end{vmatrix}$

$= 1(45 - 48) - 2(36-42) + 3\ (32-35)$

$= -3 + 12 - 9 = 0$

Notes –

(1) किसी भी द$^{\text{जी}}$ क्रम के सारणिक में कुल अवयवों की संख्या में होती है। दूसरे क्रम के सारणिक में 2^2 अर्थात 4 अवयव होते हैं तथा तीसरे क्रम के सारणिक में 3^2 यानि अवयव होते हैं।

Total number of elements in a determinant of r^{th} order is n^2. Number of elements in a second order determinant is 2^2 (4) and for a 3^{rd} order determinant number of elements is 3^2 (9).

(2). किसी भी सारणिक का विस्तार करने के लिए हम किसी भी एक पंक्ति या एक स्तंभ ले सकते हैं। हर हालात में सारणिक का मूल्य एक सा होगा।

We can expand a determinant by taking any one row or one column. In every case, the value of the determinant will remain the same.

3. विस्तार करते समय हम हर अवयव से पहले चिन्ह निकालते हैं जो कि ;–1द्ध$^{\text{पर}}$ के बराबर होता है। प तथा र उस पंक्ति तथा स्तंभ को दर्शाते हैं जिनमें वह अवयव है। उदाहरण के लिए मान लो कि अवयव तीसरी पंक्ति तथा दूसरे स्तंभ है तो उसके आगे चिन्ह ;–1द्ध^{32} यानि –होगा.

Example 4. Evaluate the determinant

$$\begin{vmatrix} b^2+c^2 & ab & ac \\ ab & c^2+a^2 & bc \\ ca & cb & a^2+b^2 \end{vmatrix}$$

Solution. Expanding along Ist row

$$\Delta = (b^2+c^2)\begin{vmatrix} c^2+a^2 & bc \\ cb & a^2+b^2 \end{vmatrix} - ab\begin{vmatrix} ab & ac \\ cb & a^2+b^2 \end{vmatrix} + ca\begin{vmatrix} ab & ac \\ c^2+a^2 & bc \end{vmatrix}$$

$$= (b^2+c^2)\,[(c^2+a^2)(a^2+b^2)-b^2c^2] - ab[ab(a^2+b^2)-abc^2] + ca[ab^2c-ac(c^2+a^2)]$$

$$= (b^2+c^2)(a^4+a^2b^2+a^2c^2) - a^2b^2(a^2+b^2-c^2) + c^2a^2(b^2-c^2-a^2)$$

$$= a^4b^2+a^2b^4+a^2c^2b^2+a^4c^2+a^2b^2c^2+a^2c^4 - a^4b^2-a^2b^4+a^2b^2c^2+a^2b^2c^2-a^2c^4-c^2a^4$$

$$= a^2b^2c^2 + a^2b^2c^2 + a^2b^2c^2 + a^2b^2c^2$$

$$= 4\,a^2b^2c^2 .$$

Example 5. Show that $\begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ bc & ca & ab \end{vmatrix} = (a-b)\,(b-c)\,(c-a)$

Solution.

$$\Delta = \begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ bc & ca & ab \end{vmatrix} = 1\begin{vmatrix} b & c \\ ca & ab \end{vmatrix} - 1\begin{vmatrix} a & c \\ bc & ab \end{vmatrix} + 1\begin{vmatrix} a & b \\ bc & ca \end{vmatrix}$$

$$= 1(ab^2-ac^2) - (a^2b - bc^2) + 1(a^2c - b^2c)$$

$$= ab^2-ac^2-a^2b+bc^2+a^2c-b^2c$$

$$= (ab^2-ac^2)-(a^2b-a^2c)-(b^2c-bc^2)$$

$$= a(b+c)(b-c)-a^2(b-c)-bc(b-c)$$

$$= (b-c)[ab+ac-a^2-bc]$$

$$= (b-c)[(ac-bc)-(a^2-ab)]$$

$$= (b-c)\,[c(a-b) - a(a-b)]$$

$$= (b-c)\,(c-a)\,(a-b)$$

$$= (a-b)\,(b-c)(c-a)$$

Hence proved.

lgxq.ku rFkk lgxq.ku[kaM (Minors and Co-factors)

किसी भी अवयव a_{ij} का सहगुणन वह सारणिक है जो i पंक्ति तथा स्तंभ j काटने के बाद बचता है। सही चिन्ह के सहगुणन को सहगुणनखंड़ कहते हैं।

Minor of an element a_{ij} is the determinant left after deleting the i^{th} row and j^{th} column from the original determinant. A minor with proper sign is called co-factor. For example in the determinant

$$\Delta = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

Minor of the element $b_3 = \begin{vmatrix} a_1 & a_2 \\ c_1 & c_2 \end{vmatrix}$

Now minor with proper sign $(-1)^{i+j}$ is the cofactors. The sign can be either positive or negative.

Now co-factor of element $b_3 = (-1)^{2+3} \begin{vmatrix} a_1 & a_2 \\ c_1 & c_2 \end{vmatrix}$

$$= \begin{vmatrix} a_1 & a_2 \\ c_1 & c_2 \end{vmatrix}$$

Example 6. Find the minors of all the elements of the determinant

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

Let minor of an element a_{ij} be represented by M_{ij}

$$\therefore \quad M_{11} = \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix}, \quad M_{12} = \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix}, \quad M_{13} = \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

$$M_{21} = \begin{vmatrix} a_{12} & a_{13} \\ a_{32} & a_{33} \end{vmatrix}, \quad M_{22} = \begin{vmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{vmatrix}, \quad M_{23} = \begin{vmatrix} a_{11} & a_{12} \\ a_{31} & a_{32} \end{vmatrix}$$

$$M_{31} = \begin{vmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \end{vmatrix}, \quad M_{32} = \begin{vmatrix} a_{11} & a_{13} \\ a_{21} & a_{23} \end{vmatrix}, \quad M_{33} = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}$$

Example 7. Find the co-factors of the elements a_{22}, a_{12}, a_{31} in the determinant

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

Co-factor of a_{22}

$$C_{22} = (-1)^{2+2} \begin{vmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{vmatrix} = (a_{11} \times a_{33} - a_{13} \times a_{31})$$

Co-factor $a_{12} = C_{12} = (-1)^{1+2} \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} = -(a_{21} \times a_{33} - a_{23} \times a_{31})$

Co-factor of $a_{31} = C_{31} = (-1)^{3+1} \begin{vmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \end{vmatrix} = (a_{12} \times a_{23} - a_{13} \times a_{22})$

Example 8. Write the minors and co-factors of each element of the first column of the following determinant and evaluate the determinant in each case.

(i) $\begin{vmatrix} 5 & 20 \\ 0 & -1 \end{vmatrix}$ (ii) $\begin{vmatrix} 1 & -3 & 2 \\ 4 & -1 & 2 \\ 3 & 5 & 2 \end{vmatrix}$

Solution. (i) $M_{11} = -1$

$C_{11} = (-1)^{1+1}(-1) = 1(-1) = -1$

$M_{21} = 20$

$C_{21} = (-1)^{2+1}(20) = (-1)(20) = -20$.

(ii) $M_{11} = \begin{vmatrix} -1 & 2 \\ 5 & 2 \end{vmatrix} = -1(2) - 5(2) = -2-10 = -12$

$C_{11} = (-1)^{1+1}\begin{vmatrix} -1 & 2 \\ 5 & 2 \end{vmatrix} = 1(-2-10) = -12$

$M_{21} = \begin{vmatrix} -3 & 2 \\ 5 & 2 \end{vmatrix} = (-3)(2) - 5(2) = -6-10 = -16$

$C_{21} = (-1)^{2+1}\begin{vmatrix} -3 & 2 \\ 5 & 2 \end{vmatrix} = (-1)(-6-10) = 16$

$M_{31} = \begin{vmatrix} -3 & 2 \\ -1 & 2 \end{vmatrix} = (-3)(2) - (-1)(2) = -6+2 = -4$

$C_{31} = (-1)^{3+1}\begin{vmatrix} -3 & 2 \\ -1 & 2 \end{vmatrix} = (1)(-6+2) = -4.$

तीसरे क्रम के सारणिक का सरोस नियम से विस्तार

(Expansion of determinant of third order by Sarrous rule)

Let $\Delta = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$

Sarraus rule is written in the following way

$$\begin{matrix} a_{11} & a_{12} & a_{13} & a_{11} & a_{12} \\ a_{21} & a_{22} & a_{23} & a_{21} & a_{22} \\ a_{31} & a_{32} & a_{33} & a_{31} & a_{32} \end{matrix}$$

उन गुणनफलों को जो सीधे रेखा ;——द्ध से जुड़ते है को घनात्मक चिहन से जोडो व उन गुणनफलों को जो कटी हुई रेखा ;३ण्णद्ध से जुडते हैं ऋणात्मक चिहनों से जोड़ो !

Add the product of elements which lie on continuous line, by a positive sign and those on the dotted line by a negative sign.

So $\Delta = a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{13}a_{22}a_{31} - a_{11}a_{23}a_{32} - a_{12}a_{21}a_{33}$

Example 9. Evaluate by Sarraus diagram

$$\begin{vmatrix} 5 & 15 & -25 \\ 7 & 21 & 30 \\ 8 & 24 & 42 \end{vmatrix}$$

Solution. Writing in form of Sarraus diagram

5	15	25	5	15
7	21	30	7	21
8	24	42	8	24

$\Delta = 5\times21\times42 + 15\times30\times8 + (-25)\times7\times24 - (-25)\times21\times8 - 5\times30\times24 - 15\times7\times42 = 4410 + 3600 - 4200 + 4200 - 3600 - 4410$

$= 0$

सहगुणनखण्ड़ो की सहायता से सारणिक का विस्तार
(Expansion of a determinant with the help of co factors).

Let $\Delta = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$

It's value can be obtained by expanding by any row or column.
For example if we expand by first row, then

$$\Delta = (-1)^{1+1} a_{11} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} + (-1)^{1+2} a_{12} \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + (-1)^{1+3} \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

$= a_{11}. C_{11} + a_{12} . C_{12} + a_{13} . C_{13}$

Similarly $\Delta = a_{21} . C_{12} + a_{22}\ C_{22} + a_{23} . C_{23}$ (Expansion by second row)

$\Delta = a_{31} . C_{31} + a_{32}\ C_{32} + a_{33} . C_{33}$ (Expansion by third row)

$\Delta = a_{11} . C_{11} + a_{21}\ C_{21} + a_{31} . C_{31}$ (Expansion by first column)

$\Delta = a_{12} . C_{12} + a_{22}\ C_{22} + a_{32} . C_{32}$ (Expansion by second column)

$\Delta = a_{13} . C_{13} + a_{23}\ C_{23} + a_{33} . C_{33}$ (Expansion by third column)

lkjf.kdksa dh foÓs"krk,W
(Properties of determinants)

सारणिकों की कद्छ क्येिषताएँ होती है जिनकी मध्य से हम, बिना विस्तार किए, उनका मुल्य निकाल सकते है ।
Determinants has same properties with the help of which, we can evaluate then without expansion

1. If rows are changed into columns and columns into rows, value of the determinant retains

Let $\Delta = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$

Expanding by first row

$\Delta = a_1 (b_2 c_2 - b_3 c_2) - b_1 (a_2 c_3 - a_3 c_2) + c_1 (a_2 b_3 - a_3 b_2)$ …(1)

Let Δ' be the determinant obtained by changing rows into columns and columns into rows of determinant.

So $\Delta' = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$

Expanding by first column, we get

$\Delta' = a_1 (b_2 c_3 - b_3 c_2) - b_1 (a_2 c_3 - a_3 c_2) + c_1 (a_2 b_3 - a_3 b_2)$ …(2)

From (1) and (2) $\Delta = \Delta'$

2. If any two rows or columns of a determinant are inter changed, the determinant retains its absolute value but changes in sign.

Let $\Delta = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$

Let Δ' be the determinant from Δ by interchanging the first and third row, then

$$\Delta' = \begin{vmatrix} a_3 & b_3 & c_3 \\ a_2 & b_2 & c_2 \\ a_1 & b_1 & c_1 \end{vmatrix}$$

Then we are to prove that $\Delta' = -\Delta$.

Expanding Δ with the first columns, we have

$\Delta = a_1 (b_2c_3 - b_3c_2) - a_2 (b_1c_3 - b_3c_1) + a_3 (b_1c_2 - b_2c_1)$ …(1)

Expanding Δ' with the first columns, we have

$\Delta' = -a_3 (b_1c_2 - b_2c_1) + a_2(b_1c_3 - b_3c_1) + a_1 (b_3c_2 - b_2c_3)$.

or $\Delta' = -a_3 (b_1c_2 - b_2c_1) + a_2(b_1c_3 - b_3c_1) - a_1 (b_2c_3 - b_3c_1)$.

$= -[a_1 (b_2c_3 - b_3c_2) a_2(b_1c_3 - b_3c_1) + a_3 (b_1c_2 - b_2c_1)]$.

$= -\Delta$.

3. If any two rows or columns of a determinant are identical, the value of the determinant is zero.

Let $\Delta = \begin{vmatrix} a_1 & a_1 & b_1 \\ a_2 & a_2 & b_2 \\ a_3 & a_3 & b_3 \end{vmatrix}$ …(1)

Expanding by first row

$\Delta = a_1 (a_2b_3 - b_2a_3) - a_1 (a_2 b_3 - b_2 a_3) + b_1 (a_2a_3 - a_2a_3)$

$= 0$

4. If all the elements of one row, or of one column, multiplied by the same quantity, the determinant is multiplied by that quantity.

Let $\Delta = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$

Let Δ' be the determinant from Δ by multiplying all the elements of the first column by k.

Then $\Delta' = \begin{vmatrix} ka_1 & b_1 & c_1 \\ ka_2 & b_2 & c_2 \\ ka_3 & b_3 & c_3 \end{vmatrix}$

Expand with first column

$\Delta' = ka_1A_1 - ka_2 A_2 + ka_2A_3$

$= k[a_1A_1 - a_2A_2 + a_3A_3]$

$= k\,\Delta$. Hence the result.

Corollary 1. $\begin{vmatrix} ka_1 & lb_1 & mc_1 \\ ka_2 & lb_2 & mc_2 \\ ka_3 & lb_3 & mc_3 \end{vmatrix} = klm \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$

$= \begin{vmatrix} ka_1 & kb_1 & kc_1 \\ la_2 & lb_2 & lc_2 \\ ma_3 & mb_3 & mc_3 \end{vmatrix}$

Corollary 2. If each element of one column or one row, is the same multiple of corresponding elements of another column or row, the determinant vanishes.

i.e., $\Delta = \begin{vmatrix} a_1 & ka_1 & c_1 \\ a_2 & ka_2 & c_2 \\ a_3 & ka_3 & c_3 \end{vmatrix} = k \begin{vmatrix} a_1 & a_1 & c_1 \\ a_2 & a_2 & c_2 \\ a_3 & a_3 & c_3 \end{vmatrix} = k \times 0 = 0 = \begin{vmatrix} a_1 & b_1 & c_1 \\ ka_1 & kb_1 & kc_1 \\ a_3 & b_3 & c_3 \end{vmatrix}$

5. If each element of one row or any column be the sum of two quantities, the determinant can be expressed as the sum of the two determinants of the same order.

i.e., $\Delta = \begin{vmatrix} a_1+\alpha_1 & b_1 & c_1 \\ a_2+\alpha_2 & b_2 & c_2 \\ a_3+\alpha_3 & b_3 & c_3 \end{vmatrix} = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} + \begin{vmatrix} \alpha_1 & b_1 & c_1 \\ \alpha_2 & b_2 & c_2 \\ \alpha_3 & b_3 & c_3 \end{vmatrix}$

$$\begin{vmatrix} a_1+\alpha_1 & b_1 & c_1 \\ a_2+\alpha_2 & b_2 & c_2 \\ a_3+\alpha_3 & b_3 & c_3 \end{vmatrix} = (a_1 + \alpha_1)(b_2c_3 - b_3c_2) - (a_2 + \alpha_2)(b_1c_3 - b_3c_1)$$

$$+ (a_2 + \alpha_3)(b_1c_2 - b_2c_1)$$

$$= [a_1(b_2c_3 - b_3c_2) - a_2(b_1c_3 - b_3c_1) + a_3(b_1c_2 - b_2c_1)]$$

$$+ \alpha_1(b_2c_3 - b_3c_2) - \alpha_2(b_1c_3 - b_3c_1) + \alpha_3(b_1c_2 - b_2c_1)]$$

$$= \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} + \begin{vmatrix} \alpha_1 & b_1 & c_1 \\ \alpha_2 & b_2 & c_2 \\ \alpha_3 & b_3 & c_3 \end{vmatrix}$$

Remember. If each element of a row or column consists of m terms, the determinant can be expressed as the sum of m determinants.

6. A determinant remains unaltered in value, by adding to all the elements of any column or of any row the same multiple of the corresponding elements of any number of other columns or of rows.

i.e., $\Delta = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = \begin{vmatrix} a_1+mb_1+nc_1 & b_1 & c_1 \\ a_2+mb_2+nc_2 & b_2 & c_2 \\ a_3+mb_3+nc_3 & b_3 & c_3 \end{vmatrix}$

R.H.S. $= \begin{vmatrix} a_1+mb_1+nc_1 & b_1 & c_1 \\ a_2+mb_2+nc_2 & b_2 & c_2 \\ a_3+mb_3+nc_3 & b_3 & c_3 \end{vmatrix}$

$= \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} + m \begin{vmatrix} b_1 & b_1 & c_1 \\ b_2 & b_2 & c_2 \\ b_3 & b_3 & c_3 \end{vmatrix} + n \begin{vmatrix} c_1 & b_1 & c_1 \\ c_2 & b_2 & c_2 \\ c_3 & b_3 & c_3 \end{vmatrix}$ (By property 5)

$= \Delta + m\times 0 + n\times 0$ (By property 4)

$= \Delta$

Example 10. Show that $\begin{vmatrix} 1 & 1 & 1 \\ a^2 & b^2 & c^2 \\ a^3 & b^3 & c^3 \end{vmatrix} = (a-b)(b-c)(c-a)(ab+bc+ca)$

$$\Delta = \begin{vmatrix} 1 & 1 & 1 \\ a^2 & b^2 & c^2 \\ a^3 & b^3 & c^3 \end{vmatrix}$$

Applying $C_2 \to C_2 - C_1$ and $C_3 \to C_3 - C_1$

$$\Delta = \begin{vmatrix} 1 & 0 & 0 \\ a^2 & b^2-a^2 & c^2-a^2 \\ a^3 & b^3-a^3 & c^3-a^3 \end{vmatrix} = \begin{vmatrix} (b+a)(b-a) & (c+a)(c-a) \\ (b-a)(b^2+a^2+ab) & (c-a)(c^2+a^2+ac) \end{vmatrix}$$

$$= (b-a)(c-a)\begin{vmatrix} b+a & c+a \\ a^2+b^2+ab & c^2+a^2+ac \end{vmatrix}$$

Applying $C_2 \to C_2 - C_1$

$$\Delta = (b-a)(c-a)\begin{vmatrix} a+b & c-b \\ a^2+b^2+ab & c^2+ac-b^2-ab \end{vmatrix}$$

$$= (b-a)(c-a)\begin{vmatrix} a+b & c-b \\ a^2+b^2+ab & (c-b)(c+b)+a(c-b) \end{vmatrix}$$

$$= (b-a)(c-a)(c-b)\begin{vmatrix} a+b & 1 \\ a^2+b^2+ab & a+b+c \end{vmatrix}$$

$$= (b-a)(c-a)(c-b)\,[\cancel{a^2}+ab+ac+\cancel{ab}+\cancel{b^2}+bc-\cancel{a^2}-\cancel{b^2}-\cancel{ab}]$$

$$= [-(a-b)]\,(c-a)(-(b-c))(ab+bc+ca) = (a-b)(b-c)(c-a)(ab+bc+ca)$$

Example 11. Evaluate $\begin{vmatrix} 1 & w & w^2 \\ w & w^2 & 1 \\ w^2 & 1 & w \end{vmatrix}$

where w is one of the imaginary cube roots of unity.

Solution. Applying $C_1 \to C_1+C_2+C_3$, we get

$$\Delta = \begin{vmatrix} 1+w+w^2 & w & w^2 \\ 1+w+w^2 & w^2 & 1 \\ 1+w+w^2 & 1 & w \end{vmatrix} = \begin{vmatrix} 0 & w & w^2 \\ 0 & w^2 & 1 \\ 0 & 1 & w \end{vmatrix} = 0 \qquad [\because\ 1+w+w^2 = 0]$$

Example 12. Without expanding, prove that

$$\begin{vmatrix} a & b & c \\ x & y & z \\ p & q & r \end{vmatrix} = \begin{vmatrix} y & b & q \\ x & a & p \\ z & c & r \end{vmatrix}$$

Solution. $\Delta = \begin{vmatrix} a & b & c \\ x & y & z \\ p & q & r \end{vmatrix}$

Changing rows into columns and columns into rows

$$\Delta = \begin{vmatrix} a & x & p \\ b & y & q \\ c & z & r \end{vmatrix}$$

Applying $C_1 \leftrightarrow C_2$

$$\Delta = (-1)\begin{vmatrix} x & a & p \\ y & b & q \\ z & c & r \end{vmatrix}$$

Applying $R_1 \to R_2$

$$= (-1)^2\begin{vmatrix} y & b & q \\ x & a & p \\ z & c & r \end{vmatrix} = \begin{vmatrix} y & b & q \\ x & a & p \\ z & c & r \end{vmatrix}.$$

Example 13. Show that $\begin{vmatrix} a & b & c \\ a^2 & b^2 & c^2 \\ a^3 & b^3 & c^3 \end{vmatrix}$

$= abc(a-b)(b-c)(c-a)$.

Solution. Taking a, b, c common from C_1, C_2 and C_3 respectively, the given determinant

$$\Delta = abc \begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^2 & b^2 & c^2 \end{vmatrix}$$

Applying $C_2 \to C_2 - C_1$ and $C_3 \to C_3 - C_1$, we get

$$= abc \begin{vmatrix} 1 & 0 & 0 \\ a & b-a & c-a \\ a^2 & b^2-a^2 & c^2-a^2 \end{vmatrix} = abc \begin{vmatrix} b-a & c-a \\ b^2-a^2 & c^2-a^2 \end{vmatrix}$$

(Expanding along R_1)

$$= abc(b-c)(c-a) \begin{vmatrix} 1 & 1 \\ b+a & c+a \end{vmatrix}$$

$$= abc(b-a)(c-a)(c+a-b-a) = abc(b-a)(c-a)(c-b)$$

$$= abc(a-b)(b-c)(c-a) .$$

Example 14. Sow that $\begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^3 & b^3 & c^3 \end{vmatrix}$

$$= (a-b)(b-c)(c-a)(a+b+c) .$$

Solution. Applying $C_2 \to C_2 - C_1$ and $C_3 \to C_3 - C_1$, we get

$$\Delta = \begin{vmatrix} 1 & 0 & 0 \\ a & b-a & c-a \\ a^3 & b^3-a^3 & c^3-a^3 \end{vmatrix} = \begin{vmatrix} b-a & c-a \\ b^3-a^3 & c^3-a^3 \end{vmatrix}$$

(Expanding along R_1)

$$= (b-a)(c-a) \begin{vmatrix} 1 & 1 \\ b^2+ab+a^2 & c^2+ca+a^2 \end{vmatrix}$$

$$= (b-a)(c-a)[(c^2+ca+a^2) - (b^2+ab+a^2)]$$

$$= (b-a)(c-a)[(c^2-b^2)+a(c-b)]$$

$$= (b-a)(c-a)(c-b)(c+b+a)$$

$$= (a-b)(b-c)(c-a)(a+b+c) .$$

Example 15. Prove that

$$\begin{vmatrix} b+c & c+a & a+b \\ q+r & r+p & p+q \\ y+z & z+x & x+y \end{vmatrix} = 2 \begin{vmatrix} a & b & c \\ p & q & r \\ x & y & z \end{vmatrix} .$$

Solution. $\Delta = \begin{vmatrix} b+c & c+a & a+b \\ q+r & r+p & p+q \\ y+z & z+x & x+y \end{vmatrix}$

$$= \begin{vmatrix} b & c+a & a+b \\ q & r+p & p+q \\ y & z+x & x+y \end{vmatrix} + \begin{vmatrix} c & c+a & a+b \\ r & r+p & p+q \\ z & z+x & x+y \end{vmatrix}$$ [Note]

$$= \begin{vmatrix} b & c+a & a \\ q & r+p & p \\ y & z+x & x \end{vmatrix} + \begin{vmatrix} b & c+a & b \\ q & r+p & q \\ y & z+x & y \end{vmatrix} + \begin{vmatrix} c & c & a+b \\ r & r & p+q \\ z & z & x+y \end{vmatrix} + \begin{vmatrix} c & a & a+b \\ r & p & p+q \\ z & x & x+y \end{vmatrix}$$

$$= \begin{vmatrix} b & c+a & a \\ q & r+p & p \\ y & z+x & x \end{vmatrix} + \begin{vmatrix} c & a & a+b \\ r & p & p+q \\ z & x & x+y \end{vmatrix}$$

[Second and third determinants vanish as two columns in each are identical]

$$= \begin{vmatrix} b & c & a \\ q & r & p \\ y & z & x \end{vmatrix} + \begin{vmatrix} b & a & a \\ q & p & p \\ y & x & x \end{vmatrix} + \begin{vmatrix} c & a & a \\ r & p & p \\ z & x & x \end{vmatrix} + \begin{vmatrix} c & a & b \\ r & p & q \\ z & x & y \end{vmatrix}$$

$$= \begin{vmatrix} b & c & a \\ q & r & p \\ y & z & x \end{vmatrix} + \begin{vmatrix} c & a & b \\ r & p & q \\ z & x & y \end{vmatrix}$$

[Second and third determinants vanish as two columns in each are identical]

$$= - \begin{vmatrix} b & a & c \\ q & p & r \\ y & x & z \end{vmatrix} - \begin{vmatrix} a & c & b \\ p & q & r \\ x & y & z \end{vmatrix}$$

[Interchanging C_2 and C_3 in first determinant and C_1 and C_2 in second]

$$= \begin{vmatrix} a & b & c \\ p & q & r \\ x & y & z \end{vmatrix} + \begin{vmatrix} a & b & c \\ p & q & r \\ x & y & z \end{vmatrix}$$

[Interchanging C_1 and C_2 in first and C_2 and C_3 in second determinant]

$$= 2\begin{vmatrix} a & b & c \\ p & q & r \\ x & y & z \end{vmatrix} .$$

Example 16. Prove that

$$\begin{vmatrix} a^2 & a^2-(b-c)^2 & bc \\ b^2 & b^2-(c-a)^2 & ca \\ c^2 & c^2-(a-b)^2 & ab \end{vmatrix} = (a-b)(b-c)(c-a)(a+b+c)(a^2+b^2+c^2)$$

Solution. Applying $C_2 \to C_2 - 2C_1 - 2C_3$, we get

$$\Delta = \begin{vmatrix} a^2 & -(a^2+b^2+c^2 & bc \\ b^2 & -(a^2+b^2+c^2 & ca \\ c^2 & -(a^2+b^2+c^2 & ab \end{vmatrix}$$

$$= -(a^2+b^2+c^2) \begin{vmatrix} a^2 & 1 & bc \\ b^2 & 1 & ca \\ c^2 & 1 & ab \end{vmatrix}$$

Interchanging C_1 and C_2

$$= (a^2+b^2+c^2) \begin{vmatrix} 1 & a^2 & bc \\ 1 & b^2 & ca \\ 1 & c^2 & ab \end{vmatrix}$$

Multiplying R_1, R_2 and R_3 by a, b and c respectively,

$$= (a^2+b^2+c^2) \times \frac{1}{abc}\begin{vmatrix} a & a^2 & abc \\ b & b^2 & abc \\ c & c^2 & abc \end{vmatrix}$$

$$= (a^2+b^2+c^2) \times \frac{abc}{abc}\begin{vmatrix} a & a^3 & 1 \\ b & b^3 & 1 \\ c & c^3 & 1 \end{vmatrix}$$

$$= (a^2+b^2+c^2) \times \begin{vmatrix} 1 & a & a^3 \\ 1 & b & b^3 \\ 1 & c & c^3 \end{vmatrix}$$

Now proceed as in Example 14

$$\Delta = (a^2+b^2+c^2)\times(a-b)(b-c)(c-a)(a+b+c)$$

$$\Delta = (a-b)(b-c)(c-a)(a+b+c)(a^2+b^2+c^2)\ .$$

Exercise 7.1

Q. 1. Evaluate the following determinants

(i) $\begin{vmatrix} 18 & 8 \\ 7 & 13 \end{vmatrix}$ (ii) $\begin{vmatrix} 15 & -12 \\ -9 & 10 \end{vmatrix}$

(iii) $\begin{vmatrix} 13 & 16 & 19 \\ 14 & 17 & 20 \\ 15 & 18 & 21 \end{vmatrix}$ (iv) $\begin{vmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{vmatrix}$

Q. 2. Write the minors and co-factors of each element of the following determinants

(i) $\begin{vmatrix} 1 & 2 & 5 \\ -4 & 3 & 4 \\ 2 & -10 & 9 \end{vmatrix}$ (ii) $\begin{vmatrix} 5 & 1 & -3 \\ 0 & 3 & 1 \\ -2 & -4 & 2 \end{vmatrix}$

Q. 3. Evaluate the following determinants by Sarraus method :

(i) $\begin{vmatrix} 2 & -3 & 4 \\ 5 & 1 & -6 \\ -7 & 8 & -9 \end{vmatrix}$ (ii) $\begin{vmatrix} 4 & 7 & 8 \\ -9 & 0 & 0 \\ 2 & 3 & 4 \end{vmatrix}$

Q. 4. Evaluate

(i) $\begin{vmatrix} x+\lambda & x & x \\ x & x+\lambda & x \\ x & x & x+\lambda \end{vmatrix}$ (ii) $\begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix}$

(iii) $\begin{vmatrix} \frac{a^2+b^2}{c} & c & c \\ a & \frac{b^2+c^2}{a} & a \\ b & b & \frac{c^2+a^2}{b} \end{vmatrix}$ (iv) $\begin{vmatrix} 19 & 17 & 45 \\ 7 & 9 & 5 \\ 9 & 3 & 4 \end{vmatrix}$

Without expanding, prove the following :

Q. 5. Prove that

$$\begin{vmatrix} x+a & b & c \\ a & x+b & c \\ b & b & x+c \end{vmatrix} = x^2(a+b+c)\ .$$

Q. 6. Prove that

$$\begin{vmatrix} 1+i & 1-i & i \\ 1-i & i & 1+i \\ i & 1+i & 1-i \end{vmatrix} = 4+7i \text{, where } i = \sqrt{-1}$$

Q. 7. Prove that $\begin{vmatrix} -2a & a+b & a+c \\ b+a & -2b & b+c \\ c+a & c+b & -2c \end{vmatrix} = 4(b+c)(c+a)(a+b)$

Q. 8. Prove that

$$\begin{vmatrix} b+c & a-b & a \\ c+a & b-c & b \\ a+b & c-a & c \end{vmatrix} = 3abc - a^3 - b^3 - c^3 .$$

Q. 9. Prove that $\begin{vmatrix} x & y & z \\ x^2 & y^2 & z^2 \\ x^3 & y^3 & z^3 \end{vmatrix} = xyz(x-y)(y-z)(z-x).$

Q. 10. Prove that $\begin{vmatrix} 1 & b+c & b^2+c^2 \\ 1 & c+a & c^2+a^2 \\ 1 & a+b & a^2+b^2 \end{vmatrix} = (a-b)(b-c)(c-a)$.

Q. 11. Prove that $\begin{vmatrix} (b+c)^2 & a^2 & bc \\ (c+a)^2 & b^2 & ca \\ (a+b)^2 & c^2 & ab \end{vmatrix} = (a^2+b^2+c^2)(a+b+c)(b-c)(c-a)(a+b)$.

Q. 12. Prove that $\begin{vmatrix} a^3 & 2ab & b^4 \\ b^2 & a^2 & 2ab \\ 2ab & b^3 & a^2 \end{vmatrix}$ is a perfect square.

Q. 13. Prove that $\begin{vmatrix} (a+b)^2 & ca & cb \\ ca & (b+c)^2 & ab \\ bc & ab & (c+a)^2 \end{vmatrix} = 2abc(a+b+c)^3$.

Q. 14. Prove that

$$\begin{vmatrix} a+b & b+c & c+a \\ b+c & c+a & a+b \\ c+a & a+b & b+c \end{vmatrix} = 2\begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix}$$

Q. 15. Prove that

$$\begin{vmatrix} 1+a & 1 & 1 \\ 1 & 1+b & 1 \\ 1 & 1 & 1+c \end{vmatrix} = abc\left(1+\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right) = abc+bc+ac+ab .$$

सारणिक के प्रयोग से रेखीय युगपत समीकरणों का हल
(Solution of linear equation using determinants)

दो या तीन युगपत समीकरणों के हल के लिए हम क्रैमर नियम ;ब्तंउउमत तनसमद्ध अपनाते हैं। यह नियम सारणिकों पर आधारित हैं।

1- दो अज्ञात मूल्यों में रेखीय समीकरणों का हल (Solution of linear equations in two unknowns) :

Let the equation be

$$a_1x + b_1y = c_1$$
$$a_2x + b_2y = c_2$$

Values of x and y are calculated by the following formula

$$x = \frac{D_1}{D} \text{ and } y = \frac{D_2}{D}$$

where

$$D = \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}, \quad D_1 = \begin{vmatrix} c_1 & b_1 \\ c_2 & b_2 \end{vmatrix}, \quad D_2 = \begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix}, D \neq 0$$

In this case

D is the determinant showing co-efficients of x and y .

D_1 is the determinant obtained by replacing elements of first column of D by constant values on the Right Hand Side of the equations.

D_2 is the determinant obtained by replacing elements of 2^{nd} column of D by constant values on the Right Hand Side of the equations.

Example 17. Solve the following set of linear equations using Crammer's rule :

$$4x-3y = 7$$
$$2x+5y = 23$$

Solution.

$$D = \begin{vmatrix} 4 & -3 \\ 2 & 5 \end{vmatrix} = 4\times5-2(-3) = 20+6 = 26$$

$$D_1 = \begin{vmatrix} 7 & -3 \\ 23 & 5 \end{vmatrix} = 7\times5-23(-3) = 35+69 = 104$$

$$D_2 = \begin{vmatrix} 4 & 7 \\ 2 & 23 \end{vmatrix} = 4\times23-2\times7 = 92-14 = 78$$

$$\therefore \quad x = \frac{D_1}{D} = \frac{104}{26} = 4$$

$$y = \frac{D_2}{D} = \frac{78}{26} = 3$$

2. In case of three equations for three unknowns, the mechanism is as given below :

$$a_1x+b_1y+c_1z = d_1$$
$$a_2x+b_2y+c_2z = d_2$$
$$a_3x+b_3y+c_3z = d_3$$

$$x = \frac{D_1}{D}, \quad y = \frac{D_2}{D}, \quad z = \frac{D_3}{D}$$

$$D = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}, \quad D_1 = \begin{vmatrix} d_1 & b_1 & c_1 \\ d_2 & b_2 & c_2 \\ d_3 & b_3 & c_3 \end{vmatrix}, \quad D_2 \begin{vmatrix} a_1 & d_1 & c_1 \\ a_2 & d_2 & c_2 \\ a_3 & d_3 & c_3 \end{vmatrix}$$

$$D_3 = \begin{vmatrix} a_1 & b_1 & d_1 \\ a_2 & b_2 & d_2 \\ a_3 & b_3 & d_3 \end{vmatrix}$$

Example 18. Solve the following systems of equations by means of determinants.

$$x+y+z-7 = 0$$

$$x+2y+3z-16 = 0$$
$$x+3y+4z-22 = 0 .$$

Solution. The given system of equation is

$$x+y+z = 7$$
$$x+2y+3z = 16$$
$$x+3y+4z = 22$$

$$D = \begin{vmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 3 & 4 \end{vmatrix} = 1(8-9)-1(4-3)+1(3-2) = -1-1+1 = -1$$

$$D_1 = \begin{vmatrix} 1 & 1 & 1 \\ 16 & 2 & 3 \\ 22 & 3 & 4 \end{vmatrix} = 7(8-9)-1(64-66)+1(48-44) = -7+2+4 = -1.$$

$$D_2 = \begin{vmatrix} 1 & 17 & 1 \\ 1 & 16 & 3 \\ 1 & 22 & 4 \end{vmatrix} = 1(64-65)-7(4-3)+(22-16) = -2-7+6 = -3$$

$$D_3 = \begin{vmatrix} 1 & 1 & 7 \\ 1 & 2 & 16 \\ 1 & 3 & 22 \end{vmatrix} = 1(44-48)-1(22-16)+7(3-2) = -4-6+7 = -3$$

Now $x = \frac{D_1}{D} = \frac{-1}{-1} = 1$

$$y = \frac{D_2}{D} = \frac{-3}{-1} = 3$$

$$z = \frac{D_3}{D} = \frac{-3}{-1} = 3$$

Hence $x = 1$, $y = 3$, $z = 3$.

Cramer's rule can be used in exactly the same way to solve the system of n equations in n unknowns. Below we state the theorem for the general case.

Theorem. Consider the systems of n linear equations in n unknowns given by

$$a_{11}x + a_{12}x_2 + \ldots + a_{1n}x_n = b_1$$
$$a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n = b_2$$
$$\vdots$$
$$a_{n1}x_1 + a_{n2}x_2 + \ldots + a_{nn}x_n = b_n$$

Let $D = \begin{vmatrix} a_{11} & a_{12} & \ldots & a_{1n} \\ a_{21} & a_{22} & \ldots & a_{2n} \\ \vdots & & & \\ a_{n1} & a_{n2} & \ldots & a_{nn} \end{vmatrix}$

Let D_j be the determinant obtained from D after replacing the j^{th} column by $\begin{vmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{vmatrix}$

and we can obtain the different values like

$$x_1 = \frac{D_1}{D}, x_2 = \frac{D_2}{D}, x_3 = \frac{D_3}{D}, \ldots x_n = \frac{D_n}{D}$$

संगत तथा असंगत समीकरण (Consistent and Inconsistent equations)

समीकरणों के निकाय पर विचार कीजिए, यदि यह निकाय हल रखता है ;अर्थात ग$_1$ए ग$_2$ए३एए गद्द के उन मानों का समुच्चय जो निकाय ;पद्ध में उ समीकरणों को संतुष्ट करता हैद्ध तो दी हुई समीकरणें संगत कहलाती हैं, अन्यथा समीकरणें असंगत कहलाती हैं।

नोट. संगत समीकरणों का निकाय या तो हल रखता है या अनन्त हल रखता है। **असंमघात समीकरणों के हल (Solution of non-homogeneous equations)-** समीकरण निकाय (i) का हल, जब $m = n$ तथा A व्युत्क्रमणीय है।

Consider the system of equations shown above. If this system has a solution, then given equations are called consistent, otherwise inconsistent.

Note : System of consistent equation has either a unique solution or infinite number of solutions.

Example 19. Solve the following equations by Cramer's rule

$$x+y+z = 1 \; ; \; x+2y+3z = 2; \; x+4y + 9z = 4.$$

Solution. Here

$$D = \begin{vmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 4 & 9 \end{vmatrix}$$

$$D_1 = \begin{vmatrix} 1 & 1 & 1 \\ 2 & 2 & 3 \\ 4 & 4 & 9 \end{vmatrix} = 0 \quad [\because \; C_1 \text{ and } C_2 \text{ are identical}]$$

$$D_2 = \begin{vmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 4 & 9 \end{vmatrix} = D,$$

$$D_3 = \begin{vmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 4 & 4 \end{vmatrix} = 0. \quad [\because \; C_2 \text{ and } C_3 \text{ are identical}]$$

$$x = \frac{D_1}{D} = \frac{0}{D} = 0$$

$$y = \frac{D_2}{D} = \frac{D}{D} = 1$$

$$z = \frac{D_3}{D} = \frac{0}{D} = 0$$

Hence $x = 0$, $y = 1$, $z = 0$.

Example 20. Using determinants, solve the following systems of equations

(a) $x-y = 1$
$x+z = -6$
$x+y-2z = 3$

(b) $2y-3z = 0$
$x+3y = -4$
$3x+4y = 3$

Solution. Rewriting the above equations

(a) $x-y+0z = 1$
$x+0y+z = -6$
$x+y-2z = 3$

(b) $0.x+2y-3z = 0$
$x+3y+0z = -4$
$3x+4y+0z = 3$.

Solution. (a) Here $D = \begin{vmatrix} 1 & -1 & 0 \\ 1 & 0 & 1 \\ 1 & 1 & -2 \end{vmatrix}$

Expanding along R_1, we have

$$D = 1(0-1)+1(-2-1) + 0(1-0)$$
$$= -1-3+0 = -4$$

$$D_1 = \begin{vmatrix} 1 & -1 & 0 \\ -6 & 0 & 1 \\ 3 & 1 & -2 \end{vmatrix} \begin{aligned} &= 1(0-1)+(12-3)+0 \\ &= -1+9 = 8 [\text{Expanding along } R_1] \end{aligned}$$

$$D_2 = \begin{vmatrix} 1 & 1 & 0 \\ 1 & -6 & 1 \\ 1 & 3 & -2 \end{vmatrix} \begin{aligned} &= 1(12-3)-1(-2-1)+0 \\ &= 9+3 = 12 [\text{Expanding along } R_1] \end{aligned}$$

$$D_3 = \begin{vmatrix} 1 & -1 & 1 \\ 1 & 0 & -6 \\ 1 & 1 & 3 \end{vmatrix} \begin{aligned} &= 1(0+6)+1(3+6)+1(1-0) \\ &= 6+9+1 = 16 [\text{Expanding along } R_1] \end{aligned}$$

$$x = \frac{D_1}{D} = \frac{8}{-4} = -2$$

$$y = \frac{D_2}{D} = \frac{12}{-4} = -3$$

$$z = \frac{D_3}{D} = \frac{16}{-4} = -4$$

(b) Here $D = \begin{vmatrix} 0 & 2 & -3 \\ 1 & 3 & 0 \\ 3 & 4 & 0 \end{vmatrix} \begin{aligned} &= 0-2(0-0)-3(4-9) \\ &= -3(-5) = 15 \end{aligned}$

$$D_1 = \begin{vmatrix} 0 & -1 & -3 \\ -4 & 3 & 0 \\ 3 & 4 & 0 \end{vmatrix} \begin{aligned} &= 0-2(0-0)-3(-16-9) \\ &= -3(-25) = 75 \end{aligned}$$

$$D_2 = \begin{vmatrix} 0 & 0 & -3 \\ 1 & -4 & 0 \\ 3 & 3 & 0 \end{vmatrix} = -3(3+12) = -45$$

$$D_3 = \begin{vmatrix} 0 & 2 & 0 \\ 1 & 3 & -4 \\ 3 & 4 & 3 \end{vmatrix} = -2(3+12) = -30$$

$\therefore$ $x = \frac{D_1}{D} = \frac{75}{15} = 5, \; y = \frac{D_2}{D} = \frac{-45}{15} = -3, \; z = \frac{D_3}{D} = \frac{-30}{15} = -2$

Hence x = 5, y = –3, z = –2

Exercise 7.2

1. Sum of three numbers is 10. If we multiply the first number by 3 and third number by 4 and subtract 5 times the second number from this sum we get 11. By adding 2 times the first number and 3 times the second number and subtract the third number from them we get 8. Find the numbers.

2. Using determinants solve the following set of linear equations.

$$x + 2y + 3z = 6$$
$$2x + 4y + z = 7$$
$$3x + 2y + z = 14$$

3. Using determinants solve the following system of equations :

(a) $2x - 4y = -3$, $4x + 2y = 9$ (b) $4x + 3y = 3$, $8x - 9y = 1$.

4. Solve the following system of equations using Cramer's rule :

(i) $x+2y = 1$, $3x + y = 4$ (CBSE, 1990) (ii) $9x + 5y = 10$, $3y - 2x = 8$. (CBSE, 1991 C)

5. Solve the following system of equations by using Cramer's rule :

(i) $x + y + z = 6$, $x-y+z = 2$, $2x+y -z =1$

(ii) $3x + y + z = 10$, $x + y-z = 0$, $5x-9y = 1$

(iii) $2x-y+3z = 9$, $x +y +z = 6$, $x-y +z =2$

(iv) $3x + y +2z =3$, $2x-3y -z = -3$, $x-2y +z =4$.

6. Solve the following system of equations by using Cramer's rule :

(i) $x-y + z -4 = 0$, $2x+y-3z = 0$, $x + y +z -2 = 0$ (CBSE, 1985)

(ii) $x+y +z =1$, $3x + 5y +6z = 4$, $9x +2y -36z = 17$. (A. I. CBSE, 1985)

Answers

Exercise 7.1

1. (i) 42 (ii) 42 (iii) 0 (iv) abc
2. (i) $M_{11} = 67$, $M_{12} = -44$, $M_{13} = 34$, $M_{21} = 68$, $M_{22} = -1$, $M_{23}, = -14$, $M_{31} = -7$, $M_{32} = 24$, $M_{33} = 11$

$C_{11} = 67$, $C_{12} = 44$, $C_{13} = 34$, $C_{21} = -68$, $C_{22} = -1$, $C_{23} = 14$, $C_{31} = -7$,$C_{32} = -24$, $C_{33} = 11$

(ii) $M_{11} = 10$, $M_{12} = 2$, $M_{13} = 6$, $M_{21} = -10$, $M_{22} = 4$, $M_{23} = -18$, $M_{31} = 10$, $M_{32} = 5$ $M_{33} = 15$

$C_{11} = 10$, $C_{12} = -2$, $C_{13} = 6$, $C_{21} = 10$, $C_{22} = 4$ $C_{23} = 18$, $C_{31} = 10$, $C_{32} = -5$, $C_{33} = 15$

3. (i) 5 (ii) 36
4. (i) $\lambda^2(3x+\lambda)$ (ii) $3\,abc - (a^3 + b^3 + c^3)$ (iii) 4 abc (iv) -2012

Exercise 7.2

1. $x = 2, y = 3, z = 5$
2. $x = 1, y = 1, z = 1$
3. (a) $x = \frac{3}{2}, y = \frac{3}{2}$ (b) $x = \frac{1}{2}, y = \frac{1}{3}$
4. (i) $x = \frac{7}{5}, y = -\frac{1}{5}$ (ii) $x = \frac{-10}{37}$, $y = \frac{92}{37}$
5. (i) $x = 1, y = 2, z = 3$ (ii) $x = 2, y = 1, z = 3$ (iii) $x = 1, y = 2, z = 3$
   (iv) $x = 1, y = 2, z = -1$
6. (i) $x = 2, y = -1, z = 1$ (ii) $x = \frac{1}{3}, y = \frac{1}{3}, z = -\frac{1}{3}$

Chapter-8

आव्यूह ;जारीद्ध
Matrices (Continued)

इस अध्याय में हम आव्यूह का परिवर्त (Transpose of a matrix)] सहखण्डज आव्यूह (Adjoint matrixद्ध, आव्यूह का व्युतक्रम (Inverse of a matrix) तथा रेखीय युगपत समीकरणों के हल में आव्यूह की उपयोगिता के बारे में अध्ययन करेंगे।

1- आव्यूह का परिवर्त (Transpose of a matrix)

परिभाषा (Definition). $m \times n$ क्रम की आव्यूह । का परिवर्त इसकी पंक्तियों को स्तम्भों में तथा स्तम्भों को पंक्तियों में बदलने पर अथवा स्तम्भों को पंक्तियों में व पंक्तियों को स्तम्भों में बदलने पर प्राप्त क्रम $n \times m$ की आव्यूह । होता है।

आव्यूह का परिवर्त A^T, A^t, A' ख्पढ़ा जाता है श। परिवर्त (A tranposeद्धश, से भी निरूपित किया जाता है।

Let $A = [a_{ij}]_{m\times n}$ Then the n×m matrix obtained from A by changing its rows into columns and its columns into rows is called the transpose of A and is denoted by the symbol A' or A^T.

Symbolically if

$$A = [a_{ij}]_{m\times n}$$

Then $A' = [b_{ij}]_{n\times m}$ where $b_{ij} = a_{ij}$

i.e., the (j, i)th element of A′ is the (i, j)th element of A.

For Example. If $A = \begin{bmatrix} 1 & 3 & 5 \\ 2 & 4 & 6 \end{bmatrix}$, then $A' = \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix}$

Note 1. The element a_{ij} in the ith row and jth column of A stands in jth row and ith column of A′.

2. The transpose of an m×n matrix is an n×m matrix.

आव्यूह के परिर्वत की क्येिषताएँ
(Properties of transpose of matrix)

1. एक आव्यूह के परिवर्त का परिर्वत स्वम् आव्यूह होता है, अर्थात $(A')' = A$ (The transpose of the transpose of a matrix is the matrix itself, i.e. $(A')' = A$).

2. माना ा कोई अदिय तथा । कोई आव्यूह है, तब

$$(kA)' = kA'$$

If A is any matrix and k is any scalar then

$$(kA)' = k. A'$$

3. A rFkk B vkO;wg ds fy,

$$(A+B)' = A' + B'$$

For two matrices A and B

$$(A+B)' = A' + B'$$

4. दो आव्यूहों A o B के गुणनफल AB का परिवर्त, व्युत्क्रम में परिवर्तों के गुणनफल के बराबर होता हैं, अर्थात

$$(AB)' = B'. A'$$

The transpose of the product of two matrices is the product in reverse order of their transpose i.e. $(AB)' = B' A'$.

समामित तथा विषम सममित आव्यूह (Symmetric and Skew Symmetric Matrices)

यदि किसी आव्यूह का परिर्वत उस आव्यूह के बराबर होता हैं तो उसे सममित आव्यूह कहते है ।

A matrix is said to be symmetric if its transpose is equal to the matrix itself i.e. $A' = A$

Let $A = [a_{ij}]$ be of order $m \times n$

Then $A' = [a_{ji}]$ is of order $n \times m$

Matrix A is symmetric if $A' = A$

This is possible only if $m = n$ (the matrix must be a square matrix) and $a_{ji} = a_{ij}$ for all i and j

For example

$$\begin{bmatrix} 1 & 3 \\ 3 & 4 \end{bmatrix}, \begin{bmatrix} a & b & c \\ b & c & a \\ c & a & b \end{bmatrix}, \begin{bmatrix} 3 & 2 & 5 \\ 2 & 4 & 0 \\ 5 & 0 & 7 \end{bmatrix}$$

are all symmetric matrices.

विषम सममित आव्यूह (Skew-Symmetric matrix)

यदि एक आव्यूह का परिवर्त उस के आव्यूह के णिात्मक के बराबर होता है तो उसे विषम सममित आव्यूह कहते हैं।

A matrix is said to be skew symmetric if its transpose is equal to its negative i.e.

$A' = -A$

The skew symmetric matrix is also a square matrix but $a_{ij} = -a_{ji}$ for all i and j

For example

$$\begin{bmatrix} 0 & 2 \\ -2 & 0 \end{bmatrix}, \begin{bmatrix} 0 & b & c \\ -b & 0 & b \\ -c & -b & 0 \end{bmatrix}, \begin{bmatrix} 0 & h & g \\ -n & 0 & f \\ -g & -f & 0 \end{bmatrix}$$ are all skew symmetric matrices.

Example 1. Find the transpose of the following matrices.

(i) $A = \begin{bmatrix} 5 & 2 & -1 \\ 1 & 0 & 3 \\ 2 & 4 & 1 \end{bmatrix}$ (ii) $A = \begin{bmatrix} 1 & 0 & -2 \\ 2 & 2 & 4 \\ 0 & 0 & 2 \end{bmatrix}$

Solution. (i) $A' = \begin{bmatrix} 5 & 1 & 2 \\ 2 & 0 & 4 \\ -1 & 3 & 1 \end{bmatrix}$

(ii) $A' = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 2 & 0 \\ -2 & 4 & 2 \end{bmatrix}$

Example 2. If $A = \begin{bmatrix} -1 & 7 & 1 \\ 2 & 3 & 4 \\ 5 & 0 & 5 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 3 & 4 \\ 3 & 2 & 2 \\ 2 & 4 & 3 \end{bmatrix}$

Verify that (i) $(A+B)' = A' + B'$ and (ii) $(AB)' = B'. A'$

Solution. $A = \begin{bmatrix} -1 & 7 & 1 \\ 2 & 3 & 4 \\ 5 & 0 & 5 \end{bmatrix}, B = \begin{bmatrix} 1 & 3 & 4 \\ 3 & 2 & 2 \\ 2 & 4 & 3 \end{bmatrix}$

$A' = \begin{bmatrix} -1 & 2 & 5 \\ 7 & 3 & 0 \\ 1 & 4 & 5 \end{bmatrix}, B' = \begin{bmatrix} 1 & 3 & 2 \\ 3 & 2 & 4 \\ 4 & 2 & 3 \end{bmatrix}$

$$\text{(i) } (A+B) = \begin{bmatrix} -1+1 & 7+3 & 1+4 \\ 2+3 & 3+2 & 4+2 \\ 5+2 & 0+4 & 5+3 \end{bmatrix} = \begin{bmatrix} 0 & 10 & 5 \\ 5 & 5 & 6 \\ 7 & 4 & 8 \end{bmatrix}$$

$$(A+B)' = \begin{bmatrix} 0 & 5 & 7 \\ 10 & 5 & 4 \\ 5 & 6 & 8 \end{bmatrix}$$

$$A' + B' = \begin{bmatrix} -1 & 2 & 5 \\ 7 & 3 & 0 \\ 1 & 4 & 5 \end{bmatrix} + \begin{bmatrix} 1 & 3 & 2 \\ 3 & 2 & 4 \\ 4 & 2 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} -1+1 & 2+3 & 5+2 \\ 7+3 & 3+2 & 0+4 \\ 1+4 & 4+2 & 5+3 \end{bmatrix} = \begin{bmatrix} 0 & 5 & 7 \\ 10 & 5 & 4 \\ 5 & 6 & 8 \end{bmatrix} = (A+B)'$$

Hence the result

$$\text{(ii) } (AB) = \begin{bmatrix} -1 & 7 & 1 \\ 2 & 3 & 4 \\ 5 & 0 & 5 \end{bmatrix} \begin{bmatrix} 1 & 3 & 4 \\ 3 & 2 & 2 \\ 2 & 4 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} -1\times1+7\times3+1\times2 & -1\times3+7\times2+1\times4 & -1\times4+7\times2+1\times3 \\ 2\times1+3\times3+4\times2 & 2\times3+3\times2+4\times4 & 2\times4+3\times2+4\times3 \\ 5\times1+0\times3+5\times2 & 5\times3+0\times2+5\times4 & 5\times4+0\times2+5\times3 \end{bmatrix}$$

$$= \begin{bmatrix} 22 & 15 & 13 \\ 19 & 28 & 26 \\ 15 & 35 & 35 \end{bmatrix}$$

$$\therefore \quad (AB)' = \begin{bmatrix} 22 & 19 & 15 \\ 15 & 28 & 35 \\ 13 & 26 & 35 \end{bmatrix}$$

$$\text{Now} \quad B'.A' = \begin{bmatrix} 1 & 3 & 2 \\ 3 & 2 & 4 \\ 4 & 2 & 3 \end{bmatrix} \begin{bmatrix} -1 & 2 & 5 \\ 7 & 3 & 0 \\ 1 & 4 & 5 \end{bmatrix}$$

$$= \begin{bmatrix} 1\times-1+3\times7+2\times1 & 1\times2+3\times3+2\times4 & 1\times5+3\times0+2\times5 \\ 3\times-1+2\times7+4\times1 & 3\times2+2\times3+4\times4 & 3\times5+2\times0+4\times5 \\ 4\times-1+2\times7+3\times1 & 4\times2+2\times3+3\times4 & 4\times5+2\times0+3\times5 \end{bmatrix}$$

$$= \begin{bmatrix} 22 & 19 & 15 \\ 15 & 28 & 35 \\ 13 & 26 & 35 \end{bmatrix} = (AB)'$$

Hence the result

Example 3. If A and B are symmetric matrices prove that AB −BA a skew-symmetric matrix.

Solution. A and B are symmetric matrices,

$\Rightarrow \quad A' = A$ and $B' = B$ …(1)

Now $(AB - BA)' = (AB)' - (BA)'$ $\quad [\because (A-B)' = A' - B']$

$= B'A' - A'B'$ $\quad [(AB)' = B'A']$

$= BA - AB$ $\quad$ [using (1)]

$= -(AB - BA)$

$\therefore$ (AB − BA) is a skew-symmetric matrix.

Example 4. Express the following matrix as the sum of a symmetric matrix and skew symmetric matrix

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$$

Solution. $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}, \; A' = \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix}$

$$A + A' = \begin{bmatrix} 1+1 & 2+3 \\ 3+2 & 4+4 \end{bmatrix} = \begin{bmatrix} 2 & 5 \\ 5 & 8 \end{bmatrix} \text{ is symmetric}$$

$$A - A' = \begin{bmatrix} 1-1 & 2-3 \\ 3-2 & 4-4 \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \text{ is skew symmetric}$$

Now
$$\frac{A+A'}{2} + \frac{A-A'}{2} = \frac{1}{2}\begin{bmatrix} 2 & 5 \\ 5 & 8 \end{bmatrix} + \frac{1}{2}\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & \frac{5}{2} \\ \frac{5}{2} & 4 \end{bmatrix} + \begin{bmatrix} 0 & -\frac{1}{2} \\ \frac{1}{2} & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1+0 & \frac{5}{2}-\frac{1}{2} \\ \frac{5}{2}+\frac{1}{2} & 4+0 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = A$$

Hence the result

Example 5. Show that A + A′ is symmetric where $A = \begin{bmatrix} 6 & 5 \\ 2 & 4 \end{bmatrix}$

$$A = \begin{bmatrix} 6 & 5 \\ 2 & 4 \end{bmatrix}, \; A' = \begin{bmatrix} 6 & 2 \\ 5 & 4 \end{bmatrix}$$

$$A + A' = \begin{bmatrix} 6+6 & 5+2 \\ 2+5 & 4+4 \end{bmatrix} = \begin{bmatrix} 12 & 7 \\ 7 & 8 \end{bmatrix} \text{ which is symmetric}$$

lg[k.Mt vkO;wg (Adjoint matrix)

माना $A = [a_{ij}]$,d $n \times n$ क्रम का वर्ग आव्यूह है तथा माना A_{ij} ;k C_{ij} सारणिक (determinant) |A| में a_{ij} का सह [k.M (cofactor) है। अतः A का सहखण्डज जो ।करण। द्वारा प्रदर्शित किया जाता है, आव्यूह $[A_{ij}]$ का परिवर्त है।

अथवा

यदि $A = [a_{ij}]$ एक $n \times n$ क्रम का वर्ग आव्यूह है तथा । का सह गुणनखण्ड (cofactor) आव्यूह $[A_{ij}]$;k $[C_{ij}]$ gS tgkWa A_{ij}, A में पर का सहगुणन है, तो सहगुणनखण्ड आव्यूह के परिवर्त आव्यूह को A का सहगुणनखण्ड आव्यूह कहते है तथा ।करण। से प्रदर्शित करते हैं। अतएव

$$;\text{fn } A = \begin{bmatrix} a_{11} & a_{12} & \ldots a_{1n} \\ a_{21} & a_{22} & a_{2n} \\ \ldots\ldots & \ldots\ldots & \ldots\ldots \\ a_{n1} & a_{n2} & a_{nn} \end{bmatrix} \text{ rks] adj } A = \begin{bmatrix} A_{11} & A_{21} & \ldots A_{n1} \\ A_{12} & A_{22} & A_{n2} \\ \ldots\ldots & \ldots\ldots & \ldots\ldots \\ A_{1n} & A_{2n} & A_{nn} \end{bmatrix}$$

The adjoint of a square matrix is the transpose of the matrix obtained by replacing each element of A by its co-factor in A .

Let A = $[a_{ij}]_{n\times n}$ be any n×n matrix. This transpose B′ of the matrix

$$B = [A_{ij}]_{n\times n},$$

where A_{ij} denotes the cofactor of the element a_{ij} in the determinant |A| is called the adjoint of the matrix A and is denoted by the symbol Adj. A.

If A be a square matrix of size n × n then A.(adj.A) = (adj.A).A = |A|. I_n where I_n is the identity matrix of the n^{th} order.

$$A.(Adj.A) = \begin{bmatrix} a_{11} & a_{12} & ...a_{1n} \\ a_{21} & a_{22} & a_{2n} \\ ...... & .... & .... \\ a_{n1} & a_{n2} & a_{nn} \end{bmatrix} \begin{bmatrix} A_{11} & A_{21} & ...A_{n1} \\ A_{12} & A_{22} & A_{n2} \\ ...... & .... & .... \\ A_{1n} & A_{2n} & A_{nn} \end{bmatrix}$$

$$= \begin{vmatrix} |A| & 0 & 0...0 \\ 0 & |A| & 0...0 \\ 0 & 0 & |A|...0 \\ \vdots & & \\ 0 & 0 & 0....|A| \end{vmatrix} = |A| \begin{vmatrix} 1 & 0 & 0...0 \\ 0 & 1 & 0...0 \\ 0 & 0 & 1...0 \\ & & \\ 0 & 0 & 0....1 \end{vmatrix} = |A|\, I_n$$

If would be noted here that

$a_{11}\, a_{11} + c_{12}\, A_{12} + \ldots + a_{1n}\, A_{1n} = |A|$

and $a_{11}.\, A_{21} + a_{12}\, A_{22} + \ldots + a_{1n}.\, A_{2n} = 0$

So for all cases where i = j, value of product = |A| and for all cases where i ≠ j value of product = 0

Similarly it can be shown that

Adj A. A = |A|. I_n

Example 6. Find adjoint of the matrix $\begin{bmatrix} 0 & 1 & 2 \\ 1 & 2 & 3 \\ 3 & 1 & 1 \end{bmatrix}$

Solution. $A = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 2 & 3 \\ 3 & 1 & 1 \end{bmatrix}$ so $|A| = \begin{vmatrix} 0 & 1 & 2 \\ 1 & 2 & 3 \\ 3 & 1 & 1 \end{vmatrix}$

$A_{11} = +\begin{vmatrix} 2 & 3 \\ 1 & 1 \end{vmatrix} = 2 - 3 = -1$ $\quad A_{22} = +\begin{vmatrix} 0 & 2 \\ 3 & 1 \end{vmatrix} = 0 - 6 = -6$

$A_{12} = -\begin{vmatrix} 1 & 3 \\ 3 & 1 \end{vmatrix} = -(1-9) = 8$ $\quad A_{23} = -\begin{vmatrix} 0 & 1 \\ 3 & 1 \end{vmatrix} = -(0-3) = 3$

$A_{13} = +\begin{vmatrix} 1 & 2 \\ 3 & 1 \end{vmatrix} = 1 - 6 = -5$ $\quad A_{31} = +\begin{vmatrix} 1 & 2 \\ 2 & 3 \end{vmatrix} = 3 - 4 = -1$

$A_{21} = -\begin{vmatrix} 1 & 2 \\ 1 & 1 \end{vmatrix} = -(1-2) = 1$ $\quad A_{32} = -\begin{vmatrix} 0 & 2 \\ 1 & 3 \end{vmatrix} = -(0-2) = 2$

$A_{33} = +\begin{vmatrix} 0 & 1 \\ 1 & 2 \end{vmatrix} = 0 - 1 = -1$

So matrix obtained by cofactors $\quad C = \begin{bmatrix} -1 & 8 & -5 \\ 1 & -6 & 3 \\ -1 & 2 & -1 \end{bmatrix}$

$$\therefore \quad \text{Adj. (A)} = C' = \begin{bmatrix} -1 & 1 & -1 \\ 8 & -6 & 2 \\ -5 & 3 & -1 \end{bmatrix}$$

Example 7. For the following matrix A, prove that

A (Adj. A) = O

$$A = \begin{bmatrix} 1 & -1 & 1 \\ 2 & 3 & 0 \\ 18 & 2 & 10 \end{bmatrix}.$$

Solution. $A = \begin{bmatrix} 1 & -1 & 1 \\ 2 & 3 & 0 \\ 18 & 2 & 10 \end{bmatrix}, \quad \therefore |A| = \begin{vmatrix} 1 & -1 & 1 \\ 2 & 3 & 0 \\ 18 & 2 & 10 \end{vmatrix}$

$$A_{11} = \begin{vmatrix} 3 & 0 \\ 2 & 10 \end{vmatrix} = 30 ; \qquad A_{12} = -\begin{vmatrix} 2 & 0 \\ 18 & 10 \end{vmatrix} = -20 :$$

$$A_{13} = \begin{vmatrix} 2 & 3 \\ 18 & 2 \end{vmatrix} = -50$$

$$A_{21} = -\begin{vmatrix} -1 & 1 \\ 2 & 10 \end{vmatrix} = 12 ; \qquad A_{22} = \begin{vmatrix} 1 & 1 \\ 18 & 10 \end{vmatrix} = -8 ;$$

$$A_{23} = -\begin{vmatrix} 1 & -1 \\ 18 & 2 \end{vmatrix} = -20$$

$$A_{31} = \begin{vmatrix} -1 & 1 \\ 3 & 0 \end{vmatrix} = -3 ; \qquad A_{32} = -\begin{vmatrix} 1 & 1 \\ 2 & 0 \end{vmatrix} = 2 ;$$

$$A_{33} = \begin{vmatrix} 1 & -1 \\ 2 & 3 \end{vmatrix} = 5$$

$$\text{Adj. } A = \begin{bmatrix} A_{11} & A_{21} & A_{31} \\ A_{12} & A_{22} & A_{32} \\ A_{13} & A_{23} & A_{33} \end{bmatrix} = \begin{bmatrix} 30 & 12 & -3 \\ -20 & -8 & 2 \\ -15 & -20 & 5 \end{bmatrix}$$

$$A\ (\text{Adj. } A) = \begin{bmatrix} 1 & -1 & 1 \\ 2 & 3 & 0 \\ 18 & 2 & 10 \end{bmatrix} \begin{bmatrix} 30 & 12 & -3 \\ -20 & -8 & 2 \\ -50 & -20 & 5 \end{bmatrix}$$

$$= \begin{bmatrix} 30+20-50 & 12+8-20 & -3-2+5 \\ 60-60+0 & 24-24+0 & -6+6+0 \\ 540-40-500 & 216-16-200 & -54+4+50 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = O.$$

Exercise 8.1

Find the transpose of the following matrices:

1. (i) $\begin{bmatrix} 1 & 0 & 2 \\ 2 & 1 & 0 \\ 3 & 2 & 1 \end{bmatrix}$ (ii) $\begin{bmatrix} 1 & -2 & 4 \\ 2 & -4 & 5 \\ 4 & 5 & -6 \end{bmatrix}$

2. If $A = \begin{bmatrix} 2 & -3 \\ 4 & 5 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix}$

verify that $(A+B)' = A' + B'$ and $(AB)' = B'.A'$

3. Express the following matrix as the sum of a symmetric and skew symmetric matrices

$$\begin{bmatrix} -1 & 7 & 1 \\ 2 & 3 & 4 \\ 5 & 0 & 5 \end{bmatrix}$$

4. Find the adjoint of matrices :

(i) $\begin{bmatrix} -2 & 5 \\ 4 & -1 \end{bmatrix}$ (ii) $\begin{bmatrix} 2 & 3 & -1 \\ 0 & 2 & 3 \\ 4 & 1 & 4 \end{bmatrix}$

5. If $A = \begin{bmatrix} 5 & -2 \\ 3 & -2 \end{bmatrix}$ verify that A. (Adj. A) = (Adj. A). A = A. I_2

6. If $A = \begin{bmatrix} 1 & -1 & 1 \\ 2 & 3 & 0 \\ 18 & 2 & 10 \end{bmatrix}$ prove that A (Adj. A) = O

vkO;wg dk O;qRØe (Inverse of a matrix)

माना $A = [a_{ij}]$ एक द कोटि की वर्ग आव्यूह है । यदि $|A| = 0$ तो वंग आव्यूह A, अव्युत्क्रमणीय आव्यूह (Singular matrix) कहलाती है ।

यदि $|A| \neq 0$, तब वर्ग आव्यूह A, व्युत्क्रमणीय आव्यूह (Non-singular matrix) कहलाती है A

Oयुत्क्रम आव्यूह (Invertible matrix). माना A ,d n क्रम का वर्ग आव्यूह है । यदि एक ऐसे वर्ग आव्यूह B का अस्तित्व हो कि BA = AB = I, जहॉ I, क्रम द का तत्समकारी आव्यूह (unit matrix) है, तो B को A का व्युत्क्रम आव्यूह कहते हैं तथा इसे लिखते है :

$$B = A^{-1}$$

Let $A = [a_{ij}]$ be a square matrix of nth order. If $|A| = 0$ then matrix A is called a singular matrix.

If $|A| \neq 0$ then matrix A is called non-singular matrix.

Definition. Let A be an n-rowed square matrix. If there exists an n-rowed square matrix B such that

$$AB = BA = I,$$

where I is the identity matrix of order n, then the matrix A is said to be invertible and B is called the inverse (or reciprocal) of A.

Note 1. Only square matrices can have inverse.

Note 2. From the definition, it is clear that if B is the inverse of A, then A is the inverse of B.

Note 3. Inverse of A is denoted by A^{-1}, thus $B = A^{-1}$ and

$$AA^{-1} = A^{-1} A = I.$$

The inverse of a square matrix if it exists, is unique

Let A be an invertible square matrix. If possible, let B and C be two inverse of A.

Then AB = BA = I

AC = CA = I (By def. of inverse)

Now B = BI = B (AC)

= (BA) C [$\therefore$ Matrix multiplication is associative]

= IC = C.

Hence the inverse of A is unique.

Theorem. The necessary and sufficient condition for a square matrix A to possess inverse is that $|A| \neq 0$ i.e., A is non-singular.

Proof. (a) The condition is necessary :

i.e., Given that A has inverse, to show that $|A| \neq 0$.

Let B be the inverse of A, then

$AB = BA = I$

$\Rightarrow \quad |AB| = |BA| = |I|$

$\Rightarrow \quad |A||B| = |B||A| = 1$

$\Rightarrow \quad |A| \neq 0.$

(b) The condition is sufficient :

i.e., Given that $|A| \neq 0$, to show that A has inverse.

$\therefore \quad |A| \neq 0.$

$\therefore \quad$ Consider $B = \dfrac{\text{Adj. } A}{|A|}$

Hence A is invertible

or A is non singular

Theorem If A and B are two non-singular square matrices of same order n then

$(AB)^{-1} = B^{-1}.A^{-1}$

Proof. $|A| \neq 0$ and $|B| \neq 0$ (Given)

$\therefore |AB| = |A|.|B| \neq 0$

or $(AB)(B^{-1}.A^{-1}) = A(BB^{-1})A^{-1}$

$= A.I_n.A^{-1} = A.A^{-1}$

$= I_n$

Again $(B^{-1}.A^{-1})(AB) = B^{-1}(A^{-1}A)B$

$= B^{-1}.I_n B = B^{-1}.B$

$= I_n$

Thus

$(AB)(B^{-1}A^{-1}) = (B^{-1}A^{-1})(AB) = I_n$

$\therefore \quad (AB)^{-1} = B^{-1}.A^{-1}$

Example 8. Find the inverse of the matrix $\begin{vmatrix} 1 & 2 & 3 \\ 2 & 4 & 5 \\ 3 & 5 & 6 \end{vmatrix}$

$A_{11} = \begin{vmatrix} 4 & 5 \\ 5 & 6 \end{vmatrix} = -1$; $A_{12} = -\begin{vmatrix} 2 & 5 \\ 3 & 6 \end{vmatrix} = 3$; $A_{13} = \begin{vmatrix} 2 & 4 \\ 3 & 5 \end{vmatrix} = -2$;

$A_{21} = -\begin{vmatrix} 2 & 3 \\ 5 & 6 \end{vmatrix} = 3$; $A_{22} = \begin{vmatrix} 1 & 3 \\ 3 & 6 \end{vmatrix} = -3$; $A_{23} = -\begin{vmatrix} 1 & 2 \\ 3 & 5 \end{vmatrix} = 1$;

$A_{31} = \begin{vmatrix} 2 & 3 \\ 4 & 5 \end{vmatrix} = -2$; $A_{32} = -\begin{vmatrix} 1 & 3 \\ 2 & 5 \end{vmatrix} = 1$; $A_{33} = \begin{vmatrix} 1 & 2 \\ 2 & 4 \end{vmatrix} = 0$

$\therefore \quad C = \begin{bmatrix} -1 & 3 & -2 \\ 3 & -3 & 1 \\ -2 & 1 & 1 \end{bmatrix}$

$\therefore \quad \text{Adj } A = C' = \begin{bmatrix} -1 & 3 & -2 \\ 3 & -3 & 1 \\ -2 & 1 & 1 \end{bmatrix}$

LkkFk gh $|A| = \begin{vmatrix} 1 & 2 & 3 \\ 2 & 4 & 5 \\ 3 & 5 & 6 \end{vmatrix} = \begin{vmatrix} 1 & 0 & 0 \\ 2 & 0 & -1 \\ 3 & -1 & -3 \end{vmatrix}$, $C_2 \to .C_2 - 2C_1$, $C_3 \to C_3 - 3C_1$ rFkk

$= \begin{vmatrix} 0 & -1 \\ -1 & -3 \end{vmatrix} = -1$

$\therefore\ A^{-1} = \dfrac{\text{Adj.A}}{|A|} = -\begin{bmatrix} -1 & 3 & -2 \\ 3 & -3 & 1 \\ -2 & 1 & 0 \end{bmatrix}$

$= \begin{bmatrix} 1 & -3 & 2 \\ -3 & 3 & -1 \\ 2 & -1 & 0 \end{bmatrix}$

Example 9. Compute the inverse of the matrix

$A = \begin{bmatrix} 3 & -2 & 3 \\ 2 & 1 & -1 \\ 4 & -3 & 2 \end{bmatrix}$

Solution. $|A| = \begin{bmatrix} 3 & -2 & 3 \\ 2 & 1 & -1 \\ 4 & -3 & 2 \end{bmatrix}$

$= 3(2-3)+2\,(4+4)+3\,(-6-4)$

$= -3+16-30 = 16-33 = -17 \neq 0$

$A_{11} = \begin{vmatrix} 1 & -1 \\ -3 & 2 \end{vmatrix} = 2-3 = -1$; $A_{12} = -\begin{vmatrix} 2 & -1 \\ 4 & 2 \end{vmatrix} = -(4+4) = -8$

$A_{13} = \begin{vmatrix} 2 & 1 \\ 4 & -3 \end{vmatrix} = -6-4 = -10$

$A_{21} = -\begin{vmatrix} -2 & 3 \\ -3 & 2 \end{vmatrix} = -(-4+9) = -5$; $A_{22} = \begin{vmatrix} 3 & 3 \\ 4 & 2 \end{vmatrix} = 6-12 = -6$

$A_{23} = -\begin{vmatrix} 3 & -2 \\ 4 & -3 \end{vmatrix} = -(-9+8) = 1$

$A_{31} = \begin{vmatrix} -2 & 3 \\ 1 & -1 \end{vmatrix} = (2-3) = 1$;

$A_{32} = -\begin{vmatrix} 3 & 3 \\ 2 & -1 \end{vmatrix} = -(-3-6) = 9$

$A_{33} = \begin{vmatrix} 3 & -2 \\ 2 & 1 \end{vmatrix} = 3+4 = 7$

$\text{Adj. A} = \begin{bmatrix} A_{11} & A_{21} & A_{31} \\ A_{12} & A_{22} & A_{32} \\ A_{13} & A_{23} & A_{33} \end{bmatrix} = \begin{bmatrix} -1 & -5 & -1 \\ -8 & -6 & 9 \\ -10 & 1 & 7 \end{bmatrix}$

$A^{-1} = \dfrac{\text{Adj.A}}{|A|} = \dfrac{1}{-17}\begin{bmatrix} -1 & -5 & -1 \\ -8 & -6 & 9 \\ -10 & 1 & 7 \end{bmatrix}$

$$= \begin{bmatrix} 1/17 & 5/17 & 1/17 \\ 8/17 & 6/17 & -9/17 \\ 10/17 & -1/17 & -7/17 \end{bmatrix}.$$

Example 10. If $A = \begin{bmatrix} 3 & -1 \\ -4 & 0 \\ 2 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & 2 \\ -1 & -2 \\ 1 & 1 \end{bmatrix}$. Find $(A'B)^{-1}$.

Solution. Here $A' = \begin{bmatrix} 3 & -4 & 2 \\ -1 & 0 & 1 \end{bmatrix}$

Now $A'B = \begin{bmatrix} 3 & -4 & 2 \\ -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ -1 & -2 \\ 2 & 1 \end{bmatrix}$

$$= \begin{bmatrix} 6+4+2 & 3+8+2 \\ -2-0+1 & -1-0+1 \end{bmatrix} = \begin{bmatrix} 12 & 13 \\ -1 & 0 \end{bmatrix}$$

Let $C = A'B$

$\therefore$ $C = \begin{bmatrix} 12 & 13 \\ -1 & 0 \end{bmatrix}$

Now $|C| = \begin{vmatrix} 12 & 13 \\ -1 & 0 \end{vmatrix} = 0+13 = 13 \neq 0$.

$\therefore$ C is a non-singular matrix and hence C^{-1} exists. $C_{11} = 0$, $C_{12} = 1$, $C_{21} = -13$, $C_{22} = 12$.

$\therefore$ $Adj.C = \begin{bmatrix} C_{11} & C_{21} \\ C_{12} & C_{22} \end{bmatrix} = \begin{bmatrix} 0 & -13 \\ 1 & 12 \end{bmatrix}$

$\therefore$ $C^{-1} = \dfrac{Adj.C}{|C|} = \dfrac{1}{13}\begin{bmatrix} 0 & -13 \\ 1 & 12 \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1/13 & 12/13 \end{bmatrix}$

Hence, $(A'B)^{-1} = C^{-1} = \begin{bmatrix} 0 & -1 \\ 1/13 & 12/13 \end{bmatrix}$.

Example 11. Let $A = \begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix}$.

Prove that $A^2-4A-5I = O$. Hence obtain A^{-1}. (CBSE, 1985C)

Solution. $A^2 = AA = \begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix}\begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix}$

$$= \begin{bmatrix} 1+4+4 & 2+2+4 & 2+4+2 \\ 2+2+4 & 4+1+4 & 4+2+2 \\ 2+4+2 & 4+2+2 & 4+4+1 \end{bmatrix} = \begin{bmatrix} 9 & 8 & 8 \\ 8 & 9 & 8 \\ 8 & 8 & 9 \end{bmatrix}$$

$$A^2-4A-5I = \begin{bmatrix} 9 & 8 & 8 \\ 8 & 9 & 8 \\ 8 & 8 & 9 \end{bmatrix} - \begin{bmatrix} 4 & 8 & 8 \\ 8 & 4 & 8 \\ 8 & 8 & 4 \end{bmatrix} - \begin{bmatrix} 5 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 5 \end{bmatrix}$$

$$= \begin{bmatrix} 9-4-5 & 8-8-0 & 8-8-0 \\ 8-8-0 & 9-4-5 & 8-8-0 \\ 8-8-0 & 8-8-0 & 9-4-5 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = O$$

$\therefore$ $A^2-4A-5I = O$.

Multiplying both sides by A^{-1}

$$A^{-1}(A^2-4A-5I) = A^{-1}(O)$$

or $$A^{-1}A^2-4A^{-1}A-5A^{-1}I = O$$

or $$(A^{-1}A)A-4(A^{-1}A)-5A^{-1} = O$$

or $$IA-4I-5A^{-1} = O \text{ or } A-4I-5A^{-1} = O$$

or $$5A^{-1} = A-4I$$

$$\therefore \quad 5A^{-1} = \begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix} - 4\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} -3 & 2 & 2 \\ 2 & -3 & 2 \\ 2 & 2 & -3 \end{bmatrix}$$

$$\therefore \quad A^{-1} = 1/5\begin{bmatrix} -3 & 2 & 2 \\ 2 & -3 & 2 \\ 2 & 2 & -3 \end{bmatrix} = \begin{bmatrix} -3/5 & 2/5 & 2/5 \\ 2/5 & -3/5 & 2/5 \\ 2/5 & 2/5 & -3/5 \end{bmatrix}.$$

Example 12. Find the inverse of the matrix $\begin{bmatrix} 1 & 0 & -2 \\ -2 & -1 & 2 \\ 3 & 4 & 1 \end{bmatrix}$ and verify that $B.B^{-1} = I$.

Solution. Let $|B| = \begin{vmatrix} 1 & 0 & -2 \\ -2 & -1 & 2 \\ 3 & 4 & 1 \end{vmatrix} = \begin{vmatrix} 1 & 0 & 0 \\ -2 & -1 & -2 \\ 3 & 4 & 7 \end{vmatrix}$, $C_3 \to C_3 + 2C_1$

Expanding by first row

$$= \begin{vmatrix} -1 & -2 \\ 4 & 7 \end{vmatrix}$$

$$= (-1)(7)-(-2)(4) = -7+8 = 1 \neq 0$$

lkFk esa]

$$C_{11} = \begin{vmatrix} -1 & 2 \\ 4 & 1 \end{vmatrix} = -9, \quad C_{12} = -\begin{vmatrix} -2 & 2 \\ 3 & 1 \end{vmatrix} = 8,$$

$$C_{13} = \begin{vmatrix} -2 & -1 \\ 3 & 4 \end{vmatrix} = -5, \quad C_{21} = -\begin{vmatrix} 0 & -2 \\ 4 & 1 \end{vmatrix} = -8,$$

$$C_{22} = \begin{vmatrix} 1 & -2 \\ 3 & 1 \end{vmatrix} = 7, \quad C_{23} = -\begin{vmatrix} 1 & 0 \\ 3 & 4 \end{vmatrix} = -4,$$

$$C_{31} = \begin{vmatrix} 0 & -2 \\ -1 & 2 \end{vmatrix} = -2, \quad C_{32} = -\begin{vmatrix} 1 & -2 \\ -2 & 2 \end{vmatrix} = 2,$$

$$C_{33} = \begin{vmatrix} 1 & 0 \\ -2 & -1 \end{vmatrix} = -1.$$

$$\therefore \quad C = \begin{bmatrix} -9 & 8 & -5 \\ -8 & 7 & -4 \\ -2 & 2 & -1 \end{bmatrix}$$

$$\therefore \quad Adj.B = C' = \begin{bmatrix} -9 & -8 & -2 \\ 8 & 7 & 2 \\ -5 & -4 & -1 \end{bmatrix}$$

$$B^{-1} = \frac{Adj.B}{|B|} = \begin{bmatrix} -9 & -8 & -2 \\ 8 & 7 & 2 \\ -5 & -4 & -1 \end{bmatrix} \qquad [\because |B| = 1]$$

or $$B^{-1} = \begin{bmatrix} -9 & -8 & -2 \\ 8 & 7 & 2 \\ -5 & -4 & -1 \end{bmatrix}$$

Now $$B.B^{-1} = \begin{bmatrix} 1 & 0 & -2 \\ -2 & -1 & 2 \\ 3 & 4 & 1 \end{bmatrix}\begin{bmatrix} -9 & -8 & -2 \\ 8 & 7 & 2 \\ -5 & -4 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} -9+0+10 & -8+0+8 & -2+0+2 \\ 18-8-10 & 16-7-8 & 4-2-2 \\ -27+32-5 & -24+28-4 & -6+8-1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I$$

Hence the result.

रेखीय युगपत समीकरणों का हल
(Solution of linear equation)

rhu ;qxir lehdj.kksa ds fuEufyf[kr fudk; dks nsf[k,

$a_1x + b_1y + c_1z = d_1$

$a_2x + b_2y + c_2z = d_2$

$a_3x + b_3y + c_3z = d_3$

These equations can be represented in the following form

$$A.X = B \quad \ldots(i)$$

where A $\rightarrow$ Matrix of the coefficients of x, y and z

X $\rightarrow$ Matrix of the three variables

B $\rightarrow$ Matrix of the constraints on the right hand side.

$$\therefore \quad A = \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix}, \quad X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}, \quad B = \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix}$$

They can be represented in the following form

$$\begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix}$$

From (1), $X = B.A^{-1}$

So to find the values of x, y and z, find the value of A^{-1} and multiply it with constraints matrix.

Example 13. Price of 3 chairs and 2 tables is Rs. 700 and the price of 5 chairs and 3 tables is Rs. 1100. Find the price of a chair and a table.

Solution. Let price of one chair be Rs. x and that of one table be Rs. y

$\therefore$ The equations are $3x+2y = 700$

$5x+3y = 1100$

or $$\begin{bmatrix} 3 & 2 \\ 5 & 3 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 700 \\ 1100 \end{bmatrix}$$

$$A = \begin{bmatrix} 3 & 2 \\ 5 & 3 \end{bmatrix}, X = \begin{bmatrix} x \\ y \end{bmatrix}, B = \begin{bmatrix} 700 \\ 1100 \end{bmatrix}$$

Now $A_{11} = 3$, $A_{12} = -5$, $A_{21} = -2$, $A_{22} = 3$

$\therefore$ Adjoint $A = \begin{bmatrix} 3 & -2 \\ -5 & 3 \end{bmatrix}$ and $A^{-1} = \dfrac{Adj.A}{|A|}$

$|A| = 3 \times 3 - 2 \times 5 = -1$

$$\therefore \quad A^{-1} = \frac{1}{-1}\begin{bmatrix} 3 & -2 \\ -5 & 3 \end{bmatrix} = \begin{bmatrix} -3 & 2 \\ 5 & -3 \end{bmatrix}$$

$$\therefore \quad \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -3 & 2 \\ 5 & -3 \end{bmatrix}\begin{bmatrix} 700 \\ 1100 \end{bmatrix} = \begin{bmatrix} -3\times700+2\times1100 \\ 5\times700-3\times1100 \end{bmatrix} = \begin{bmatrix} 100 \\ 200 \end{bmatrix}$$

$\therefore$ x = Rs. 100, y = Rs. 200.

Example 14. Solve the following system of equations by matrix method

$$2x + 8y + 5z = 5$$
$$x + y + z = -2$$
$$x + 2y - z = 2$$

Solution. The system can be represented by the matrix equation AX = B

where $A = \begin{bmatrix} 2 & 8 & 5 \\ 1 & 1 & 1 \\ 1 & 2 & 1 \end{bmatrix}$ $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$, $B = \begin{bmatrix} 5 \\ -2 \\ 2 \end{bmatrix}$

$$|A| = \begin{vmatrix} 2 & 8 & 5 \\ 1 & 1 & 1 \\ 1 & 2 & -1 \end{vmatrix} = 2(-1-2) - 8(-1-1) + 5(2-1) = -6 + 16 + 5 = 15 \neq 0$$

$\therefore$ A is non-singular

The system has the unique solution $X = A^{-1}B$.

Now $A_{11} = \begin{vmatrix} 1 & 1 \\ 2 & -1 \end{vmatrix} = -3$; $A_{12} = -\begin{vmatrix} 1 & 1 \\ 1 & -1 \end{vmatrix} = 2$

$A_{13} = \begin{vmatrix} 1 & 1 \\ 1 & 2 \end{vmatrix} = 1$.

$A_{21} = -\begin{vmatrix} 8 & 5 \\ 2 & -1 \end{vmatrix} = 18$; $A_{22} = \begin{vmatrix} 2 & 5 \\ 1 & -1 \end{vmatrix} = -7$

$A_{23} = -\begin{vmatrix} 2 & 8 \\ 1 & 2 \end{vmatrix} = 4$

$A_{31} = \begin{vmatrix} 8 & 5 \\ 1 & 1 \end{vmatrix} = 3$; $A_{32} = -\begin{vmatrix} 2 & 5 \\ 1 & 1 \end{vmatrix} = 3$;

$A_{31} = \begin{vmatrix} 2 & 8 \\ 1 & 1 \end{vmatrix} = -6$

$$Adj. = A_{21} = \begin{bmatrix} A_{11} & A_{21} & A_{31} \\ A_{12} & A_{22} & A_{32} \\ A_{13} & A_{23} & A_{33} \end{bmatrix} = \begin{bmatrix} -3 & 18 & 3 \\ 2 & -7 & 3 \\ 1 & 4 & -6 \end{bmatrix}$$

$$A^{-1} = \frac{Adj.\ A}{|A|} = \frac{1}{15}\begin{bmatrix} -3 & 18 & 3 \\ 2 & -7 & 3 \\ 1 & 4 & -6 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{-3}{15} & \frac{18}{15} & \frac{3}{15} \\ \frac{2}{15} & \frac{-7}{15} & \frac{3}{15} \\ \frac{1}{15} & \frac{4}{15} & \frac{-6}{15} \end{bmatrix}$$

$$\therefore X = A^1 B = \begin{bmatrix} \frac{-3}{15} & \frac{18}{15} & \frac{3}{15} \\ \frac{2}{15} & \frac{-7}{15} & \frac{3}{15} \\ \frac{1}{15} & \frac{4}{15} & \frac{-6}{15} \end{bmatrix} \begin{bmatrix} 5 \\ -2 \\ 2 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{-15-36+6}{15} \\ \frac{10+14+6}{15} \\ \frac{5-8-12}{15} \end{bmatrix} = \begin{bmatrix} -3 \\ -2 \\ -1 \end{bmatrix}$$

$$\therefore \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -3 \\ 2 \\ 1 \end{bmatrix} \qquad \Rightarrow x = -3,\ y = 2,\ z = -1.$$

संगत तथा असंगत समीकरण (Consistent and inconsistent equations)

(i) If $|A| \neq 0$, the system is consistent and has a unique solution given by $X = A^{-1} B$.

(ii) If $|A| = 0$, the system has either no solution or infinite number of solutions. To find this, determine (Adj A). B

(a) If (Adj A). B $\neq 0$, the system is inconsistent and has no solutions.

(b) If (Adj A). B $= 0$, the system is consistent and has infinite number of solutions.

Example 15. Solve the equations by matrix method

$5x+3y+z = 16$

$2x+y+3z = 19$

$x+2y+4z = 25.$

Solution. The system can be represented by the matrix equation AX = B.

where $A = \begin{bmatrix} 5 & 3 & 1 \\ 2 & 1 & 3 \\ 1 & 2 & 4 \end{bmatrix}$, $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$, $B = \begin{bmatrix} 16 \\ 19 \\ 25 \end{bmatrix}$

$$|A| = \begin{vmatrix} 5 & 3 & 1 \\ 2 & 1 & 3 \\ 1 & 2 & 4 \end{vmatrix} \begin{aligned} &= 5(4-6) - 3(8-3) + 1(4-1) \\ &= -10 - 15 + 3 = -22 \neq 0 \end{aligned}$$

$\therefore$ A is non-singular.

$\therefore$ The system has the unique solution $X = A^{-1} B$

$A_{11} = -2, \quad A_{12} = -5, \quad A_{13} = 3,$

$A_{21} = -10, \quad A_{22} = 19, \quad A_{23} = -7$

$A_{31} = 8, \quad A_{32} = -13, \quad A_{33} = -1.$

$$\therefore \text{ Adj. } A = \begin{bmatrix} A_{11} & A_{21} & A_{31} \\ A_{12} & A_{22} & A_{32} \\ A_{13} & A_{23} & A_{33} \end{bmatrix} = \begin{bmatrix} -2 & -10 & 8 \\ -5 & 19 & -13 \\ 3 & -7 & -1 \end{bmatrix}$$

$$A^{-1} = \frac{\text{Adj.} A}{|A|} = \frac{1}{-22}\begin{bmatrix} -2 & -10 & 8 \\ -5 & 19 & -13 \\ 3 & -7 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{2}{22} & \frac{10}{22} & -\frac{8}{22} \\ \frac{5}{22} & -\frac{19}{22} & \frac{13}{22} \\ -\frac{3}{22} & \frac{7}{22} & \frac{1}{22} \end{bmatrix}$$

Solution is given by $X = A^{-1} B$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} \frac{2}{22} & \frac{10}{22} & -\frac{8}{22} \\ \frac{5}{22} & -\frac{19}{22} & \frac{13}{22} \\ -\frac{3}{22} & \frac{7}{22} & \frac{1}{22} \end{bmatrix}\begin{bmatrix} 16 \\ 19 \\ 25 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{32+190-200}{22} \\ \frac{80-361+325}{22} \\ \frac{-48+133+25}{22} \end{bmatrix}$$

or $$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} \frac{22}{22} \\ \frac{44}{22} \\ \frac{110}{22} \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 5 \end{bmatrix}$$

$\therefore \quad x = 1, y = 2, z = 5.$

Example 16. Gaurav purchases 3 pens, 2 bags and 1 instrument box and pays Rs 41. From the same shop , Dheeraj purchases 2 pens, 1 bag and 2 intrument boxes and pays Rs. 29, while Ankur purchases 2 pens, 2 bags and 2 instrument boxes and pays Rs. 44. Translate the problem into a system of equations. Solve the system of equations by matrix method and hence find the cost of one pen, one bag and one instrument box.

Solution. Let the price of 1 pen be Rs. x, price of 1 bag be Rs. y and prince of 1 instrument box be Rs. z respectively. Then

$$3x+2y+z = 41$$
$$2x+y+2z = 29$$
$$2x+2y+2z = 44.$$

The system can be represented by the matrix equation AX = B.

where $$A = \begin{bmatrix} 3 & 2 & 1 \\ 2 & 1 & 2 \\ 2 & 2 & 2 \end{bmatrix}, X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}, B = \begin{bmatrix} 41 \\ 29 \\ 44 \end{bmatrix}$$

$$|A| = \begin{vmatrix} 3 & 2 & 1 \\ 2 & 1 & 2 \\ 2 & 2 & 2 \end{vmatrix} \begin{aligned} &= 3(2-4)-2(4-4)+1(4-2) \\ &= 3(-2)-2(0)+1(2) \\ &= -6-0+2 = -4 \neq 0 \end{aligned}$$

$\Rightarrow$ The system has a unique solution $X = A^{-1} B$

$$= -\frac{1}{4}\begin{bmatrix} -82-58+132 \\ 0+116-176 \\ 82-58-44 \end{bmatrix} = -\frac{1}{4}\begin{bmatrix} -8 \\ -60 \\ -20 \end{bmatrix} = \begin{bmatrix} 2 \\ 15 \\ 5 \end{bmatrix}$$

$\therefore$ x = 2, y = 15, z = 5.

Hence, the cost of 1 pen is Rs. 2.
The cost of 1 bag is Rs. 15.
and the cost of 1 instrument box is Rs. 5.

Example 17. Solve by matrix method, the equations

$$x+y = 0$$
$$y+z = 1$$
$$z+x = 3.$$

Solution. The system can be represented by the matrix equation AX = B

where $$A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix}, X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}, B = \begin{bmatrix} 0 \\ 1 \\ 3 \end{bmatrix}$$

$$|A| = \begin{vmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{vmatrix} = 1\,(1-0) - 1(0-1) + 0(0-1) = 2 \neq 0.$$

$\therefore$ A is non-singular

$\therefore$ The system has the unique solution $X = A^{-1} B$

$A_{11} = 1$, $A_{12} = 1$, $A_{13} = -1$
$A_{21} = -1$, $A_{22} = 1$, $A_{23} = 1$
$A_{31} = 1$, $A_{32} = -1$, $A_{33} = 1$

$$\text{Adj. } A = \begin{bmatrix} A_{11} & A_{21} & A_{31} \\ A_{12} & A_{22} & A_{32} \\ A_{13} & A_{23} & A_{33} \end{bmatrix} = \begin{bmatrix} 1 & -1 & 1 \\ 1 & 1 & -1 \\ -1 & 1 & 1 \end{bmatrix}$$

$$A^{-1} = \frac{\text{Adj.A}}{|A|} = \frac{1}{2}\begin{bmatrix} 1 & -1 & 1 \\ 1 & 1 & -1 \\ -1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} \end{bmatrix}$$

Solution is given by $X = A^{-1} B$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} \end{bmatrix}\begin{bmatrix} 0 \\ 1 \\ 3 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{-1+3}{2} \\ \frac{1-3}{2} \\ \frac{1+3}{2} \end{bmatrix} = \begin{bmatrix} \frac{2}{2} \\ -\frac{2}{2} \\ \frac{4}{2} \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}$$

$\therefore$ $x = 1, y = -1, z = 2.$

Use matrix method to examine the following systems of equations for consistency or inconsistency.

Example 18. $3x-2y = 5$
$6x-4y = 9$

Example 19. $4x-2y = 3$
$6x-3y = 5$

Example 20. $6x+4y = 2$
$9x+6y = 3$

Example 21. $2x+3y = 5$
$6x+9y = 10.$

Solutions

18. The system can by represented by the matrix equation AX = B.

$$\begin{bmatrix} 3 & -2 \\ 6 & -4 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 5 \\ 9 \end{bmatrix}$$

where $A = \begin{bmatrix} 3 & -2 \\ 6 & -4 \end{bmatrix}, X = \begin{bmatrix} x \\ y \end{bmatrix}, B = \begin{bmatrix} 5 \\ 9 \end{bmatrix}$

$$|A| = \begin{vmatrix} 3 & -2 \\ 6 & -4 \end{vmatrix} = -12 + 12 = 0$$

$\therefore$ A is singular.

$\Rightarrow$ Either the given system has no solution or an infinite number of solutions.

Now $A_{11} = -4,$ $A_{12} = -6$
$A_{21} = 2,$ $A_{22} = 3$

$$\text{Adj. } A = \begin{bmatrix} A_{11} & A_{21} \\ A_{12} & A_{22} \end{bmatrix} = \begin{bmatrix} -4 & 2 \\ -6 & 3 \end{bmatrix}$$

$$(\text{Adj. } A)B = \begin{bmatrix} -4 & 2 \\ -6 & 3 \end{bmatrix}\begin{bmatrix} 5 \\ 9 \end{bmatrix} = \begin{bmatrix} -20+18 \\ -30+27 \end{bmatrix}$$

$$= \begin{bmatrix} -2 \\ -3 \end{bmatrix} \neq O.$$

$\therefore$ The given system has no solution and is, therefore inconsistent.

19. The system can be represented by the matrix equation AX = B.

$$\begin{bmatrix} 4 & -2 \\ 6 & -3 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 3 \\ 5 \end{bmatrix}$$

where $A = \begin{bmatrix} 4 & -2 \\ 6 & -3 \end{bmatrix}, X = \begin{bmatrix} x \\ y \end{bmatrix}, B = \begin{bmatrix} 3 \\ 5 \end{bmatrix}$

$$|A| = \begin{vmatrix} 4 & -2 \\ 6 & -3 \end{vmatrix} = -12 + 12 = 0$$

$\therefore$ A is singular

So either the system has no solution or an infinite number of solutions.

Now $A_{11} = -3, A_{12} = -6, A_{21} = 2, A_{22} = 4$

$$\therefore \quad \text{Adj } A = \begin{bmatrix} A_{11} & A_{21} \\ A_{12} & A_{22} \end{bmatrix} = \begin{bmatrix} -3 & 2 \\ -6 & 4 \end{bmatrix}$$

$$(\text{Adj. } A).\ B = \begin{bmatrix} -3 & 2 \\ -6 & 4 \end{bmatrix}\begin{bmatrix} 3 \\ 5 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \neq O$$

So the given system has no solution and is, therefore inconsistent.

20. The system can be represented by the matrix equation AX = B

$$\begin{bmatrix} 6 & 4 \\ 9 & 6 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$$

A. X = B

$$\text{Now } |A| = \begin{vmatrix} 6 & 4 \\ 9 & 6 \end{vmatrix} = 36 - 36 = 0$$

$\therefore$ A is singular.

$\Rightarrow$ Either the system has no solution or an infinite number of solutions.

$A_{11} = 6, \qquad A_{12} = -9$

$A_{21} = -4, \qquad A_{22} = 6$

$$\text{Adj. } A = \begin{bmatrix} A_{11} & A_{21} \\ A_{12} & A_{22} \end{bmatrix} = \begin{bmatrix} 6 & -4 \\ -9 & 6 \end{bmatrix}$$

$$(\text{Adj. } A)\ B = \begin{bmatrix} 6 & -4 \\ -9 & 6 \end{bmatrix}\begin{bmatrix} 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 12 & -12 \\ -18 & 18 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} = O$$

Since (Adj. A) B = O.

The given system is consistent and has an infinite number of solutions. To obtain the infinite solution, take y = k and solve any one of the given equations as follows

$$6x + 4y = 2$$

$$\Rightarrow \quad 6x = 2-4y \text{ or } x = \frac{1}{6}(2-4k) \qquad [\because y = k]$$

i.e., $x = \frac{1}{3}(1-2k)$.

Hence $x = \frac{1}{3}(1-2k)$, $y = k$, where k is any real number gives us infinite number of solutions.

21. The system can be represented by the matrix equation AX = B.

$$\begin{bmatrix} 2 & 3 \\ 6 & 9 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 5 \\ 10 \end{bmatrix}$$

where $A = \begin{bmatrix} 2 & 3 \\ 6 & 9 \end{bmatrix}, X = \begin{bmatrix} x \\ y \end{bmatrix}, B = \begin{bmatrix} 5 \\ 10 \end{bmatrix}$

Here $|A| = \begin{vmatrix} 2 & 3 \\ 6 & 9 \end{vmatrix} = 18 - 18 = 0$

$\therefore$ A is singular

$\Rightarrow$ Either the system has no solution or an infinite number of solutions

$A_{11} = 9$ $A_{12} = -6$

$A_{21} = -3,$ $A_{22} = 2$

$$\text{Adj. } A = \begin{bmatrix} A_{11} & A_{21} \\ A_{12} & A_{22} \end{bmatrix} = \begin{bmatrix} 9 & -3 \\ -6 & 2 \end{bmatrix}$$

$$(\text{Adj. } A)\ B = \begin{bmatrix} 9 & -3 \\ -6 & 2 \end{bmatrix}\begin{bmatrix} 5 \\ 10 \end{bmatrix} = \begin{bmatrix} 45-30 \\ -30+20 \end{bmatrix} = \begin{bmatrix} 15 \\ -10 \end{bmatrix} \neq O$$

The given system has no solution and is, therefore, inconsistent.

System of homogeneous linear equation

The system of equations

$a_1x + b_1y + c_1 z = 0$

$a_2x + b_2y + c_2z = 0$

$a_3x + b_3y + c_3z = 0$

is a system of homogeneous linear equations and written as AX = O

x = y = z = 0 is always a solution and is called the trivial solution.

Let us see whether the system has non-trivial solutions.

Suppose A is non-singular. Then A^{-1} exists and we get

$A^{-1}(AX) = O$

$(A^{-1}A)X = O$ or $I.X = O$

$\therefore$ $X = O$

Hence, if A is non-singular, the system has only trivial solution.

In order that the system AX = O has non-trivial solution, it is necessary | A | = O.

Theorem. A system of a n homogeneous linear equations in n unknowns has non-trivial solutions if and only if the coefficient matrix is singular.

We illustrate a method for finding non-trivial solution by an example.

Example 22. Find non-trivial solution of the system

$3x+2y+7z = 0$

$4x-3y-2z = 0$

$5x+9y+23z = 0$

Solution. The coefficient matrix

$$A = \begin{bmatrix} 3 & 2 & 7 \\ 4 & -3 & -2 \\ 5 & 9 & 23 \end{bmatrix}$$

$$|A| = \begin{vmatrix} 3 & 2 & 7 \\ 4 & -3 & -2 \\ 5 & 9 & 23 \end{vmatrix} \begin{matrix} = 3(-69+18) - 2(92+10) + 7(36+15) \\ = -153 - 204 + 357 = -357 + 357 = 0 \end{matrix}$$

$\therefore$ A is singular

$\therefore$ The given system of equations has non-trivial solution.

From first equation, we get

$$3x = -2y - 7z \qquad \text{or } x = \frac{-2y-7z}{3}$$

putting this value of x in the second equation, we have

$$4\left(\frac{-2y-7z}{3}\right) - 3y - 2z = 0$$

$$\Rightarrow \qquad \frac{-8y-28x-9y-6z}{3} = 0$$

$$-17y - 34z = 0 \qquad \text{or } y = -2z.$$

Putting this value of y in the first equation.

$$3x-4z+7z = 0 \qquad \text{or } 3x = -3z \text{ i.e., } x = -z.$$

Hence x = –k, y = –2k and z = k constitute the general solution of equations, where k is any real number. Since we can give any arbitrary values to k, therefore the given system of equations has an infinite number of solutions.

Criterion of Consistency regarding solution of homogeneous linear equations given by AX = O, where A is a square matrix.

(i) If $|A| \neq 0$, then the system has only trivial solution.

(ii) If $|A| = 0$, the system has infinitely many solutions.

Note that if $|A| = 0$, then (Adj. A) B = O as B = O.

Exercise 8.2

1. Find the inverse of the following matrices.

(i) $\begin{bmatrix} 1 & 2 & 4 \\ 5 & 7 & 8 \\ 9 & 10 & 12 \end{bmatrix}$ (ii) $\begin{bmatrix} 2 & 5 & 3 \\ 3 & 1 & 2 \\ 1 & 2 & 1 \end{bmatrix}$ (iii) $\begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 2 \\ 3 & 3 & 4 \end{bmatrix}$

2. If $A = \begin{bmatrix} 2 & 5 \\ 1 & 6 \end{bmatrix}$ find A^{-1} and verify that

$A^{-1} = \frac{-1}{7}A + \frac{8}{7}I$

3. If $A = \begin{bmatrix} 4 & 5 \\ 2 & 1 \end{bmatrix}$ show that $A - 3I = 2[I + 3A^{-1}]$

4. Find the inverse of the matrix $\begin{bmatrix} 2 & 0 & -1 \\ 5 & 1 & 0 \\ 0 & 1 & 3 \end{bmatrix}$ and verify that

$A^{-1}, A = A. A^{-1} = I$

5. Given $A = \begin{bmatrix} 0 & -1 & 2 \\ 2 & -2 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & 1 \\ 1 & 0 \\ 1 & 1 \end{bmatrix}$

Find the product C = AB and find C^{-1}

6. If $A = \begin{bmatrix} 0 & 1 & -1 \\ 2 & 0 & 0 \\ 1 & -1 & 0 \end{bmatrix}$ show that $(A^{-1})^2 + A = I$

7. If $A = \begin{bmatrix} 1 & 3 & 4 \\ 3 & -1 & 6 \\ -1 & 5 & 1 \end{bmatrix}$ prove that $(A')^{-1} = (A^{-1})'$

8. Solve the following system of equations by matrix method

(a) $4x - 3y = 5$, $3x - 5y = 1$

(b) $x + 2y = 4$, $2x + 5y = 9$

(c) $2x - y + 4z = 18$, $-3x + z = -2$, $-x + y = 0$

(d) $x - 2y + 3z = 5$, $4x + 3y + 4z = 7$, $x + y - z = -4$

9. Solve the following set of homogeneous equations.

(i) $x + y + z = 0$, $x - y - 5z = 0$, $x + 2y + 4z = 0$

(ii) $3x - y + z = 0$, $-15x + 6y - 5z = 0$, $5x - 2y + 2z = 0$

10. Two cities A and B are 70 km apart. Two cars start at the same time, one from A and other from B. If they move in the same direction, they meet after 7 hours but if they move in opposite directions, they meet after 1 hour. Find the speeds of the two cars.

Answers 8.1

1. (i) $\begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 2 & 0 & 1 \end{bmatrix}$ (ii) $\begin{bmatrix} 1 & 2 & 4 \\ -2 & -4 & 5 \\ 4 & 5 & -6 \end{bmatrix}$

3. $\begin{bmatrix} -1 & \frac{9}{2} & 3 \\ \frac{9}{2} & 3 & 2 \\ 3 & 2 & 5 \end{bmatrix} + \begin{bmatrix} 0 & \frac{5}{2} & -2 \\ \frac{-5}{2} & 0 & 2 \\ 2 & -2 & 0 \end{bmatrix}$

4. (i) $\begin{bmatrix} -1 & 5 \\ 4 & -2 \end{bmatrix}$ (ii) $\begin{bmatrix} 5 & -13 & 11 \\ 12 & 12 & -6 \\ -8 & 10 & 4 \end{bmatrix}$

Answers 8.2.

(1) (i) $\frac{1}{24}\begin{bmatrix} -4 & -16 & 12 \\ -12 & 24 & -12 \\ 13 & -8 & 3 \end{bmatrix}$ (ii) $\frac{1}{4}\begin{bmatrix} -3 & 1 & 7 \\ -1 & -1 & 5 \\ 5 & 1 & -13 \end{bmatrix}$, (iii) $\frac{1}{7}\begin{bmatrix} -6 & -1 & 5 \\ 2 & 5 & -4 \\ 3 & -3 & 1 \end{bmatrix}$ 5, $\begin{bmatrix} \frac{1}{3} & -\frac{1}{3} \\ \frac{1}{3} & \frac{1}{6} \end{bmatrix}$

8. (a) $x = 2, y = 1$ (b) $x = 2, y = 1$ (c) $x = 2, y = 2, z = 4$ (d) $x = -2, y = 1, z = 3$

9. $x = 2k, y = -3k, z = k$ where k is any real number

10. $x = y = z = 0$.

Chapter-9

रेखीय नियोजन–~~1~~ I बिन्दु रेखीय विधि
(Linear Programming Graphical Method)

एक वस्तु के उत्पादन के लिए कई तरह के संसाधनों की जरूरत पड़ती है जैसे कच्चा माल, मशीनें, मानव संसाधन तथा अन्य तरह के माल आदि। ये संसाधन सीमित मात्रा में उपलब्ध होते हैं जबकि इनकी मांग ज्यादा होती है। इसलिए इनका सही ढ़ंग से उपयोग किया जाना चाहिए ताकि लागत में कमी की जा सके तथा लाभ में वृद्धि की जा सके। रेखीय नियोजन एक ऐसी विधि है जो हमें bu~~y~~ संसाधनों का सही ढ़ंग से उपयोग करने में सहायता करती है।

For manufacturing a product, a number of resources like raw material, machines, manpower and other types of materials. These resources are available in limited quantity but their demand is more. S~~O~~o they are used be optimally utilized so that cost could be minimised and profits could be maximised. Linear programming is a technique which helps us in the optimal utilisation of these resources.

रेखीय नियोजन की विशेषताएँ –
Characteristics of Linear Programming

1. हर रेखीय नियोजन के प्रश्न में एक उद्देश्य होता है जो कि साफ तरह से पहचानने तथा मापने योग्य होना चाहिए। Every linear programming problem has an objective which should be clearly identifiable and measurable. For example objective can be maximisation of sales, profits and minimisation of costs and so on.

2. सभी उत्पाद तथा संसाधन भी पहचानने तथा नापने योग्य होने चाहिए।
All the products and resources should also be clearly identifiable and measurable.

3. संसाधन सीमित मात्रा में उपलब्ध हों।
Resources are available in limited quantity.

4. उद्देश्य तथा संसाधनों की सीमित मात्रा दिखाने वाले सम्बन्धों को सीमाओं के समीकरणों या असमीकरणों से दिखाया जाता है। ये सम्बन्ध रेखीय प्रकृति का होता है।
The relationship representing objectives and resources limitation are represented by constraint in equalities or equations. These relationships are linear in nature.

The above characteristics will be clear from the following example :

A firm is engaged in manufacturing two products A and B. Each unit of A requires 2 kg. of raw material and 4 labour hours for processing while each unit of B requires 3 kg. of raw material and 3 labour hours. The weekly availability of raw material and labour hours is limited to 60 kg. and 96 hours respectively. One unit of product A sold for Rs. 40 while are unit of B is sold for Rs. 35.

In the above problem, first we define the objective. Since we are given data on sales pri~~n~~ce per unit of two products, so our objective is to maximise the soles.

Let x_1 be the number of units of A and x_2 be the number of units of B to be produced. So
Sales revenue ~~frim~~ from sale of x_1 unit of A = $40x_1$ and
Sales revenue from sale of x_2 units of B = $35x_2$. Thus

$$\text{Total sales } = 40x_1 + 35x_2 \text{ and}$$

Our objective becomes

Maximise Z $= 40x_1 + 35x_2$

After this we come to the constraint in-equalities, from the given date, we find that quantity of raw material required to produce x_1 kg of

A = $2x_1$ [∵ each unit of A requires 2 kg of raw material so x_1 units of A require $2x_1$ kg of raw material]

In the same way, quantity of raw material required to produce x_2 units of B = $3x_2$

So new total raw material requirement $= 2x_1$ ~~=~~ $+ 3x_2$

But here we have a constraint in the form of quantity of raw material available. Since the maximum quantity of raw material available is limited to 60 kg, so we can not use more than 60 kg of raw material in any case. Mathematically we can write it as

$$2x_1 + 3x_2 \leq 60$$

In other words we can ~~ray~~ say that total quantity of raw material consumed is less than and equal to (shown by the symbol ≤) 60

Similarly we can produced to expressed the labour constraints in the following way :

Labour hours required to produce to produce x_1 units of A = $3x_1$ and

Labour hours required x_2 units of B = $3x_2$.

Since the total number of labour hours available per week is limited to 96, we can express this constraint as

$$3x_1 + 3x_2 \leq 96$$

In the last we express the non-negativity condition, i.e. $x_1, x_2 \geq 0$

Which is self clear because number of units of product A or B can be zero or positive only.

Now the above problem can be summarised as

~~Max.~~ Max. $Z = 40x_1 + 35x_2$

Subject to

$$2x_1 + 3x_2 \leq 60$$
$$3x_1 + 3x_2 \leq 96$$
$$x_1, x_2 \geq 0$$

सामान्यतः एक रेखीय नियोजन समस्या को हम निम्नलिखित तरह से दिखा सकते हैं

Generally we can express a linear programming in the following way :

Maximise or minimise $Z = C_1x_1 + C_2x_2 + \ldots + C_nx_n$ objective function

Subject to

$$\left.\begin{array}{l} a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n \leq b_1 \text{ or } \geq b_1 \\ a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n \leq b_2 \text{ or } \geq b_2 \\ \ldots\ldots\ldots\ldots \\ \ldots\ldots\ldots\ldots \\ \ldots\ldots\ldots\ldots \\ a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n \leq b_n \text{ or } \geq b_m \end{array}\right\} \text{Constraints}$$

~~Constraints~~

$$x_1, x_2, x_3, \ldots, x_n \geq 0$$

where C's are the profit or cost co~~-~~efficient~~s~~ of decisions variables x's, a's are the resource co-efficients and b's are the resource values.

रेखीय नियोजन समस्या का हल (Solution of Linear Programming problems)

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रेखीय नियोजन समस्याओं के हल की दो विधियाँ हैं – बिन्दु रेखीय विधि तथा सिम्पलैक्स विधि

There are two methods of solving the linear programming problems - Graphical method and Simplex method.

बिन्दु रेखीय विधि

Graphical Method

~~Bl~~ bl विधि से हम केवल ~~b~~nks चरों वाली समस्याओं का हल कर सकते हैं एक को ग ले लेते हैं व दूसरे को ल ले लेते हैं। किसी भी बिन्दु रेखा पर हम केवल nks~~bks~~ ही चरों के मूल्यों को दिखा सकते हैं। इस विधि के निम्नलिखित पग है:

1) बिन्दु रेखीय कागज पर एक समतल सीधी रेखा खींचते है जिसे ग–अक्ष रेखा कहते हैं।
2) एक लम्बवत सीधी रेखा खींचते है जो इस समतल रेखा पर 90° का कोण बनाती है। इस लम्बवत रेखा को ल–अक्ष रेखा कहते हैं।
3) रूकावटों वाले असमीकरण में बदल कर बिंदुओं को मिलाकर एक सरल रेखा खीचतें हैं।
4) सभी असमीकरणों का समान सम्माव्य क्षेत्रफल ज्ञात किया जाता है। इसी में से हमें 2 के अधिकतम या न्युनतम मूल्य की प्राप्ति होती है।

By this method, we can solve problems involving two variables only, one variable is taken as x and the second as y. On any graph we can show only two variables. The steps involved in this method are following :

1) We draw a horizontal straight line on a graph paper which is called x-axis.
2) We draw a vertical straight line which is perpendicular to this horizontal line. This perpendicular line is called Y-axis.

~~2)~~3) Constraints inequalities are converted into equations and plot their points on the graph.
~~3)~~4) Common feasible area of all the inequalities is found out. From this area, we get the maximum or minimum values of Z.

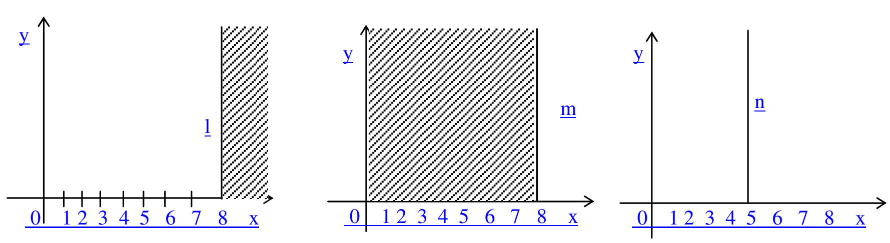
The following illustrations will make it more clear :–

Example 1. Find the feasible area of the following :

(i) $x \leq 8$ (ii) $x \geq 6$ (iii) $x = 5$

Solution.

(i) (ii) (iii)



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Shaded portion shows the feasible area.

In the first case, feasible area is to the right of the vertical line *l*. There is no limit to its maximum value. In the second case, the feasible area is to the left of the line m forwards y-axis and in third case, feasible area is along the line n.

To find feasible area of an equation having two variables only

Example 2. Find the feasible area of the following :

(i) $2x + y \leq 6$ (ii) $2x + y \leq 6$, $x, y \geq 0$

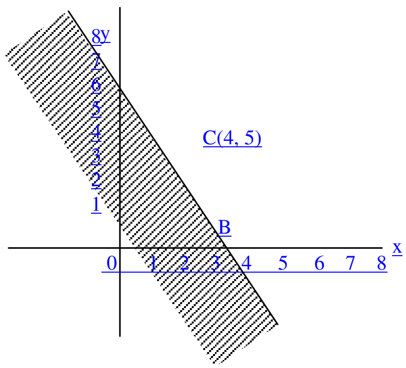
Solution (i). For plotting the lines, change the inequalities into equations so $2x + y = 6$

Find two points to be plotted on the graph

When $x = 0$ $y = 6$

When $y = 0$ $x = 3$

So we get two points (0, 6) and (3, 0) which when plotted on the graph and joined by a line gives us the line of the above equation.

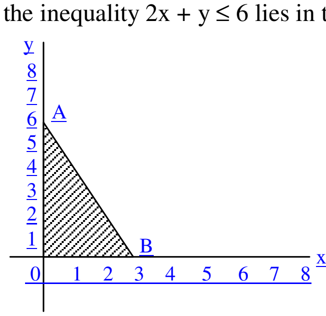


Now since there is no mention of the signs of x and y. So feasible area extends infinitely to the left of line joining points A(0, 6) and B(3, 0). Any point taken to the right of line joining A and B does not satisfy this condition of inequality. For example, let us consider point C(4, 5) Substituting the values of x = 4 and y = 5 in the L.H.S. of the inequality we get

$$2x + y = 2 \times 4 + 5 = 13$$

which is not less than 6. Hence this point does not lie in the feasible area of the inequation.

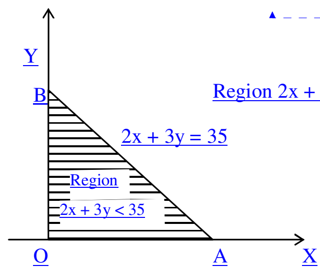
(ii) In this case, since we are given $x, y \geq 0$, it means that neither x nor y can have negative sign. So the feasible area of the inequality $2x + y \leq 6$ lies in the region OAB.



Example 3. Draw the graph of the in equation $2x + 3y \leq 35$.

Solution. The solution of the above problem is the set of pairs (x, y) satisfying $2x + 3y \leq 35$

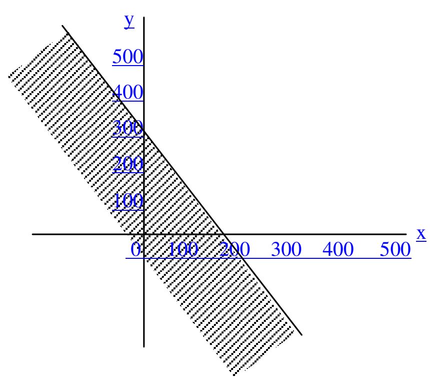
Draw the straight line $2x + 3y = 35$.



This meets x-axis (y = 0) at A($\frac{35}{2}$, 0) and y-axis (x = 0) at B(0, $\frac{35}{3}$). The regions are as shown in the figure. The solution of the above problem is the set of those points (x, y) whose co-ordinates are integers (can be zero) and lying in the region OAB (shaded).

Example 4. Draw the graph of he in equation $\frac{x}{200}+\frac{y}{300} \le 1$. Which of the following points lie in the graph (i) 300, 0 (ii) 200, 400 (iii) 150, 250

Solution. $\frac{x}{200}+\frac{y}{300} \le 1$



or $\quad 3x + 2y \le 600$

Changing into equation

$$3x + 2y = 600$$

when x = 0, y = 300

y = 0, x = 200

From the graph, it is clear that

(i) point (300, 0) does not lie in the graph.

(ii) Point (200, 400) also does not lie in graph but

(iii)Point (150, 250) lies in the graph.

Example 5. Draw the diagram of solution set of the linear constraints

$2x + 3y \le 6$

$x + 4y \le 4$

$x \ge 0,\ y \ge 0.$

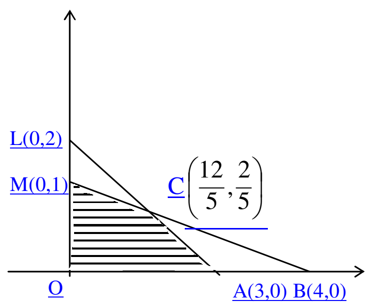
Solution. The given constraints are

$2x + 3y \le 6$
$x + 4y \le 4$
$x \ge 0, \; y \ge 0.$

Consider a set of rectangular cartesian axes OXY in the plane. Each point has co-ordinates of the type (x, y) and conversely. It is clear that any point which satisfies $x \ge 0$, $y \ge 0$ lies in the first quadrant.

Let us draw the graph of $2x + 3y = 6$
For $x = 0$, $3y = 6$, i.e., $y = 2$
For $y = 0$, $2x = 6$ i.e., $x = 3$

∴ line $2x + 3y = 6$ meets OX in A(3, 0) and OY in L(0, 2)
Again let us draw the graph of
$x + 4y = 4$
For $x = 0$, $4y = 4$ i.e., $y = 1$
For $y = 0$, $x = 4$
∴ line $x + 4y = 4$ meets OX in B(4, 0) and OY in M(0, 1)



Since feasible region is the region which satisfies all the constraints,
∴ feasible region is the quadrilateral OACM.
The corner points are
O(0, 0), A(3, 0), C$\left(\frac{12}{5}, \frac{2}{5}\right)$, M(0, 1)

Note 1. Students must note that graphs are to be drawn on the graph paper.

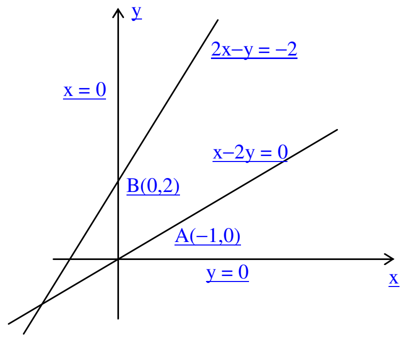
2. Point C can also be calculated by solving the equation $2x + 3y = 6$, $x + 4y = 4$. It helps them in verifying the result obtained from the graph.

3. $2x + 3y \le 6$ represents the region on and below the line AL. Similarly $2x + 3y \ge 6$ will represent the region on and above the line AL.

4. In order to have a clear picture of the region below or above the line, it is better to note that if the point (x_1, y_1) satisfies the in-equation then the region containing this point is the required region.

Example 6. Verify that the solution set of the following linear constraints is empty:
$x - 2y \ge 0$, $2x - y \le -2$, $x \ge 0$, $y \ge 0$.

Solution. The straight line $x - 2y = 0$ passes through O.
The straight line $2x - y = -2$ meets x-axis at A(−1, 0) and y-axis at B(0, 2).
We have the following figure.
Since no portion satisfies all the four constraints,
∴ the solution set is empty.



Solution of LPP by graphical method

(A) Maximisation case –

Example 7. A furniture dealer deals in only two items : tables and chairs. He has Rs. 5000.00 to invest and a space to store at most 60 pieces. A table costs him Rs. 250.00 and a chair Rs. 50.00. He can sell a table at a profit of Rs. 50.00 and a chair at a profit of Rs. 15.00. Assuming that he can sell all the items that he buys, how should he invest his money in order that he may maximize his profit ?

Solution. We formulate the problem mathematically.

Max. possible investment = Rs. 5000.00

Max. storage space = 60 pieces of furniture

	Cost	Profit
Table :	Rs. 250.00	Rs. 50.00
~~Cost~~Chair :	Rs. 50.00	Rs. 15.00

Let x and y be the number of tables and chairs respectively. Then we have the following constraints :

$x \geq 0$ …(1) $y \geq 0$ …(2)

$250x + 50y \leq 5000$ i.e. $5x + y \leq 100$ …(3)

and $x + y \leq 60$ …(4)

Let Z be the profit, then $Z = 50x + 15y$ …(5)

We are to maximize Z subject to constraint (1), (2) (3) and (4).

Let us graph the constraints given in (1), (2), (3) and (4).

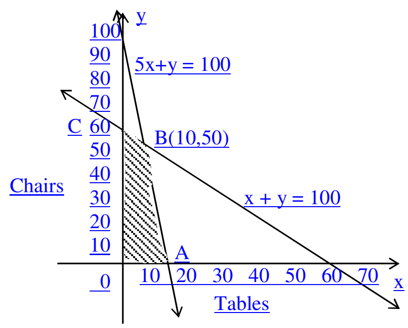
Explanation. Draw the straight lines x = 0, (y-axis), y = 0 (x-axis).

Draw the straight line x + y = 60. This meets the x-axis at (60, 0) and y-axis at (0, 60).

Draw the straight line 5x + y = 100. This meets x-axis at (20, 0) and y-axis at (0, 100).

The shaded region consists of points, which are the intersections of four constraints. This region is called feasible solution of the linear programming problem.

The vertices of the figure OABC show the possible combinations of x and y one of which gives us the maximum value. Now we consider the points one by one.



Feasible region

Example 8. A company manufactures two types of telephone sets, one of which is cordless. The cord type telephone set requires 2 hours to make, and the cordless model requires 4 hours. The company has at most 800 working hours per day to manufacture these models and the packing department can pack at the most 300 telephone sets per day. If the company sells, the cord type model for Rs. 300 and the cordless model for Rs. 400, how many telephone sets of each type should it produce per day to maximize its sales ?

Solution. Let the number of cord type telephone sets be x and the number of cordless type telephone sets be y.

Clearly we have : $x \geq 0$; $y \geq 0$ …(i)

Also $2x + 4y \leq 800$ …(ii)

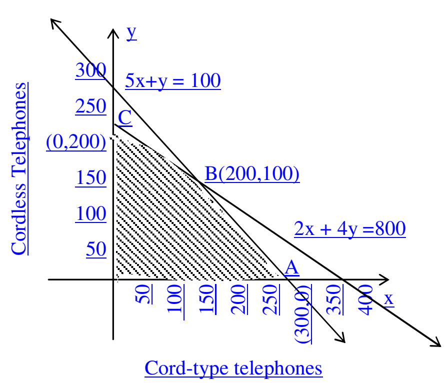
$x + y \leq$ ~~400~~300 …(iii)

Let Z be sale function, $Z = 300x + 400y$

The problem reduces to maximize sale function subject to the conditions :

$$\left.\begin{array}{ll} x \geq 0 \; ; & y \geq 0 \\ 2x + 4y \leq 800 \; ; & x + y \leq 300 \end{array}\right\} \quad \ldots(A)$$

Let us draw the graph of system (A) and solution set of these inequations is the shaded region OABC and so the feasible region is the shaded region whose corner points are O(0, 0), A(300, 0), B(200, 100), C(0, 200). Since the maximum value of the sale function occurs only at the boundary point (s) and so we calculate the sale function at every point of the feasible region.



Boundary points of the feasible region	S = 300 x + 400 y
O(0, 0)	$S = 300 \times 0 + 400 \times 0 = 0$
A(300, 0)	$S = 300 \times 300 + 400 \times 0 =$ Rs. 90, 000
B(200, 100)	$S = 300 \times 200 + 400 \times 100 =$ Rs. 100000
C(0, 200)	$S = 300 \times 0 + 400 \times 200 =$ Rs. 80, 000

Hence the maximum sale is Rs. 100000 at B(200, 100) and so company should produce 200 cord type and 100 cordless telephone sets.

Example 9. Solve the following LPP

Maximize Z = 10x + 12y

Subject to the constraints :

$$x + y \le 5$$
$$4x + y \ge 4$$
$$x + 5y \ge 5$$
$$x \le 4$$
$$y \le 3$$

Solution. It is a problem of mixed constraints. Constraints having greater than or equal to (≥) sign will have their feasible area to the right of their line while the constraints having less than a equal to (≤) sign will have their area to the left of their line.

The given linear constraints are :

$x + y \le 5$ ...(1)

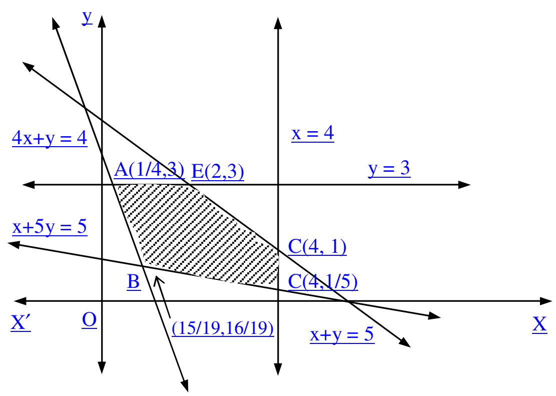
$4x + y \ge 4$ ...(2)

$x + 5y \ge 5$ ...(3)

$x \le 4$ ...(4)

$y \le 3$ ...(5)

Let us draw the graph of inequations (1), (2), (3), (4) and (5). The graph (or solution set) of these inequations is the shaded area (a polygon ABCDE) with vertices $\left(\frac{1}{4}, 3\right)$; $\left(\frac{15}{19}, \frac{16}{19}\right)$; $\left(4, \frac{1}{5}\right)$; (4, 1) ; (2, 3) as shown in the figure below :



This shaded area is bounded by the five lines x + y = 5, 4x + y = 4, x + 5y = 5, x = 4 and y = 3.

Boundary points of the feasible region	Z = 10x + 12y	
A $\left(\frac{1}{4}, 3\right)$	$10 \times \frac{1}{4} + 12 \times 3$	= 38.5
B$\left(\frac{15}{19}, \frac{16}{19}\right)$	$10 \times \frac{15}{19} + 12 \times \frac{16}{19}$	= 18
C $\left(4, \frac{1}{5}\right)$	$10 \times 4 + 12 \times \frac{1}{5}$	= 42.4
D(4, 1)	$10 \times 4 + 12 \times 1$	= 52
E(2, 3	$10 \times 2 + 12 \times 3$	= 56

Since maximum value of Z is Rs. 56 at E, so optimum solution is x = 2, y = 3

Example 10. - If a young man rides his motor-cycle at 25 km per hour, he has to spend Rs. 2 per km on petrol ; if he rides it at a faster speed of 40 km per hour, the petrol cost increases to Rs. 5 per km. He has Rs. 100 to spend on petrol and wishes to find what is the maximum distance he can travel within one hour. Express this as a linear programming problem and then solve it.

Solution.— Let the young man ride x km at the speed of 25 km per hour and y km at the speed of 40 km per hour. Let f be the total distance covered, which is to be maximized.

∴ f = x + y is the objective function.

Cost of travelling per km is Rs. 2 at the speed of 25 km per hour and cost of travelling per km is Rs. 5 at the speed of 40 km per hour.

∴ total cost of travelling = 2x + 5y

Also Rs. 100 are available for petrol

∴ $2x + 5y \leq 100$

Time taken to cover x km at the speed of 25 km per hour = $\frac{x}{25}$ hour

Time taken to cover y km at the speed of 40 km per hour = $\frac{y}{40}$ hour

Total time available = 1 hour

∴ we have $\frac{x}{25} + \frac{y}{40} \leq 1$

or $8x + 5y \leq 200$

Also $x \geq 0$, $y \geq 0$

∴ we are to maximize

$$f = x + y$$

subject to the constraints

$$2x + 5y \leq 100$$
$$8x + 5y \leq 200$$
$$x \geq 0,\ y \geq 0.$$

Consider a set of rectangular cartesian axes OXY in the plane.

It is clear that any point which satisfies $x \geq 0$, $y \geq 0$ lies in the first quadrant.

Let us draw the graph of the line $2x + 5y = 100$

For $x = 0$, $5y = 100$ or $y = 20$

For $y = 0$, $2x = 100$ or $x = 50$

∴ line meets OX in A(50, 0) and OY in L(0, 20)

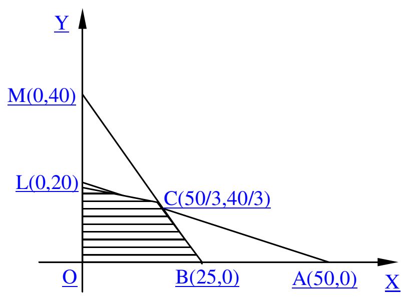
Again we draw the graph of the line

$$8x + 5y = 200.$$

For $x = 0$, $5y = 200$ or $y = 40$

For $y = 0$, $8x = 200$ or $x = 25$

∴ line meets OX in B(25, 0) and OY in M(0, 40)



Since feasible region is the region which satisfies all the constraints,

∴ feasible region is the quadrilateral OBCL. The corner points are O(0, 0), B(25, 0), $C\left(\frac{50}{3}, \frac{40}{3}\right)$, L(0, 20)

At O(0, 0) $f = 0 + 0 = 0$

At B(25, 0), $f = 25 + 0 = 25$

At $C\left(\frac{50}{3}, \frac{40}{3}\right)$, $f = \frac{50}{3} + \frac{40}{3} = 30$

At L(0, 20), $f = 0 + 20 = 20$

∴ maximum value of f = 30 at $\left(\frac{50}{3}, \frac{40}{3}\right)$.

∴ the young man covers the maximum distance of 30 km when he rides $\frac{50}{3}$ km at the speed of 25 km per hour and $\frac{40}{3}$ km at the speed of 40 km per hour.

Example 11. A farmer decides to plant up to 10 hectares with cabbages and potatoes. He decided to grow at least 2, but not more than 8 hectares of cabbage and at least 1, but not more than 6 hectares of potatoes. If he can make a profit of Rs. 1500 per hectare on cabbages and Rs. 2000 per hectare on potatoes how should he plan his farming so as to get the maximum profit ? (Assuming that all the yield that he gets is sold.)

Solution. Suppose the farmer plants x hectares with cabbages and y hectares with potatoes.

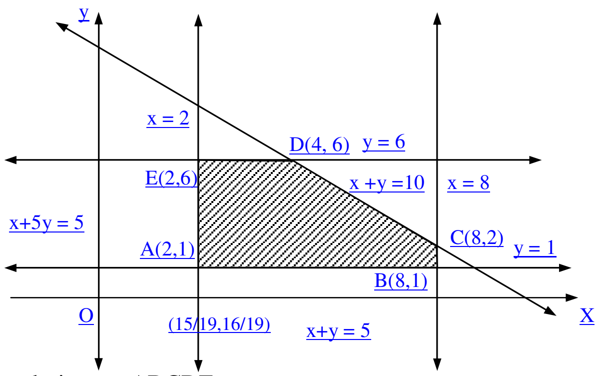
Then the constraints are

$2 \leq x \leq 8$ …(1) $1 \leq y \leq 6$ …(2)

$x + y \leq 10$ …(3)

and $P = 1500x + 2000y$ ...(4)

We draw the lines $x = 2$, $x = 8$, $y = 1$, $y = 6$ and $x + y = 10$.



The vertices of the solution set ABCDE are

A(2, 1), B(8, 1), C(8, 2), D(6, 4) and E(2, 6)

Now at A(2, 1), $P = 1500(2) + 2000(1) = 3000 + 2000 = 5000$

at B(8, 1), $P = 1500(8) + 2000(1) = 12000 + 2000 = 14000$

at C(8, 2), $P = 1500(8) + 2000(2) = 12000 + 4000 = 16000$

at D(4, 6), $P = 1500(4) + 2000(6) = 6000 + 12000 = 18000$

at E(2, 6), $P = 1500(2) + 2000(6) = 3000 + 12000 = 15000$.

Hence in order to maximise profit the farmer plants 4 hectares with cabbages and 6 hectares with potatoes.

Example ~~19~~12. An aeroplane can carry a maximum of 200 passengers. A profit of Rs. 400 is made on each first class ticket and a profit of Rs. 300 is made on each economy class ticket. The airline reserves at least 20 seats for first class. However, at least four times as many passengers prefer to travel by economy class than by the first class. Determine how many each type of tickets must be sold in order to maximise the profit for the airline. What is the maximum profit ?

Solution. Let the number of first class tickets and Economy class tickets sold by the Airline be x and y respectively.

Maximum capacity of passengers is 200 i.e. $x + y \le 200$ ...(i)

At least 20 seats of first class are reserved $\Rightarrow x \ge 20$ ...(ii)

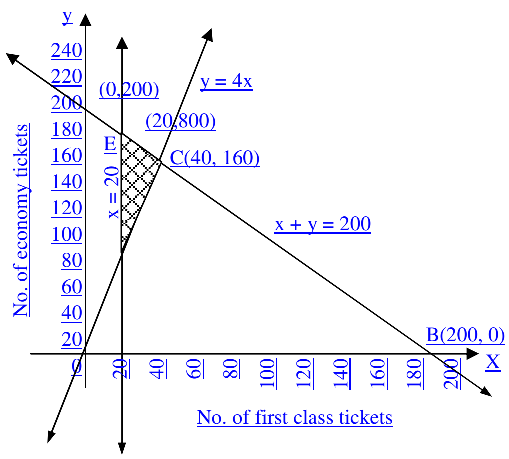
At least 4 x seats of Economy class are reserved

$\Rightarrow \quad y \ge 4x$ ...(iii)

Let P the profit function, $P = 400x + 300y$

$\therefore$ the problem reduces to maximize P subject to the constraints $x \ge 20$; $y \ge 4x$ and $x + y \le 200$.

Let us find out the solution set of the inequations $x \ge 20$; $y \ge 4x$ and $x + y \le 200$.



The triang~~le~~ular shaded region CDE is the feasible region and its vertices are :

C(40, 160), D(20, 80), E(20, 180)

Since the maximum or minimum value occurs at the boundary point (s) and we calculate the profit function P at every vertex of the feasible region.

Boundary points of the feasible region	P = 400x + 300y
C(40, ~~100~~160)	P = 400 × 40 + 300 × 160 = Rs. 64,000 (Maximum profit)
D(20, 80)	P = 400 × 20 + 300 × 80 = Rs. 32,000
E(20, 180)	P = 400 × 20 + 300 × 180 = Rs. 62,000

∴ Maximum profit Rs. 64000 is obtained at C(40, 160) and so Airline should sell 40 tickets of first and 160 tickets of economy class. **[Ans.]**

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(B) ~~Minimusation~~ Minimisation ~~ease~~Case

,sls iz'uksa esa mís'; Qyu Z dh dher fuEure djuh gksrh gSA izk;% ykxr] nwjh] [kpkZ] [krjk oxSjkg ds iz'uksa esa gekjk mís'; mu dk ewY; fuEure j[kus dk gksrk gSA

In such questions, value of the objective function is to be minimised. Generally, in the questions involving cost, distance, expenses risk etc. our objective to keep their value least.

Note : izk;% vf/kdre fLFkfr esa ge de ;k cjkcj (≤) okyh :dkoVsa iz;ksx djrs gS rFkk fuEure fLFkfr esa ge T;knk ;k cjkcj (≥) okyh :dkoVsa iz;ksx djrs gSA dHkh&dHkh ge feyh tqyh :dkoVsa (≥ rFkk ≤ nksuksa) dk Hkh iz;ksx djrs gSaA

Generally in case of maximisation, we use the constraints of less than or equal to (≤) type and in case of minimisation, we use constraints of greater than a equal to (≥) type. But ~~same~~ some times, we also use mixed constraints (both ≥ and ≤).

Example 13. A gardener uses two types of fertilizers I and II. Type I consists of 10% nitrogen and 6% phosphoric acid while type II consists of 5% nitrogen and 10% phosphoric acid. He requires at least 14 kg. of both nitrogen and phosphoric acid for his crop. If the type I fertilizer costs Rs. 0.60 per kg and type II costs Rs. 0.40 per kg., how many kilograms of each fertilizer he should use so as to minimise the total cost. Also find the minimum cost.

Solution. Let x be the quantity (in kg) of type I fertilizer and ~~to~~ y be the quantity (in kg) of type II fertilizer to be used by the gardener.

So the objective function is

$$\text{~~Minimize~~ Minimise } Z = 0.60\,x = 0.40\,y$$

Subject to the constraints

$0.1\,x + 0.05\,y \geq 14$ or $2x + y \geq 280$

$0.06x + 0.10y \geq 14$ or $3x + 5y \geq 700$

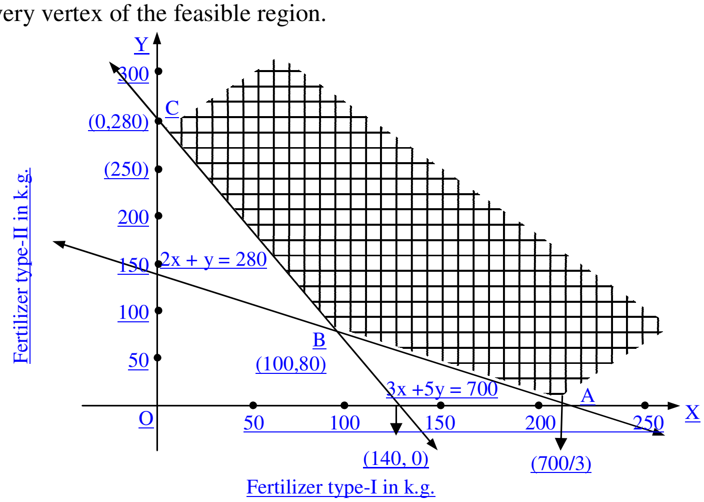
Changing the inequalities into equalities, we get 2x + y = ~~200~~ 280 and 3x ~~=~~ + 5y = 700

x	0	140
y	280	0
(x, y)	(0, 280)	(140, ~~14~~0)

x	0	$\frac{700}{3}$
y	140	0
(x, y)	(0, 140)	$(\frac{700}{3}, 0)$

~~Non~~ Now solution set of (A) is the shaded region ABC and so it is feasible region. $A(\frac{700}{3}, 0)$, B(100, 80), C(0, 280) are the boundary points of the feasible region.

Since the minimum cost occurs only at boundary points and so let us calculate the value of C at every vertex of the feasible region.



Example 14. Minimize

C = x + y subject to

$3x + 2y \geq 12$

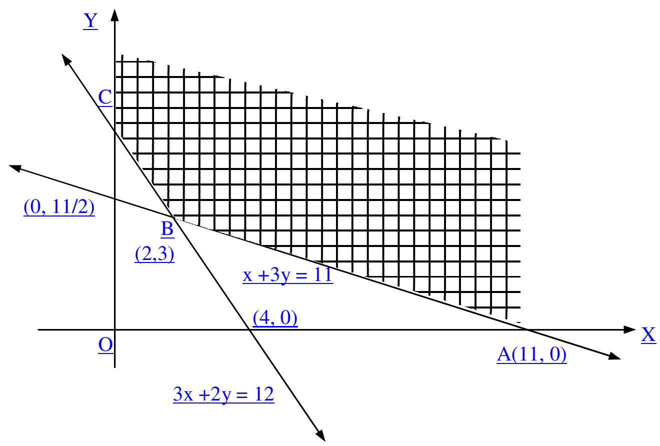
$x + 3y \geq 11$

$x \geq 0$

$y \geq 0$

Solution. Let us solve graphically the following inequations ;

$$\left.\begin{array}{r} 3x + 2y \geq 12 \\ x + 3y \geq 11 \\ x \geq 0 \\ y \geq 0 \end{array}\right\} \qquad \ldots(A)$$



Changing the inequalities into equations, we get

$3x + 2y = 12$

x	0	4
y	6	0

$x + 3y = 11$

x	11	0
y	0	$\frac{11}{3}$

Shaded region ABC is the required feasible region of the above stated inequalities.

Boundary points of the feasible region are :

A(11, 0), B(2, 3), C(0, 6).

Since minimum or maximum always occurs only at boundary point (s) and so will calculate the cost (C) at every boundary point of the feasible region.

Boundary point of the feasible region	C = x + y
A(11, 0)	11 +0 = Rs. 11
B(2, 3)	2 + 3 = Rs. 5 (Mini. Cost)
C(0, 6)	0 + 6 = Rs. 6

Hence minimum cost is at the point B(2, 3) and minimum cost is Rs. 5.

Example 15. Find the maximum and minimum values of the function $2 = 3x + y$
Subject to the constraint

$x + y \leq 2$

$4x + y \leq 25$

$x, y \geq\leq 20$

Solution. Let us first change the inequalities into equation

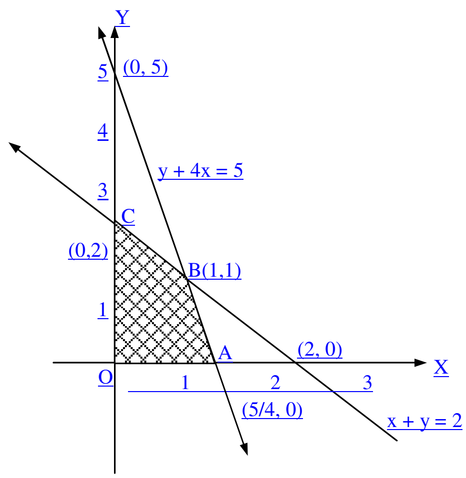
$x + y = 2$

X	0	2
y	2	0

$4x + y = 5$

x	0	$\frac{5}{4}$
y	5	0

Solution set of the above inequations is the required shaded feasible region OABC whose boundary points are O(0, 0), A($\frac{5}{4}$, 0), B(1, 1), C(0, 2).



Since the minimum or maximum value of C occurs only at the boundary point (s) and so let us calculate the value of Z at every vertex of the feasible region OABC.

Boundary points of the feasible region	$Z = 3x + y$
O(0, 0)	$Z = 3 \times 0 + 0 =$ Re. 0 (Minimum cost)
$A(\frac{5}{4}, 0)$	$Z = 3 \times \frac{5}{4} + 0 =$ Rs. 3.75
B(1, 1)	$Z = 3 \times 1 + 1 =$ Rs. 4
C(0, 2)	$Z = 3 \times 0 + 2 =$ Rs. 2

So the maximum value is Rs. 4 and the minimum value Rs. 0 for maximisation x = 1, y = 1 and for minimization x = 0, y = 0.

Example 16. Minimise $P = 2x + 3y$, subject to the conditions

$x \geq 0, \;\; y \geq 0, \;\; 1 \leq x + 2y \leq 10.$

Solution.

We have : $x \geq 0$ ...(1) $\quad y \geq 0$ ...(2)

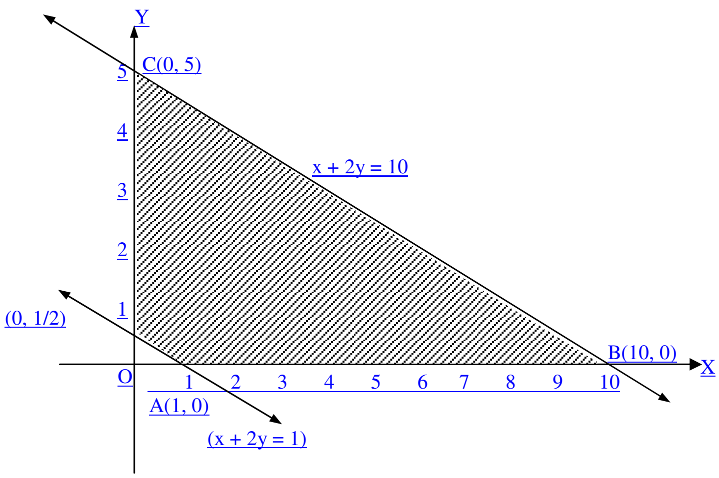
$x + 2y \geq 1$ ...(3) $\quad x + 2y \leq 10$ ...(4)

and $P = x + 3y$

We find out the solution set (convex polygon), where (1) – (4) are true,

For this, we draw the graph of he lines

$x = 0, y = 0, x + 2y = 1, x + 2y = 10.$



The shaded portion is the feasible region of the constraints.

Now

at A(1, 0), $P = 2(1) + 3(0) = 2$

at B(10, 0), $P = 2(10) + 3(0) = 20$

at C(0, 5), $P = 2(0) + 3(5) = 15$

at $D(0, \frac{1}{2})$, $P = 2(0) + 3(\frac{1}{2}) = \frac{3}{2}$.

Since the minimum value is at D, so the optimal solution is $x = 0, \ y = \frac{1}{2}$.

Example 17. Find the maximum and minimum value of

$$Z = x + 2y$$

~~Subject~~ subject to

$$2x + 3y \le 6$$
$$x + 4y \le 4$$
$$x, y \ge 0$$

Solution. We are maximize and minimize

$$Z = x + 2y$$

Subject of the constraints $2x + 3y \le 6$

$$x + 4y \le 4$$
$$x, y \ge 0$$

First ~~quadrant.~~

~~We~~ we draw the graph of the line $2x + 3y = 6$.

For $x = 0, \ 3y = 6$, or $y = 2$

For $y = 0, \ 2x = 6$, or $x = 3$

$\therefore$ line meets OX in A(3, 0) and OY in L(0,2).

draw the graph of line $x + 4y = 4$

For $x = 0, \ 4y = 4$ or $y = 1$

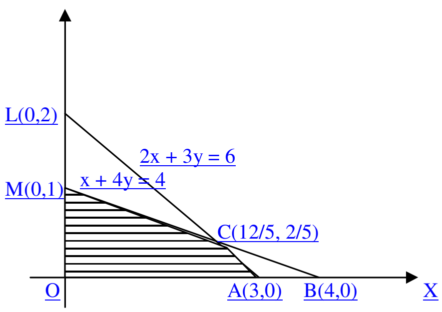
For $y = 0, \ x = 4$

$\therefore$ line meets OX in B(4, 0) and OY in M(0,1).

Since feasible region is the region which satisfies all the constraints.

$\therefore$ ~~OACm~~ OACM is the feasible region. The corner points are

$$O(0, 0),\ A(3, 0),\ C\left(\frac{12}{5}, \frac{2}{5}\right),\ M(0, 1)$$



At O(0, 0), f = 0 + 0 = 0

At A(3, 0), f = 3 + 0 = 3

At C $\left(\frac{12}{5}, \frac{2}{5}\right)$, $f = \frac{12}{5} + \frac{4}{5} = \frac{16}{5} = 3.2$

At M(0, 1), f = 0 + 2 = 2

$\therefore$ Minimum value = 0 at (0,0)

and maximum value = 3.2 at $\left(\frac{12}{5}, \frac{2}{5}\right)$.

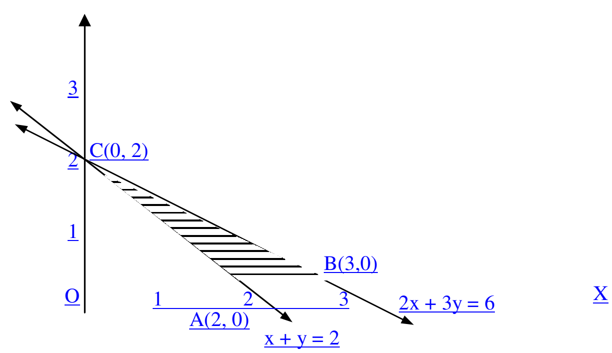
Example 18. Find the maximum and minimum value of the function

$$Z = 7x \ \text{~~=~~} + 7y$$

Subject to the constraints

$$x + y \geq \text{~~1~~}2$$
$$2x + 3y \leq \text{~~1~~}\ 6$$
$$x, y \geq 0$$

Solution. We have : $x \geq 1$ …(1) $y \geq 0$ …(2)



$x + y \geq 2$ ...(3) $2x + 3y \leq 6$ …(4)

and $P = 7x + 7y$

We fined out the solution set (convex polygon), (1) − (4) are all true. For this, we draw the graph of the lines

$x = 0,\ y = 0,\ x + y = 2$ and $2x + 3y = 6$.

The shaded portion is the feasible region of the constraints.

Now at A(2, 0), $P = 7(2) + 0 = 14$

at B(3, 0), $P = 7(3) + 0 = 21$

at C(0, 2), $P = 7(0) + 7(2) = 14$.

(i) Since max. value is 21, so optimal solution for maximisation is $x = 3,\ y = 0$

(ii) Since min. values is 14, so optimal solution for minimization is (a) $x = 2,\ y = 0$ and (ii) $x = 0,\ y = 2$. So we have multiple optimal solutions is case of minimization of this problem.

Exercise 9.1

Draw the diagrams of the solution sets of the following (1 − 3) linear constraints :

1. $3x + 4y \geq 12,\ 4x + 7y \leq 28,\ x \geq 0,\ y \geq 1$.

2. $x + y \leq 5,\ 4x + y \geq 4,\ x + 5y \geq 5,\ x \leq 4,\ y \leq 3$.

3. $x + y \geq 1,\ y \leq 5,\ x \leq 6,\ 7x + 9y \leq 63,\ x, y \geq 0$.

4. Verify that the solution set of the following constraints is empty :

$3x + 4y \geq 12,\ x + 2y \leq 3,\ x \geq 0,\ y \geq 1$.

5. Verify that the solution set of the following constraints :

$x - 2y \geq 0,\ 2x - y \leq -2$

is not empty and is unbounded.

6. Draw the diagram of the solution set of the linear constraints

(i) $3x + 2y \leq 18$ (ii) $2x + y \geq 4$

$x + 2y \leq 10$ $3x + 5y \geq 15$

$x \geq 3,\ y \geq 0$ $x \geq 0,\ y \geq 0$

7. Exhibit graphically the solution set of the linear constraints

$x + y \geq 1$

$y \leq 5$

$x \leq 6$

$7x + 9y \leq 63$

$x, y \geq 0$

8. Verify that the solution set of the following linear constraints is empty :

(i) $x - 2y \geq 0$ (ii) $3x + 4y \geq 12$

$2x - y \leq -2$ $x + 2y \leq 3$

$x \geq 0,\ y \geq 0$ $y \geq 1,\ x \geq 0$

9. Verify that the solution set of the following linear constraints is unbounded :

$3x + 4y \geq 12$

$y \geq 1$

$x \geq 0$

10. Find the linear constraints for which the shaded area in the figures below is the solution set :

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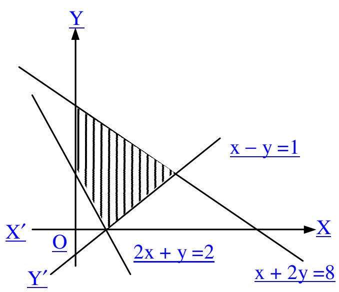

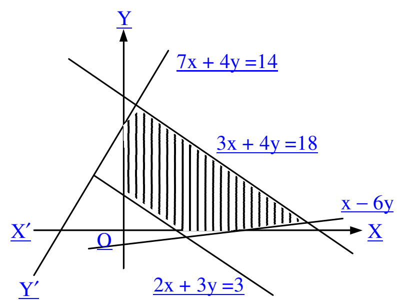

11. Find the maximum and minimum value of $2x + y$ subject to the constraints

$$2x + 3y \le 30$$
$$x - 2y \le 8$$
$$x \ge 0,\ y \ge 0$$

12. Solve by graphical method :

(i) Minimize $Z = 3x_1 + 2x_2$

Subject to the constraints

$$-2x_1 + x_2 \le 1$$
$$x_1 \le 2$$
$$x_1 + x_2 \le 3$$
$$x_1, x_2 \ge 0$$

(ii) Find the maximum value of

$z = 5x_1 + 3x_2$

subject to the constraints

$$3x_1 + 5x_2 \le 15$$
$$5x_1 + 2x_2 \le 10$$
$$x_1, x_2 \ge 0$$

(iii) Find the maximum value of

$z = 2x_1 + 3x_2$

subject to the constraints

$$x_1 + x_2 \le 1$$
$$3x_1 + x_1 \le 4$$
$$x_1, x_2 \ge 0$$

(iv) Find the minimum value of $3x + 5y$ subject to the constraints

$$-2x + y \le 4$$
$$x + y \ge 3$$
$$x - 2y \le 2$$
$$x, y \ge 0$$

13. A dealer wishes to purchase a number of fans and sewing machines. He has only Rs. 5760 to invest and has space of at most 20 items. A fan costs him Rs. 360 and a sewing machine Rs. 240. His expectation is that he can sell a fan at a profit of Rs. 22 and a sewing machine at a profit of Rs. 18.

Assuming he can sell all the items that he can buy, how should he invest his money in order to maximise his profit ? Also find the maximum profit.

14. A dietician wishes to mix two types of food in such a way that the vitamin contents of the mixture contains at least 8 units of vitamin A and 10 units of vitamin C. Food I contains 2 units per kg. of vitamin A and I unit per kg. of vitamin C, while the food II contains 1 unit per kg. of Vitamin A and 2 units per kg. of vitamin C. It costs Rs. 5 per kg. to purchase food I and Rs. 7 per kg. to purchase food II Find the minimum cost of such mixture and the quantity of the each of the foods.

15. A manufacturer produces nuts and bolts for industrial machinery. It takes 1 hour of work on machine A and 3 hours on machine ~~A and 3 on machine B~~ to produce a package of nuts while it takes 3~~2~~ hours on machine A and 1 hour on machine B to produce a package of bolts. He earns a profit of Rs. 2.50 per package on nuts and Rs. 1.00 per package on bolts. How many packages of each should he produce each day so as maximise his profit, if he operates his machines for at the most 12 hours a day ?

16. A sports factory prepares cricket bats and hockey sticks. A cricket ~~nat~~ bat takes 2 hours of machine time and 3 hours of craftsman's time. A hockey stick take 3 hours of machine time and 2 hours of craftsman's time. The factory has 90 hours of machine time and 85 hours of craftsman's time. What number of bats and sticks must be made if the factory is to work at full capacity ? If the profit on a bat is Rs. 3 and on a stick it is Rs. 4, find the maximum profit.

17. A trader deals in sewing machines and transistors. It has capacity to store at the most 30 pieces and he can invest Rs. 4500. A machine costs him Rs. 250 each and transistor costs him Rs. 100 each. The profit on machine Rs. 40 and on a transistor it is Rs. 25. Find number of sewing machines and transistors to take max. profit.

18. Every gram of wheat provides 0.1 g of proteins and 0.25 g of carbohydrates. The corresponding values for rice are 0.05 g and 0.5 g respectively. Wheat costs Rs. 2 per kg and rice Rs. 8. The minimum daily requirements of protein and carbohydrates for an average child are 50 g and 200 g respectively. In what quantities should wheat and rice be mixed in the daily diet to provide the minimum daily requirements of protein and carbohydrates at minimum cost.

19. A toy company manufactures two types of dolls, a basic version-doll A and a deluxe version-doll B. Each doll of type B takes twice as long to produce as one of type A, and the company would have time to make a maximum of 2,000 per day if it produced only the basic version. The supply of plastic is sufficient t~~-~~o produce 1,500 dolls per day (both A and B combined). The deluxe version requires a fancy dress of which there are only 600 per day available. If the company makes a profit of Rs. 3.00 and Rs. 5.00 per doll respectively on doll A and B ; how many of each should be produced pe~~-~~r day in order to maximize profit ?

20. Smita goes to the market to purchase battons. She needs at least 20 large battons and at least 30 small battons. ~~tHe~~ The shopkeeper sells battons in two forms (i) boxes and (ii) cards. A box contains t~~h~~en large and five small battons and a card contains two large and five small battons. Find the most economical way in which she should purchase the battons, if a box costs 25 paise and a card 10 paise only.

21. Vikram has two machines with which he can manufacture either bottles or tumblers. The first of the two machines has to be used for one minute and the second for two minutes in order to manufacture a bottle and the two machines have to be used for one minute each to manufacture a tumbler. During an hour the two machines can be operated for at the most 50 and 54 minutes respectively. Assuming that he can sell as many bottles and tumblers as the can produce, find how many of bottles and tumblers he should manufacture so that his profit per hour is maximum being given that ~~t~~he gets a profit of ten paise per bottle and six paise pe rtumbler.

22. A company produces two types of presentation goods A and B that require gold and silver. Each unit of type A requires 3 gms. of silver and 1 gm of gold while that of B requires 1 gm. of silver and 2 gm. of gold. The company can produce 9 gms. of silver and 8 ~~ems~~gms. of gold. If each unit of type A brings a profit of Rs. 40 and that of type B Rs. 50 determine t~~e~~he number of units of each type that the company should produce to maximize the profit. What is the maximum profit ?

सिम्पलैक्स विधि
SIMPLEX METHOD

रेखीय नियोजन की समस्याओं के हल के बिन्दु रेखीय विधि की मुख्य सीमा यह है कि इस विधि के द्वारा हम केवल दो चरों वाले प्रश्नों का हल निकाल सकते हैं। वास्तविक जीवन में हमें दो से ज्यादा चरों वाले प्रश्नों का हल खोजना पड़~~ता~~ सकता है। ऐसी स्थिति में हम ~~ऐसी~~इस विधि का प्रयोग नही कर सकते। सिम्पलैक्स विधि एक ऐसी विधि है जिससे हम इन प्रश्नों का हल निकाल सकते हैं।

इस विधि में हम क्रमबद्ध तरीके से सर्वोतम हल निकालते हैं। जैसा कि हम बिन्दु रेखीय विधि में देख चुके हैं – संभाव्य क्षेत्र के कोनों से हमें विभिन्न संभाव्य समाधान मिलते हैं। सिम्पलैक्स विधि इन संभाव्य समाधानो में से सर्वोत्तम समाधान निकालने में सहायता करती है।

Main limitation of graphical method in solving linear programming problem is that using this method, we can solve problems involving two variables only. In real life, we may have to solve the problems involving more than two variables. In such situation, we can't use this method. Simplex method is a technique with the help of which we can find solutions to such problems.

In this technique, we find the optimum solution systematically. As we have seen in graphical method, the vertices of the feasible region gives us feasible solutions. Simplex method helps us in finding the best solution from these feasible solutions.

सिम्पलैक्स विधि का प्रयोग करने की शर्ते (Conditions for application of Simplex Method)

इस विधि का प्रयोग करने के लिए दो मुख्य शर्ते हैं जो कि पूरी होनी चाहिए :–

1. हर व्यवरोध असमीकरणों को दायें पक्ष धनात्मक होना चाहिए। अगर किसी में ये मुल्य ऋणात्मक हैं तो दोनों पक्षों को (–1) से गुणा करने पर इसे धनात्मक बनाया जाता है। उदाहरण के तौर पर यदि दिया हुआ व्यवरोध है $2x_1 - 5x_2 \geq -10$ तो इसे परिवर्तित कर हम लिख सकते हैं $-2x_1 - 5x_2$ ≤~~≥~~ 10 ध्यान रहे कि (–1) से गुणा करने के बाद असमीकरण का चिन्ह बदल जाता है।
2. निर्णय वाले चर जैसे x_1, x_2 आदि मुल्य भी णिात्मक नही होंने चाहिए । यदि किसी निर्णय वाले चर का चिन्ह सीमा रहित लिखा हुआ है तो इसे दो घनात्मक चिन्हों वाले चरो के अंतर से दिखाया जाता है उदाहरण के तौर पर यदि x_3 का चिन्ह सीमा रहित है तो इसका अर्थ यह है कि x_3 का चिन्ह णिात्मक भी हो सकता है तथा घनात्मक भी, इसलिए $x_3 = x_4 - x_5$ के रुप में दिखा सकते हैं ।

To apply this techniques the following two conditions must be satisfied

1. R.H.S of every constraint is equality must be non-negative. If it is negative in any in equality, it is made positive by multiplying both sides of inequality by (–1). For example if we are given the constraint 2x, – ~~1x_2~~ $5x_2 \geq -10$. Then we can rewrite it as $-2x + 5x \leq 10$) Note that when we multiply both sides by (–1), the sign of inequality changes.
2. Decision variables like x_1, x_2 would also be non-negative. If it is given that any decision variable is unrestricted in sign, it is expressed as difference of two non-negative variables, For example if it is give that x_3 is unrestricted in sight, we can write it as $x_3 = x_4 - x_5$.

Steps in valued in simplex method

1. सर्व–प्रथम उदद्देश्य फलन लिखिए । यह अधिकतम या निम्नतम में से एक होता है ।
First write the objective function. It is either maximisation or minimisation.
Ex. Max $Z = c_1x_1 + c_2 x_2 + c_3 x_3 + \ldots +$ ~~c_4~~ $c_n x_n$
or Min $Z = c_1x_1 + c_2x_2 + c_3x_3 + \ldots +$ ~~$c_4 x_n$~~ c_nx_n
2. व्यवरोध असमीकरणों को सही चिन्हों (≥ or ≤) के साथ लिखो
Write the constraint inequalities with proper signs.
Ex. $a_{11} x_1 + a_{12} x_2 + a_{13} x_3 + \ldots + a_{1n} x_n \leq b_1$ OR
$a_{11} x_1 + a_{12} x_2 + a_{13} x_3 + \ldots + a_{1n} x_n \geq b_1$

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and so an.

3. इन असमीकरणो को ~~सही~~ शिथिल चरों तथा कृत्रिम चरो की मदद से समीकरणों में परिर्वतित करो । शिथिल चर किसी भी संसाधन की बची हुई मात्रा दिखाते है । कृत्रिम चर का कोई वास्तविक मूल्य नही होता इन्हें हम र्सिफ सर्वोतम हल निकालने में सहायता करने के लिए प्रयोग करते है । यदि महत्वपूर्ण टिप्पणी यदि असमीकरण कम या बराबर ;≤द्ध वाले है तो सिर्फ शिथिल चर ही प्रयोग किये जाएंगे लेकिन यदि असमीकरण ज्यादा या बराबर ;≥द्ध वाले है तो शिथिल तथा कृत्रिम दोनों चरों का प्रयोग किया जाएगा ।

There in equalities are converted into equation by introducing slack and artificial variable. Slack variables now the left over quantity of the resources. Artificial variables have no real value. They are introduced first to solve the problem.

Important Note. If the in equations are of less than or equal to (≤) ~~on~~ sign only slack variables are introduced. But if the in equations are of greater than or equal (≥ sign), both slack and artificial variables are introduced.

Ex. Two given constraints are

$2x_1 + 3x_2 \le 60$ and $4x_1 + x_2 \ge 40$

we will rewrite them as

$2x_1 + 3x_2 + S_1 = 60$ and $4x_1 + x_2 - S_2 + A_1 = 40$

In less then constraints, slack variables (S) will have positive sign and in more than constraints, they will have negative sign

4. इन समीकरणों को एक आव्यूह की तरह दिखाया जाएगा जिसकी रुपरेखा निम्नलिखित प्रकार से है

These equations are then presented in the form of a matrix where format is shown below with the help of an example.

Example Maximise $Z = 22\,x_1 + 18\,x_2$

Subject to $x_1 + x_2 \le 20$

$360\,x_1 + 240\,x_2$ ~~=~~ ≤ 5760

change the in equations into equations

$x_1 + x_2 + S_1 = 20$

$360\,x_1 + 240x_2 + S_2 = 5760$

Table 1.

Basic ~~c_i~~ c_j	x_1	x_2	S_1	S_2	b
S_1 0	1	1	1	0	20 } Constraint values
S_2 0	360	240	0	1	5760
~~c_i~~ C_j	22	18	0	0	Co-efficient values from constraint equation
Zj	0	0	0	0	
Cj – Zj	22	18	0	0	

~~C_i~~ C_j is the contribution/unit of each variable shown in the objective function. Slack variables have zero contribution. Z_j shows the total contribution of various variables at any given stage. The row showing $(C_j - Z_j)$ is called the index row. This row shows how much profit is foregone by not producing one unit of a product etc.

Remember. An optimal solution is searched when all the values of in–dex ~~×~~ row become zero or negative. Now for finding the optimal solution, we consider two cases –(i) Maximisation case (ii) Minimisation case.

Maximisation Case. Let us reconsider the above example.

(1) ~~lEHkkO~~vkjfHkd lEHkkO; gy fudkyuk (Finding the initial feasible solution) our problem is

Max. $Z = 22\,x_1 + 18\,x_2$

Subject to $x_1 + x_2 \le 20$

$360 x_1 + 240 x_2 \le 5760$

or $x_1 + x_2 + S_1 = 20$ …(1)

$360 x_1 +$ ~~x_{10}~~ $240 x_2 + S_2 = 5760$ …(2)

initially, put $x_1 = 0$, $x_2 = 0$. So from (1) $S_1 = 20$ and from (2) $S_2 = 5760$.

In the initial solution, we assume that we are not producing any quantity of either of the products. So the resources remain fully unutilized. That whey in equation (1) we get $S_1 = 20$ and in (2) we get $S_2 = 5760$.

This solution is shown in the above table (in the term of matrix.)

(2) $(Z_j - C_j)$ पंक्ति के सबसे ज्यादा घनात्मक मूल्य ढूढों। जिस स्तंभ में वह मुल्य होगा, उस स्तंभ का चर अब समाधान में शामिल होगा। व्यवरोधी मुल्यों को इस स्तंभ के मुल्यों से विभाजित करो (b_{ij}/a_{ij}) इस से हमें कई अनुपात मिलेगें। इन में से जिस अनुपात का सबसे कम घनात्मक मुल्य है उस की पंक्ति ढूढों। उस पंक्ति में जो भी चर है वह समाधान से बाहर हो जाएगा व इसकी जगह पर उपर वाला चर आ जाएगा।

Find the highest positive value in the row $(Zj - C_j)$. The variable of the column to which this value corresponds will ent~~re~~er the solution. Divide the constraint values (b's) by the element of this column to find the ratio (b_{ij}/a_{ij}). Choose the ratio which has minimum positive value and find the row of this ratio. The basic variable of this row will leave the solution and th~~is~~e above variable will replace this variable consider our example

Basic	c_i	x_1	x_2	S_1	S_2	b	b/a
S_1	0	1	1	1	0	20	$20 \div 1 = 20$
S_2	0	360	240	0	1	5760	$5760 \div 360 = 16$
							$\rightarrow$ Outgoing variable
	c_i	22	18	0	0		
	Zj	0	0	0	0		
	Cj – Zj	22	18	0	0		

Incoming variable

So now x_1 will replace S_2

इस स्तंभ को हम मुख्य स्तंभ (Key column) कहते है तथा पंक्ति को मुख्य पंक्ति (Key column) कहते है। जो अवयव मुख्य स्तंभ मुख्य पंक्ति दोनों में है उसे मुख्य अवयव कहते हैं।

The element which belongs to both key column and key row is called key element. Now divide all elements of key row by key column like

Key row = 360 240 0 1 5760

Divide all the values by 360, we get

$$\frac{360}{360} \quad \frac{240}{360} \quad \frac{0}{360} \quad \frac{1}{360} \quad \frac{5670}{360}$$

or 1 2/3 0 1/360 16

इसके बाद आव्यूह क्रिया की मदद से मुख्य स्तंभ के बाकी सभी अवयवों को शून्य बना दिया जाता है।

After using matrix operations, all other elements of key row are made equal to zero like

1st row	1	1	1	0	20
2nd row	1	2/3	0	1/360	16

$R_1 \rightarrow R_1 - R_2$ to get

1st row	0(1 – 1)	1/3(1 – 2/3)	$1^{(1-0)\,0-1/136}$	–1/360	$4^{(20-16)}$
2nd row	1	2/3	0	1/360	16

इन सब पक्तियों के बाद नया आव्यूह निम्नलिखित होगा।

After all these changes, the new matrices will be as follow :-

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Table 2

		x_1	x_2	S_1	S_2	b_1	Ratio
0	S_1	0	1/3	1	−1/360	4	12 Key row
22	x_1	1	2/3	0	1/360	16	24 ~~Key row~~
	c_i	22	18	0	0	$0 \times 4 + 22 \times 16 = 352$ Total profit at this stage	
	Zj	~~0~~	22	~~0~~	44/3		
	~~Cj – Zj~~	0 - —	~~0~~22/360				
		~~22~~	~~18~~	~~0~~	~~0~~		
	$C_j - Z_j$	0	10/3	0	−22/360		

Key column

क्योंकि अभी भी $(C_j - Z_j)$ पंक्ति में एक घनात्मक मूल्य बचा हुआ है, सो अभी सर्वोत्तम हल नहीं मिला है। अब हम पग 2 और 3 को फिर दोहरायेगें और ऐसा तब तक करते रहेगें जब तक कि $(C_j - Z_j)$ पंक्ति के सभी मूल्य शून्य या ऋणात्मक नहीं हो जाते।

Because still are positive value remains in the $(C_j - Z_j)$ row, so we have get to obtain optimal solution. Now we will repeat steps 2 and 3 and repeat them till all the values in the $(C_j - Z_j)$ row become be zero or negative.

The new table will be as follows :

Table 3

Basic c_I		x_1	x_2	S_1	S_2	
18	x_2	0	1	3	−1/120	12
22	x_1	1	0	−2	1/120	8
	c_I	22	18	0	0	$18 \times 12 + 22 \times 8 = 392$
	Zj	22	18	10	4/120	
Cj – Zj		0	0	−10	−4/120	

Now there is no positive value in the index row, so we have obtained optimal solution. the optimal solution is $x_1 = 8$, $x_2 = 12$ and maximum profit Z = Rs. 392 (obtained from the resources column)

Example. A firm produces three products A, B and C, each of which passes through three departments : Fabrication, Finishing and Packaging. Each unit of product A requires 3, 4 and 2; a unit of product B requires 5, 4 and 4, while each unit of product C requires 2, 4 and 5 hours respectively in the three departments. Every day, 60 hours are available in the fabrication department, 72 hours in the finishing department and 100 hours in the packaging department.

The unit contribution of product A is Rs 5, of product B is Rs. 10, and of product C is Rs. 8.

Required :

(a) Formulate the problem as an LPP and determine the number of units of each of the products, that should be made each day to maximise the total contribution. Also determine if any capacity would remain unutilised.

Solution. Let x_1, x_2 and x_3 represent the number of units of products A, B and C respectively. The given problem can be expressed as a LPP as follows :

Maximise $Z = 5x_1 + 10x_2 + 8x_3$ — Contribution

Subject to

$3x_1 + 5x_2 + 2x_3 \le 60$ Fabrication hours

$4x_1 + 4x_2 + 4x_3 \le 72$ Finishing hours

$2x_1 + 4x_2 + 5x_3 \le 100$ Packaging hours

$x_1, x_2, x_3 \ge 0$

Introducing slack variables, the augmented problem can be written as

Maximise $Z = 5x_1 + 10x_2 + 8x_3 + 0S_1 + 0S_2 + 0S_3$

Subject to

$3x_1 + 5x_2 + 2x_3 \leq + S_1 = 60$
$4x_1 + 4x_2 + 4x_3 \leq + S_2 = 72$
$2x_1 + 4x_2 + 5x_3 \leq + S_3 = 100$
$x_1, x_2, x_3, S_1, S_2, S_3 \geq 0$

The solution to the problem using simplex algorithm is contained in Tables 1 to 3

Simplex Table 1 : Initial Solution

Basic		x_1	x_2	x_3	S_1	S_2	S_3	b_i	b_i/a_{ij}	
S_1	0	3	2 5*	5*2		1	0	60	12	Outgoing
S_2	0	0						72	18	variable
S_3	0	4	4	4	0	1	0	100	25	(key row)
		2	4	5	0	0	1			
c_i		5	10	8	0	0	0			
Zj		0	0	0	0	0	0			
Cj – Zj		5	10	8	0	0	0			

(Key column)

*5 is the key element.

Simplex Table 2 : Non-optimal Solution

Basic		x_1	x_2	x_3	S_1	S_2	S_3	b_i	b_i/a_{ij}	
x_2	10	3/5	1	2/5	1/5	0	0	12	30	Outgoing
S_2	0	8/5	0	12/5	–4/5	1	0	24	10	variable
S_3	0	–2/5	0	17/5	–4/5	0	1	52	260/17	(key row)
c_i		5	10	8	0	0	0			
Zj		6	10	4	2	0	0			
Cj – Zj		–1	0	8 4		–2	8			
		0	0							

Incoming variable
(Key column)

Simplex Table 3 : Optimal Solution

Basics		x_1	x_2	x_3	S_1	S_2	S_3	b_i
x_2	10	1/3	1	0	1/3	–1/6	0	8
x_3	8	2/3	0	1	–1/3	5/12	0	10
S_3	0	–8/3	0	0	1/3	–17/12	1	18
C_i		5	10	8	0	0	0	
Solution		0	8	10	0	0	18	
Δ_j		–11/3	0	0	–2/3	–5/3	0	

According to the Simplex Tableau 3, the optimal solution is : $x_1 = 0$, $x_2 = 8$, $x_3 = 10$. Thus, it calls for producing 8 and 10 units of products B and C respectively, each day. This mix would yield a contribution of $5 \times 0 + 10 \times 8 + 8 \times 10$ = Rs. 160. S_3 being equal to 18, an equal number of hours shall remain unutilized in the packaging department.

Example 20. Solve the following L.P.P.

Maximise $Z = 40000\,x_1 = 55000x_2$

Subject to

$1000\,x_1 + 1500x_2 \leq 20000$
$x_1 \leq 12$
$x_2 \geq 5$
$x_1, x_2 \geq 0$

Solution. By changing the inequations into equations by adding surplus and artificial variables, the form of the problem is changed as :

Maximise $Z = 40000x_1 =$ ~~$15000x_2$~~ $55000x_2 + 0.S_1 + 0S_2 + 0S_3 - M.A.$

Subject to

$$1000x_1 + 1500x_2 + S_1 = 2000$$
$$x_1 + S_2 = 12$$
$$x_2 - S_3 + A_1 = 5$$
$$x_1, x_2, S_1, S_2, S_3, A_1 \geq 0$$

The solution to this problem is shown in tables ~~3~~ 1 to 3

Simplex ~~table~~ Table 1 : Initial solution

Basic	x_1	x_2	S_1	S_2	S_3	A_1	b	b_{ij}/a_{ij}
S_1 0	1000	1500	1	0	0	0	20000	40/3
S_2 0	1	0	0	1	0	0	12	-
A_1 -M	0	1*	0	0	–1	1	5	5 key row)
C_j	40000	55000	0	0	0	–M		
Z_j	0	–M	0	0	M	–M	–5M	
$C_j - Z_j$	40000	55000+M	0	0	–M	0		

Incoming variable
(Key column)

*Key element

Simplex Ttable 2 : Non-~~Initial~~ optimal solution

	x_1	x_2	S_1	S_2	S_3	A_1	b	b_{ij}/a_{ij}
S_1 0	1000	0	1	0	1500*	–1500	12500	12500/1500 → key row
S_2 0	1	0	0	1	0	0	12	–
x_2 55000	0	1	0	0	–1	1	5	–
C_j	40000	55000	0	0	0	–M	275000	
Z_j	0	55000	0	0	–55000	55000	~~–5M~~	
$C_j - Z_j$	40000	0	0	0	55000	–M–55 000		

(Key column)

*Key element

Simplex ~~table~~ Table 3 : Non-Optimal Solution

	X_1	x_2	S_1	S_2	S_3	A_1	b	b_{ij}/a_{ij}
S_3 0	2/3	0	1/1500	0	1	–1	25/3	25/2
S_2 0	1	0	0	1	0	0	12	12 key row
x_2 55000	2/3	1	1/1500	0	0	0	40/3	20
C_j	40000	55000	0	0	0	–M	–	
Z_j	11000/3	55000	110/3	0	0	0	2200000/3	
$C_j - Z_j$	10000/3	0	110/3	0	0	–M	–	

(Key column)

Simplex ~~table~~ Table 4 : Optimal Solution

	x_1	x_2	S_1	S_2	S_3	A_1	b
S_3 0	0	0	1/1500	–2/3	1	–1	1/3

x_1 40000	1	0	0	1	0	0	12
x_2 55000	0	1	1/1500	–2/3	0	0	16/3
C_j	40000	55000	0	0	0	–M	–
Z_j	40,000	55000	110/3	10000/3	0	0	2320000/3
$C_j - Z_j$	0	0	–110/3	–10000/3	0	–M	–

So optimal solution is $x_1 = 12$, $x_2 = 16/3$ and $Z = 2320000/3$

Minimisation Case

रेखीय नियोजन की समस्या में उद्देश्य फलन का निम्नतम मूल्य निकालने की विधि के पग अधिकतम मूल्य निकालने की विधि जैसे ही हैं। कुछ आधारभूत अन्तरों का ध्यान रखना जरूरी है, जो निम्नलिखित हैः

1. आरंभिक हल की सारिणी में, निर्देश iafDr ($C_j - Z_j$ iafDr) में हम अधितम ऋणात्मक मूल्य लेगें ना कि अधिकतम घनात्मक मूल्य। जिस स्तंभ में ये मूल्य होगा उसे मुख्य स्तंभ कहेगें।
2. निम्नतम मूल्य निकालने वाले प्रश्नों में यदि हम कृत्रिम चरों का प्रयोग करते तो उनका वजन ड होगा जबकि अधिकतम मुल्य वाले प्रश्नों में उनका वजन +M होता है।
3. सर्वोत्तम हल निकालते वक्त, ये कृत्रिम चर समाधान में सं निकल जाते है। यदि अन्तिम सारिणी में भी ये pj ekStwn रहते हैं तो दी हुई समस्या का कोई संभाव्य हल नहीं है।
4. सर्वोत्तम हल हम तब पाते हैं जब निर्देश पंक्ति के सभी मान शुन्य या घनात्मक हो जाए।

Steps involved in finding the minimum value of objective functions are same as in case of maximisation. Same fundamental differences should be taken case of which are as follows :

1. In the table showing initial solution, we will take highest negative value not the highest positive value. The column which has this value is the key column.
2. In problems of minimisation, if we use artificial variables they will have a weight of +M whereas in problems of maximization, they have negative weight –M.
3. While going for optimal solution, these artificial variables leave the solution. If they are in the solution in the final table, it means that the given problem has no feasible solution.
4. When all the values in the index row are zero a positive, optimal solution is reached.

Example 21. To improve the productivity of land, a framer is advised to use at least 4800 kg. of phosphate fertilizer and not less than 7200 kg. of nitrogen fertilizer. There are two sources to object these fertilizers mixture A and B. Both of these are available in bags of 100 kg. each and their cost per bag are Rs. 40 and Rs. 24 respectively. Mixture A contains 20 kg. phosphate and 80 kg. nitrogen while their respective quantities in mixture B are 80 kg. and 50 kg.

Formulate this as an LPP and determine how many bags of each type of mixture should the farmer buy in order to obtain the required fertilizer at minimum cost.

Solution. Let x_1 be number of bags of mixture A and x_2 be the number of bags of mixture B. So now the problem can be written as

Minimise $Z = 40x_1 + 24x_2$ Total cost

Subject to

$20x_1 + 50x_2 \geq 4800$ Phosphate Requirement

$80x_1 + 50x_2 \geq 7200$ Nitrogen Requirement

$x_1, x_2 \geq 0$

After introducing the slack + artificial variables, the above problem can be rewritten as :

Minimise $Z = 40x_1 + 24x_2 + 0S_1 = 0S_2 + MA_1 + MA_2$

Subject to

$20x_1 + 50x_2 - S_1 + A_1 = 4800$

$80x_1 + 50x_2 - S_2 + A_2 = 7200$

Simplex Table 1 : Initial Solution

Basis		x_1	x_2	x_3	S_1	S_2	S_3	b_i	b_i/a_{ij}
A_1	M	20	50*	−1	0	1	0	4800	96 key row
A_2	M	80	50	0	−1	0	1	7200	144
C_j		40	24	0	0	M	M		
Z_j		100 M	100 M	~~0~~–M	~~0~~–M	M	M	12000M	
$C_j - Z_j$		40–100M	24–100M	M	M	0	0		

Key column

* Key element

Simplex Table 2 : Non-optimal Solution

Basis		x_1	x_2	~~x_3~~S_1	~~S_1~~S_2	~~S_2~~A_1	~~S_3~~A_2	b_i	b_i/a_{ij}
x_2	24	2/5	1	–1/50	0	1/50	0	96	240
A_2	M	60*	0	1	–1	–1	1	2400	40 key row
C_j		40	24	0	0	M	M		
Z_j		$\frac{48}{5}$ +60 M	24	$M - \frac{24}{50}$	0–M	$-M+\frac{24}{50}$	M	2304 + 2400 M	
$C_j - Z_j$		~~40~~ $\frac{152}{5}$ – ~~10~~ ~~0M~~60M	~~24–100M~~ 0	$\frac{12}{25}$ –M	M	$2M - \frac{12}{25}$	0		

Key column

Simplex Table 3 : Non-optimal Solution

Basis		x_1	x_2	S_1	S_2	A_1	A_2	b_i	b_i/a_{ij}
x_2	24	0	1	–2/75	1/150	2/75	–1/150	80	–3000
x_1	40	1	0	1/60*	–1/60	–1/60	1/60	40	2400 key row
C_j		40	24	0	0	M	M		
Z_j		40	24	2/75~~0~~	–38/75	–2/75	–38/75	3520	
$C_j - Z_j$		0	0	–2/75 key column	38/75	$M + \frac{2}{75}$	$M+ \frac{38}{75}$		

Simplex Table 4 : Optimal Solution

Basis		x_1	x_2	S_1	S_2	A_1	A_2	b_i
x_2	24	8/5	1	0	–1/50	0	1/50	144
S_1	0	60	0	1	–1	–1	1	2400
C_j		40	24	0	0	M	M	
Z_j		192/5	24	0	–12/25	M	12/25	3456
$C_j - Z_j$		8/5	0	0	12/25	M	M–12/25	

Since all the values of the index row are zero or positive, so we have got optimal solution. The optimal solution $x_2 = 144$, $x_1 = 0$ and Z = Rs. 3456

Example 22. A finished product must weigh exactly 150 grams. The two raw materials used in manufacturing the product are A, with a cost of Rs. 2 per unit and B with a cost of Rs. 8 per unit. At least 14 units of B not more than 20 units of A must be used. Each unit of A and B weighs 5 and 10 grams respectively.

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How much of each type of raw material should be used for each unit of the final product in order to minimise the cost ? Use Simplex method. (M.~~com~~Com, Delhi, 1985)

Solution. The given problem can be expressed as LPP as

Minimise $Z = 2x_1 + 8x_2$

Subject to $5x_1 + 10x_2 = 150$

$x_1 \leq 20$

$x_2 \geq 14$

$x_1, x_2 \geq 0$

Substituting $x_2 = 14 + x_3$ and introducing necessary slack and artificial variables, we have,

Minimise $Z = 2x_1$ ~~=~~ + $8x_3 + 112 + MA_1 + 0x_4$

Subject to

$5x_1 + 10x_3 + A_1 = 10$

$x_1 + x_4 = 20$

$x_1, x_3, x_4, A_1 \geq 0$

The solution is contained in Tables 3.51 through 3.53.

~~TABLE~~ Table 5 Simplex Table 1 : Non-optimal Solution

Basis	x_1	x_3	A_1	x_4	b_i	b_i/a_{ij}
A_1 M	5	10*	1	0	10	1 key row
x_4 0	1	0	0	1	20	∞
C_j	2	8	M	0		
Z_j	+5M	+10M	M	0	10 m	
$C_j - Z_j$	2 – 5M	8 – 10M	0	0		

key column

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~~TABLE~~ Table 6 Simplex Table 2 : Non-optimal Solution

Basis	x_1	x_3	A_1	x_4	b_i	b_i/a_{ij}
x_3 8	1/2*	1	1/10	0	1	2 key row
x_4 0	1	0	0	1	20	20
C_j	2	8	M	0	-	
Z_j	4	8	8/10	0	8	
$C_j - Z_j$	–2	0	M – (8/10)	~~8~~0	-	

key column

~~TABLE~~ Table 7 Simplex Table ~~2~~ 3 : Optimal Solution

Basis	x_1	x_2	A_1	x_4	b_i
x_1 2	1	2	1/5	0	2
x_4 0	0	–2	–1/5	1	18
C_j	2	8	M	0	-
Z_j	2	4	2/5	0	4
$C_j - Z_j$	0	4	M–(2/5)	0	-

Thus, the optimal solution is : $x_1 = 2$ units, $x_2 = 14 + 0 = 14$ units, total cost $= 2 \times 2 + 8 \times 14$ = Rs. 116.

Example 23. A company produces three products, P_1, P_2 and P_3 from two raw materials A and B, and labour L. One, unit of product P_1 requires one unit of A, 3 units of B and 2 units of L. One unit of product P_2 requires 2 units of A and B each, and 3 units of L, while one units of P_3 needs 2 units of A, 6 units of B and 4 units of L. The company has a daily availability of 8 units of A, 12 units of B and 12 units of L. It is further known that the unit contribution margin for the products is Rs. 3, 2 and 5 respectively for P_1, P_2 and P_3.

Formulate this problem as a linear programming problem, and then solve it to determine the optimum product mix. Is the solution obtained by you unique ? Identify an alternate optimum solution, if any.

If x_1, x_2 and x_3 be the output of the products P_1, P_2 and P_3, respectively, we may express the linear programming formulation as follows :

Maximise $Z = 3x_1 + 2x_2 + 5x_3$ Contribution

Subject to

$x_1 + 2x_2 + 2x_3 \le 8$ Material A

$3x_1 + 2x_2 + 6x_3 \le 12$ Material B

$2x_1 + 3x_2 + 4x_3 \le 12$ Labour

$x_1, x_2, x_3 \ge 0$

Introducing slack variables S_1, S_2 and S_3, we may write the problem as follows :

Maximise $Z = 3x_1 + 2x_2 + 5x_3 + 0S_1 + 0S_2 + 0S_3$

Subject to

$x_1 + 2x_2 + 2x_3 + S_1 = 8$

$3x_1 + 2x_2 + 6x_3 + S_2 = 12$

$2x_1 + 3x_2 + 4x_3 + S_3 = 12$

$x_1, x_2, x_3, S_1, S_2, S_3 \ge 0$

Simplex Table 8: Non-optimal Solution

Basis		x_1	x_2	x_3	S_1	S_2	S_3	b_i	b_i/a_{ij}
S_1	0	1	2	2	1	0	0	8	4
S_2	0	3	2	6*	0	1	0	12	2 ←
S_3	0	2	3	4	0	0	1	12	3
c_j		3	2	5	0	0	0		
Z_j		0	0	0	0	0	0		
Δ_j		3	2	5	0	0	0		
				↑					

Simplex Table 9 : Non-optimal Solution

Basis		x_1	x_2	x_3	S_1	S_2	S_3	b_i	b_i/a_{ij}
S_1	0	0	4/3	0	1	−1/3	0	4	–
x_3	5	1/2*	1/3	1	0	1/6	0	2	4 ←
S_3	0	0	5/3	0	0	−2/3	1	4	–
c_j		3	2	5	0	0	0		
solution		5/2	5/3	5	4	5/6	4	Z = 10	
Δ_j		½	1/3	0	0	−5/6	0		
		↑							

Simplex Table 10 : Optimal Solution

Basis		x_1	x_2	x_3	S_1	S_2	S_3	b_i	b_i/a_{ij}
S_1	0	0	4/3	0	1	−1/3	0	4	3
x_1	3	1	2/3	2	0	1/6	0	4	6
S_3	0	0	5/3*	0	0	−2/3	1	4	12/5 ←
c_j		3	2	5	0	0	0		
solution		3	2	6	4	1	0	Z = 12	
Δ_j		0	0	−1	0	−1	0		

↑

The solution contained in Table 10 is optimal with $x_1 = 4$, $x_2 = x_3 = 0$ and Z = 12. However, it is not unique since x_2, a non-basic variable, has Δ_j equal to zero. The problem, thus, has an alternate optimal solution. To obtain this, we revise the solution in Table 10 with x_2 as the entering variable. It is given in Simplex Table 11.

Simplex Table 11 : Alternate Optimal Solution

Basis		x_1	x_2	x_3	S_1	S_2	S_3	b_i
S_1	0	0	0	0	1	1/5	–4/5	4/5
x_1	3	1	0	2	0	3/5	–2/5	12/5
x_2	2	0	1	0	0	–2/5	3/5	12/5
c_j		3	2	5	0	0	0	
Solution		12/5	12/5	0	4/5	0	0	Z = 12
Δ_j		0	0	–1	0	–1	0	

रेखीय नियोजन में युग्मता
(Duality in Linear Programming)

हर एक रेखीय नियोजन की समस्या के साथ एक और रेखीय नियोजन की समस्या होती है जो इससे सम्बन्धित होती है तथा इसी से प्राप्त की जाती है। प्रथम समस्या को मौलिक व दूसरी समस्या को उसका युग्म कहते हैं।

मौलिक से युग्म प्राप्त करने के नियम :

1) मौलिक समस्या के उद्देश्य फलन में चरों के गुणांक युग्म में व्यवरोध के माप हो जाते हैं तथा मौलिक समस्या में व्यवरोध के माप युग्म में उद्देश्य फलन में चरों के गुणांक बन जाते हैं।
2) यदि मौलिक में उद्देश्य अधिकतम मान निकालने का है तो युग्म में यह निम्नतम में यह उद्देश्य न्यूनतम मान निकालने का हो जाएगा तथा यदि मौलिक में न्यूनतम है तो युग्म में अधिकतम का हो जाएगा।
3) मौलिक में व्यवरोधों के गुणांको वाला प्रथम स्तंभ युग्म में प्रथम पंक्ति बन जाएगा। इसी तरह से दूसरा स्तंभ दूसरी पंक्ति बन जाएगा।
4) व्यवरोधों के असमीकरणों की दिशा भी परिवर्तित हो जाएगी। यदि मौलिक में दिशा ≤ है तो युग्म में ≥≤ हो जाएगी।

इसके अतिरिक्त निम्नलिखित बातें भी ध्यान में रखनी चाहिए:–

i) युग्म में किसी भी चर का मान ऋणात्मक नहीं होना चाहिए।

पपद्ध यदि युग्म अधिकतम मान के लिए है तो व्यवरोध ≤ तरह के होने चाहिए तथा यदि युग्म न्यूनतम मान के लिए है तो व्यवरोध ≥ तरह के होने चाहिए। एक युग्म में कभी भी दोनों तरह के व्यवरोध नहीं हो सकते।

For every linear programming problem there is another linear programming problem which is related to it and which is obtained from it. First problem is called primal and second is called its dual.

Rules for obtaining dual from primal :–

1) Co-efficients of variables in objective function of primal become constraint values in the dual and constraint values in the primal becomes co-efficients of variables in the objective function.
2) If he primal is of maximisation type, dual is of minimisation type and if primal is of primal of minimisation dual is of maximisation.
3) Co-efficient of first column of constraints of primal because co-efficient of first row of dual, second column becomes second row and so on.
4) Direction of constraint in equations is also changed. It is primal they are of ≤ type, in dual they will be ≥ type.

Besides these, the following things should also be kept in mind :

(i) All the variables in the dual must be non-negative.

(ii) If the dual is of minimisation objective, all the constraints ~~most~~ must be of ≤ type and if it is of minimisation, all the constraints must be of ≥ type. In any dual, we can't have mixed constraints.

Mathematically, change from primal to dual can be shown, with the help of an example.

Example 24. For the LPP given in Example 3.1 reproduced below, write the dual.

Maximise $Z = 40x_1 + 35x_2$

Subject to

$$2x_1 + 3x_2 \leq 60$$
$$4x_1 + 3x_2 \leq 96$$
$$x_1, x_2 \geq 0$$

Solution.

In accordance with above, its dual shall be

Minimise $G = 60y_1 + 96y_2$

Subject to

$$2y_1 + 4y_2 \geq 40$$
$$3y_1 + 3y_2 \geq 35$$
$$y_1, y_2 \geq 0$$

PRIMAL DUAL

c x b′ y

Maximise $Z = [40 \;\; 35]\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ Minimise $G = [60 \;\; 96]\begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$

Subject to

a x b a′ y c′

$$\begin{bmatrix} 2 & 3 \\ 4 & 3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \leq \begin{bmatrix} 60 \\ 96 \end{bmatrix} \qquad \begin{bmatrix} 2 & 4 \\ 3 & 3 \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} \geq \begin{bmatrix} 40 \\ 35 \end{bmatrix}$$

Obtaining Dual of LPP with Mixed Restrictions

Sometimes a given LPP has mixed restrictions so that the inequalities given are not all in the right direction. In such a case, we should convert the inequalities in the wrong direction into those in the right direction. Similarly, if an equation is given in respect ~~to~~ of a certain constraint, it should also be converted into inequality. To understand fully, consider the following examples.

Example 25. Write the dual of the following LPP.

Minimise $Z = 10x_1 + 20x_2$

Subject to

$$3x_1 + 2x_2 \geq 18$$
$$x_1 + 3x_2 \geq 8$$
$$2x_1 - x_2 \leq 6$$
$$x_1, x_2 \geq 0$$

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Solution. Here, the first two inequalities are in the right direction (being ≥ type with a minimisation type of objective function) while the third one is not. Multiplying both sides by −1, this can be written as $-2x_1 + x_2 \ge -6$. Now, we can write the primal and dual as follows :

	Primal		Dual
Minimise	$Z = 10x_1 + 20x_2$	Maximise	$G = 18y_1 + 8y_2 - 6y_3$
Subject to		Subject to	
	$3x_1 + 2x_2 \ge 18$		$3y_1 + y_2 - 2y_3 \le 10$
	$x_1 + 3x_2 \ge 8$		$2y_1 + 3y_2 + y_3 \le 20$
	$-2x_1 + x_2 \ge -6$		
	$x_1, x_2 \ge 0$		$y_1, y_2, y_3 \ge 0$

Example 26. Obtain the dual of the LPP given here :

Solution.

Maximise $Z = 8x_1 + 10x_2 + 5x_3$

Subject to

$$x_1 - x_3 \le 4$$
$$2x_1 + 4x_2 \le 12$$
$$x_1 + x_2 + x_3 \ge 2$$
$$3x_1 + 2x_2 - x_3 = 8$$
$$x_1, x_2, x_3 \ge 0$$

We shall first consider the constraints.

Constraints 1 and 2 : Since they are both of the type ≤, we do not need to modify them.

Constraint 3 : This is of type ≥. Therefore, we can convert it into ≤ type by multiplying both sides by −1 to become $-x_1 - x_2 - x_3 \le -2$.

Constraint 4 : It is in the form of an equation. An equation, mathematically, can be represented by a pair of inequalities: one of ≤ type and the other of ≥ type. The given constraint can be expressed as

$$3x_1 + 2x_2 - x_3 \le 8$$
$$3x_1 + 2x_2 - x_3 \ge 8$$

The second of these can again be converted into type ≤ by multiplying by −1 on both sides. Thus it can be written as $-3x_1 - 2x_2 + x_3 \le -8$.

Now we can write the primal and the dual as follows :

Primal		Dual
Maximise $Z = 8x_1 + 10x_2 + 5x_3$	Minimise	$G = 4y_1 + 12y_2 - 2y_3 + 8y_4 - 8y_5$
Subject to	Subject to	
$x_1 - x_3 \le 4$		$y_1 + 2y_2 - y_3 + 3y_4 - 3y_5 \ge 8$
$2x_1 + 4x_2 \le 12$		$4y_2 - y_3 + 2y_4 - 2y_5 \ge 10$
$-x_1 - x_2 - x_3 \le -2$		$-y_1 - y_3 - y_4 + y_5 \ge 5$
$3x_1 + 2x_2 - x_3 \le 8$		$y_1, y_2, y_3, y_4, y_5 \ge 0$
$-3x_1 - 2x_2 + x_3 \le -8$		
$x_1, x_2, x_3 \ge 0$		

One point needs mention here. We know that corresponding to a n-variable, m-constraint primal problem, there would be m-variable, n-constraint dual problem. For this example involving three variables and four constraints, the dual should have four variables and three constraints. But we observe that the dual that we have obtained contains five variables. The seeming inconsistency can be resolved by expressing $(y_4 - y_5) = y_6$, a variable unrestricted in sign. Thus, although, y_4 and y_5 are both non-negative, their difference could be greater than, less than, or equal to zero. The dual can be rewritten as follows :

Minimse $G = 4y_1 + 12y_2 - 2y_3 + 8y_6$

Subject to

$$y_1 + 2y_2 - y_3 + 3y_6 \geq 8$$
$$4y_2 - y_3 + 2y_6 \geq 10$$
$$-y_1 - y_3 - y_6 \geq 5$$
$$y_1, y_2, y_3 \geq 0,\ y_6 \quad \text{unrestricted in sign}$$

Thus, whenever a constraint in the primal involves an equality sign, its corresponding dual variable shall be unrestricted in sign. Similarly, an unrestricted variable in the primal would imply that the corresponding constraint shall bear the = sign.

Example 27. Obtain the dual of the following LPP :

Maximise $Z = 3x_1 + 5x_2 + 7x_3$

Subject to

$$x_1 + x_2 + 3x_3 \leq 10$$
$$4x_1 - x_2 + 2x_3 \geq 15$$
$$x_1, x_2 \geq 0,\ x_3 \text{ unrestricted in sign}$$

Solution. First of all, we should convert the second restriction into the type ≤. This results in $-4x_1 + x_2 - 2x_3 \leq -15$.

Next, we replace the variable x_3 by the difference of two non-negative variables, say, x_4 and x_5. This yields the primal problem corresponding to which dual can be written, as shown against it.

	Primal		Dual
Maximise	$Z = 3x_1 + 5x_2 + 7x_4 - 7x_5$	Minimise	$G = 10y_1 - 15y_2$
Subject to		Subject to	
	$x_1 + x_2 + 3x_4 - 3x_5 \leq 10$		$y_1 - 4y_2 \geq 3$
	$-4x_1 + x_2 - 2x_4 + 2x_5 \leq -15$		$y_1 + y_2 \geq 5$
	$x_1, x_2, x_4, x_5 \geq 0$		$3y_1 - 2y_2 \geq 7$
			$-3y_1 + 2y_2 \geq -7$
			$y_1, y_2 \geq 0$

The fourth constraint of the dual can be expressed as $3y_1 - 2y_2 \leq 7$. Now, combining the third and the fourth constraints, we get $3y_1 - 2y_2 = 7$. The dual can be expressed as follows :

Minimise $G = 10y_1 - 15y_2$

Subject to

$$y_1 - 4y_2 \geq 3$$
$$y_1 + y_2 \geq 5$$
$$3y_1 - 2y_2 = 7$$
$$y_1, y_2 \geq 0$$

The symmetrical relationship between the primal and dual problems, assuming the primal to be a 'maximisation' problem is depicted in the Chart.

Primal	Dual
Maximization	Minimisation
No. of variables	No. of constraints
No. of constraints	No. of variables
≤ type constraint	Non-negative variable
= type constraint	Unrestricted variable
Unrestricted variable	= type constraint
Objective function coefficient for j^{th} variable	RHS constant for the j^{th} constraint
RHS constant for i^{th} constraint	Objective function coefficient for j^{th} variable
Coefficient (a_{ij}) for j^{th} variable in i^{th} constraint	Coefficient (a_{ij}) for i^{th} variable in j^{th} constraint

Comparing the Optimal Solutions of the Primal and Dual

Since the dual of a given primal problem is derived from and related to it, it is natural to expect that the (optimal) solutions to the two problems shall be related to each other in the same way. To understand this, let us consider the following primal and dual problems again and compare their optimal solutions.

	Primal		Dual
Maximise	$Z = 40x_1 + 35x_2$	Minimise	$G = 60y_1 + 96y_2$
Subject to		Subject to	
	$2x_1 + 3x_2 \le 60$		$2y_1 + 4y_2 \ge 40$
	$4x_1 + 3x_2 \le 96$		$3y_1 + 3y_2 \ge 35$
	$x_1, x_2 \ge 0$		$y_1, y_2 \ge 0$

The simplex tableau containing optimal solution to the primal problem is reproduced (from Table 3.4) in Table 4.1.

Table 1 Simplex Tableau : Optimal Solution

Basis		x_1	x_2	S_1	S_2	b_i
x_2	35	0	1	2/3	−1/3	8
x_1	40	1	0	−1/2	½	18
c_j		40	35	0	0	
Z_j		40	35	10/3	25/3	1000
Δ_j		0	0	−10/3	−25/3	

Now, let us consider the solution to the dual problem which is augmented, by introducing surplus and artificial variables, as follows :

Minimise $G = 60y_1 + 96y_2 + 0S_1 + 0S_2 + MA_1 + MA_2$

Subject to

$$2y_1 + 4y_2 - S_1 + A_1 = 40$$
$$3y_1 + 3y_2 - S_2 + A_2 = 35$$
$$y_1, y_2, S_1, S_2, A_1, A_2 \ge 0$$

The solution to it is contained in Tables 4.2 through 4.4.

Simplex Table 1 : Initial Solution

Basis		y_1	y_2	S_1	S_2	A_1	A_2	b_i	b_i/a_{ij}
A_1	M	2	4*	−1	0	1	0	40	10←
A_2	M	3	3	0	−1	0	1	35	35/3
c_j		60	96	0	0	M	M		
Solution		5M	7M	−M	−M	M	M		
Δ_j		60 −5M	96 −7M	M	M	0	0		

Simplex Table 2 : Non-optimal Solution

Basis	y_1	y_2	S_1	S_2	A_1	A_2	b_i	b_i/a_{ij}
y_2 96	1/2	1	−1/4	0	1/4	0	10	20
A_2 M	3/2*	0	3/4	−1	−3/4	1	5	10/3←
c_j	60	96	0	0	M	M		
Solution	$48+\frac{3}{2}M$	96	$-24+\frac{3}{4}M$	−M	$24-\frac{3}{4}M$	M		
Δ_j	$12-\frac{3M}{2}$	0	$24-\frac{3}{4}M$	M	$\frac{7M}{4}-24$	0		
	↑							

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Simplex Table 3 : Optimal Solution

Basis	y_1	y_2	S_1	S_2	A_1	A_2	b_i
y_2 96	0	1	−1/2	1/3	1/2	−1/3	25/3
y_1 60	1	0	1/2	−2/3	−1/2	2/3	10/3
c_j	60	96	0	0	M	M	
Solution	60	96	−18	−8	18	8	
Δ_j	0	0	18	8	M−18	M−8	

Before comparing the solutions, it may be noted that there is a correspondence between variables of the primal and the dual problems. The structural variable x_1 in the primal, corresponds to the surplus variable S_1 in the dual, while the variable x_2 corresponds to S_2, the other surplus variable in the dual. In a similar way, the structural variables y_1 and y_2 in the dual correspond to the slack variables S_1 and S_2 respectively of the primal.

A comparison of the optimal solutions to the primal and the dual, and some observations follow.

(a) The objective function values of both the problems are the same. This with $x_1 = 18$ and $x_2 = 8$, Z equals $40 \times 18 + 35 \times 8 = 1000$. Similarly, with $y_1 = 10/3$ and $y_2 = 25/3$, the value of G would be $60 \times 10/3 + 96 \times 25/3 = 1000$.

(b) The numerical value of each of the variables in the optimal solution to the primal is equal to the value of its corresponding variable in the dual contained in the Δ_j row. Thus, in the primal problem, $x_1 = 18$ and $x_2 = 8$, whereas in the dual $S_1 = 18$ and $S_2 = 8$ (in the Δ_j row).

Similarly, the numerical value of each of the variables in the optimal solution to the dual is equal to the value of its corresponding variable in the primal, as contained in the Δ_j row of it. Thus, $y_1 = 10/3$ and $y_2 = 25/3$ in the dual, and $S_1 = 10/3$ and $S_2 = 25/3$ (note that we consider only the absolute values) in the primal. Of course, we do not consider artificial variables because they do not correspond to any variables in the primal, and are introduced for a specific, limited purpose only.

Clearly then, if feasible solutions exist for both the primal and the dual problems then both problems have optimal solutions of which objective function values are equal. A peripheral relationship between them is that if one problem has an unbounded solution, its dual has no feasible solution.

Further, the optimal solution to the dual can be read from the optimal solution of the primal, and vice versa. The primal and dual need not both be solved, therefore, to obtain the solution. This offers a big computational advantage in some situations. For instance, if the primal problem is a minimisation one involving, say 3 variables and 7 constraints, its solution would pose a big problem because a large number of surplus and artificial variables would have to be introduced. The number of iterations required for obtaining the answer would also be large. On the counter, the dual, with 7 variables and 3 constraints can be solved comparatively much more easily.

ifjogu leL;ka,s
(Transportation Problems)

यदि एक कंपनी एक ही वस्तु दो या अधिक कारखानों में बनाती है तथा उसके दो या अधिक मुख्य गोदाम हैं जहाँ से ग्राहकों को उस वस्तु की पूर्ति की जाती है तो कंपनी को ये निर्णय लेना होता है कि किस कारखाने से किस गोदाम को कितनी मात्रा भेजनी चाहिए ताकि परिवहन लागत कम से कम आए । इस प्रकार की समस्याओं को परिवहन समस्याएं कहा जाता हैं । यूँ तो इस समस्या के हल के लिए हमारे पास और भी विधि है लेकिन रेखीय नियोजन भी इस समस्या का हल निकालने में मदद कर सकती है ।

If a company manufactures one products in two or more factories and has two or more main go-downs from where the product can be supplied to the customers, then the company has to decide how much quantity of each factory should be transported to each of the godown so that total transportation cost is minimised. Though we have other method to solve this problem, yet linear programming can also help in solving the transportation problems.

For example, a company has three plants P_1, P_2, P_3, and three warehouses W_1, W_2 and W_3. Now various entities and costs can be shown in the form of the following matrix.

From \ To	W_1	W_2	W_3	Supply
P_1	x_{11} C_{11}	x_{12} C_{12}	x_{13} C_{13}	S_1
P_2	x_{21} C_{21}	x_{22} C_{22}	x_{23} C_{23}	S_2
P_3	x_{31} C_{31}	x_{32} C_{32}	x_{33} C_{33}	S_3
Demand	D_1	D_2	D_3	

It is assumed that total supply = total demand.

In the above matrix c_{ij} represents transportation cost /unit from factory i to warehouse j and x_{ij} represents quantity (in units) transported from factory i to warehouse j.

Now

Objective function is Minimise $Z = x_{11} c_{11} + x_{12} c_{12} + x_{13} c_{13} + x_{21} c_{21} + x_{22} c_{22} + x_{23} c_{23} + x_{31} c_{31} + x_{32} c_{32} + x_{33} c_{33}$

Subject to
$$\left.\begin{aligned} x_{11} + x_{12} + x_{13} &= S_1 \\ x_{21} + x_{22} + x_{23} &= S_2 \\ x_{31} + x_{32} + x_{33} &= S_3 \end{aligned}\right\} \text{Supply constraints}$$

$$\left.\begin{aligned} x_{11} + x_{21} + x_{31} &= D_1 \\ x_{12} + x_{22} + x_{32} &= D_2 \\ x_{13} + x_{23} + x_{33} &= D_3 \end{aligned}\right\} \text{Demand constraints.}$$

$x_{ij} \geq 0$ for i = 1, 2, 3 and j = 1, 2, 3 .

As we can see that if we use simplex method to solve the above problem, having 9 decision variables and 6 constraints, it will be a long process and so this method is not generally used to solve transportation problems. So we shall confine ourselves to graphical method for solving these problems. In other words, we will have only two decision variables (say x and y).

Example 28. A company manufacturing a product has two plants P_1 and P_2 having weekly capacities 100 and 60 units respectively. The cars are transported to three godowns w_1, w_2 and w_3 whose weekly requirements are 70, 50 and 40 units respectively. The transportation costs (Rs./unit) are as given below :

$P_1–w_1$ 5 , $P_1–w_2$ 4 , $P_1–w_3$.3, $P_2–w_1$ 4, $P_2–w_2$.2

$P_3–w_2$ 5.

Solve the above transportation problem so as to minimise total transportation costs.

Solution. First consider plant P_1. Let x and y be the units transported from P_1 to w_1 and w_2. Now we complete the matrix in the following form

From \ To	w_1 Cost/Unit	w_1 Qty.	w_2 Cost/Unit	w_2 Qty.	w_3 Cost/Unit	w_3 Qty.	Supply
P_1	5	x	4	y	3	(100-x-y)	100
P_2	4	(70-x)	2	(50-y)	5	(x+y-60)	60
Demand		70		50		40	160

Now total cost = $5x+4(70–x) + 4y + 2(50–y) +3(100–x–y) +5(x+y–60)$

$= 5x+280 –4x+4y+100–2y+300–3x–3y+5x+5y–300$

$= 3x+4y+380$

So objective function is

Min. $Z = 3x+4y+380$

Subject to the constraints

(i) In first row $100–x–y \geq 0$ so $x+y \leq 100$

(ii) In 2nd row $70–x \geq 0$, $50 –y \geq 0$ and $x+y – 60 \geq 0$

So $x \leq 70$, $y \leq 50$ and $x+y \geq 60$

So we have 4 inequations

(i) $x+y \leq 100$ (ii) $x \leq 70$ (iii) $y \leq 50$ and (iv) $x+y \geq 60$.

Plotting these values on the graph, we get the following feasible region.

The feasible region lies in the area covered by the polygon ABCDE. We also know that optimal solution lies at one of the vertices. So now we find the values of x, y and z at these points.

Points	**x**	**y**	**Z = 3x+4y+380**
A	60	0	60×3+4×0+380 = 560
B	70	0	70×3+4×0+380 = 590
C	70	30	70×3+30×4+380 = 710
D	50	50	50×5+50×4+380 = 830
E	10	50	10×5+50×4+380 = 630

Since the minimum value of Z is Rs. 560 at A, so optimal values of x and y are x = 60, y = 0.

So the optimal transportation schedule is

From P_1, 60 units will be transported to w_1 and 40 units to w_3 .

From P_2, 10 units will be transported to w_1 and 50 units to w_2 .

Exercise 9.2

Solve the following linear programming using simplex method.

1. Maximise $Z = 7x_1 + 14x_2$

Subject to the constraints

$3x_1 + 2x_2 \le 36$

$x_1 + 4x_2 \le 10$

$x_1, x_2 \ge 0$

2. Maximise $Z = 20x_1 + 30x_2 + 5x_3,$

Subject to

$4x_1 + 3x_2 + x_3 \le 40$

$2x_1 + 5x_2 \le 28$

$8x_1 + 2x_2 \le 36$

and $x_1, x_2, x_3 \ge 0$

3. Maximise $Z = 10x_1 + 20x_2,$

Subject to

$2x_1 + 5x_2 \ge 50$

$4x_1 + x_2 \le 28$

$x_1 x_2 \ge 0$

4. Minimise $Z = 6x_1 + 4x_2$

Subject to

$3x_1 + 0.5x_2 \ge 12$

$2x_1 + x_2 \ge 16$

$x_1, x_2 \ge 0$

5. Using two-phase Method, solve the following problem :

Minimise $150x_1 + 150x_2 + 100x_3,$

Subject to

$2x_1 + 3x_2 + x_3 \ge 4$

$3x_1 + 2x_2 + x_3 \ge 3$

and $x_1, x_2, x_3 \ge 0$

6. Solve the following LPP :

Minimise $Z = 100x_1 + 80x_2 + 10x_3,$

Subject to

$100x_1 + 7x_2 + x_3 \ge 30$

$120x_1 + 10x_2 + x_3 \ge 40$

$70x_1 + 8x_2 + x_3 \ge 20$

and $x_1, x_2, x_3 \ge 0$

7. A pharmaceutical company produces two popular drugs A and B which are sold at the rate of Rs. 9.60 and Rs. 7.80, respectively. The main ingredients are x, y and z and they are required in the following proportions :

Drugs	x%	y%	z%
A	50	30	20
B	30	30	40

The total available quantities (gm) of different ingredients are 1,600 in x, 1,400 in y and 1,200 in z. The costs (Rs) of x, y and z per gm are Rs. 8, Rs. 6 and Rs. 4, respectively.

Estimate the most profitable quantities of A and B to produce, using simplex method.

8. A factory produces three different products viz. A, B and C, the profit (Rs) per unit of which are 3, 4 and 6, respectively. The products are processed in three operations viz. X, Y and Z and the time (hour) required in each operation for each unit is given below :

Operation	Products		
	A	B	C
X	4	1	6
Y	5	3	1
Z	1	2	3

The factory works 25 days in a month, at rate of 16 hours a day in two shifts. The effective working of all the processes is only 80% due to assignable causes like power cut and breakdown of machines. The factory has 3 machines in operation X, 2 machines in operation Y and one machine in operation Z. Find out the optimum product mix for the month.

9. A factory engaged in the manufacturing of pistons, rings and valves for which the profits per unit are Rs. 10, 6 and 4, respectively, wants to decide the most profitable mix. It takes one hour of preparatory work, ten hours of machining and two hours of packing and allied formalities for a piston. Corresponding requirements for rings and valves are 1, 4 and 2 , and 1, 5 and 6 hours, respectively. The total number of hours available for preparatory work, packing and allied formalities are 100, 600 and 300, respectively. Determine the most profitable mix, assuming that what all produced can be sold.

10. A pharmaceutical company has 100 kg of material A, 180 kg of material B and 120 kg of material C available per month. They can use these materials to make three basic pharmaceutical products namely 5-10-5, 5-5-10 and 20-5-10, where the numbers in each case represent the percentage by weight of material A, material B and material C respectively, in each of the products and the balance represents inert ingredients. The cost of raw material is given below :

Ingredient	Cost per kg (Rs)
Material A	80
Material B	20
Material C	50
Inert ingredient	20

Selling price of these products is Rs. 40.50, Rs. 43 and Rs. 45 per kg respectively. There is a capacity restriction of the company for the product 5-10-5, that is, they cannot produce more than 30 kg per month. Formulate a linear programming model for maximising the monthly profit.

Determine how much of each of the products should they produce in order to maximise their monthly profits.

11. The Clear-Vision Television Company manufactures models A, B and C which have profits Rs. 200, 300 and 500 per piece, respectively. According to the production license the maximum weekly production requirements are 20 for model A, 15 for B and 8 for C. The time required for manufacturing these sets is divided among following activities.

Activity	Time per piece (hours)			Total time available
	Model A	Model B	Model C	
Manufacturing	3	4	5	150
Assembling	4	5	5	200
Packaging	1	1	2	50

Formulate the production schedule as an LPP and calculate number of each model to be manufactured for yielding maximum profit.

12. A company produces two products, A and B. The sales volume of product A is at least 60 percent of the total sales of the two products. Both the products use the same raw material of which the daily availability is limited to 100 tonnes. Products A and B use this material at the rate of 2 tonnes per unit and 4 tonnes per unit, respectively. The sales price for the two products are Rs. 20 and Rs. 40 per unit.

(a) Construct a linear programming formulation of the problem

(b) Find the optimum solution by simplex method.

(c) Find an alternative optimum, if any.

Write the dual of the following linear programming problems

13. Maximise $Z = 10y_1 + 8y_2 - 6y_3$

Subject to

$$3y_1 + y_2 - 2y_3 \le 10$$
$$-2y_1 + 3y_2 - y_3 \ge 12$$
$$y_1, y_2, y_3 \ge 0$$

14. Maximise $Z = x_1 - x_2 + x_3$

Subject to

$$x_1 + x_2 + x_3 \le 10$$
$$2x_1 - x_3 \le 2$$
$$2x_1 - 2x_2 + 3x_3 \le 6$$
$$x_1, x_2, x_3 \ge 0$$

15. Maximise $Z = 3x_1 - 2x_2$

Subject to

$$x_1 \le 4$$
$$x_2 \le 6$$
$$x_1 + x_2 \le 5$$
$$-x_2 \le -1$$
$$x_1, x_2 \ge 0$$

16. Minimise $Z = 4x_1 + x_2$

Subject to

$$3x_1 + x_2 = 2$$
$$4x_1 + 3x_2 \ge 6$$
$$x_1 + 2x_2 \le 3$$
$$x_1, x_2 \ge 0$$

17. Maximise $Z = 3x_1 + 4x_2 + 7x_2$

Subject to

$$x_1 + x_2 + x_3 \le 10$$
$$4x_1 - x_2 - x_3 \ge 15$$
$$x_1 + x_2 + x_3 = 7$$
$$x_1, x_2 \ge 0,\ x_3 \text{ unrestricted in sign.}$$

18. Solve the following transportation problems :

Transportation cost (Rs./unit)

To From	W_1	W_2	W_3	Supply
F_1	6	3	2	100
F_2	4	2	3	50
Demand	60	50	40	150

19. A brick manufacturer has two depots A and B with stocks of 30,000 and 20,000 bricks respectively. He receives orders from three builders P, θ and R for 11000, 20000 and 15000 bricks respectively. The distance in kms. From these depots to the builder's location are given in the following matrix :

Transportation cost (Rs./unit)

To / From	A	B
P	40	20
Q	20	60
R	30	40

How should the brick manufacturer fulfill the orders so that the total transportation costs are minimised ?

Answers

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Exercise 9.1.

7. Polygon with vertices (1,0), (6,0), (6, $\frac{7}{3}$), $\left(\frac{18}{7}\right)$, (0,5), (0, 1)

10. (i) $x \geq 0,\ 2x + y \geq 2,\ x - y \leq 1,\ x + 2y \leq 8,\ y \geq 0$

(ii) $x \geq 0,\ 2x + 3y \geq 3,\ x - 6y \leq 3,\ 3x + 4y \leq 18,\ -7x + 4y \geq 14,\ y \geq 0$

11. Max. = 26 at (12, 2) and min. = 0 at (0, 0)

12. (i) $x_1 = 2,\ x_2 = 1,\ Z = 8$ (ii) $x_1 = \frac{20}{19},\ x_2 = \frac{45}{19},\ Z = \frac{235}{19}$

(iii) $x_1 = 0,\ x_2 = 1,\ Z = 3$ (iv) $x = \frac{8}{3},\ y = \frac{1}{3},\ Z = \frac{29}{3}$

13. 8 fans and 12 sewing machines, Z = Rs. 392

14. Food I-2kg, Food II –4kg, Z = Rs. = 38

15. 3 packages of each, Z = Rs. 10.5, 16.15 bats, 20 sticks, Rs. 125

17. 10 sewing machines, 20 transistors , Z = Rs. 900

18. quantity of wheat = 400 gm, quantity of rice = 200 gm, Z = Rs. 2.40

19. 1000 of A, 500 of B, Z = Rs. 5500

20. 5 cards, 1 box, Z = Rs. 0.75

21. 4 bottles, 46 tumblers, Z = Rs. 3.16

22. A = 2, B = 3, Z = Rs. 230

Exercise 9.2

1. $x_1 = 10,\ x_2 = 0,\ Z = 70,\ Z{-}x_1 = 0,\ x_2 = 5.6,\ x_3 = 23.2,\ Z = 284$ 3 – $x_1 = 0,\ x_2 = 58,\ Z = 760$

4. $x_1 = 8,\ x_2 = 0,\ Z = 48$ 5 – $x_1 = 1/5,\ x_2 = 6/5,\ x_3 = 0,\ Z = 210$ 6 – $x_1 = 1/3,\ x_2 = 0,\ x_3 = 0,\ Z = 100/3$

7. A = 2000, B = 2000, Z = 10000 8 – A = 800/7, B = 0, C = 480/7, Z = 5280/7

8. Pistons = 100/3, Rings = 200/3, valves = nil, Z = 2200/3.

10. 5 –10-5 = 30, 5-5-10 = 1185, 20-5-10 = 0, Z = Rs. 20625

11. A = 50/3, B = 15, C = 8, Z = 35500/3

12. (a) max. $Z = 20x_1 + 40x_2$ subject to $2x_1 + 4x_2 \leq 100,\ -8x_1 + 24x_2 \leq 0,\ x_1, x_2 \geq 0$

(b) $x_1 = 30,\ x_2 = 10,\ Z = 1000$ (c) $x_1 = 10,\ x_2 = 0,\ Z = 1000$

13. Min. $G = 10x_1 + 12x_2$ subject to $3x_1 + 2x_2 \geq 10,\ x_1 - 3x_2 \geq 8,\ 2x_1 - x_2 \leq 6,\ x_1, x_2 \geq 0$

14. Min $G = 10y_1 + 2y_2 + 6y_3$ Subject to $y_1 + 2y_2 + 2y_3 \geq 1,\ y_1 - 2y_3 \geq -1,\ y_1 - y_2 + 3y_3 \geq 3,\ y_1, y_2, y_3 \geq 0$

15. Min. $G = 4y_1 + 6y_2 + 5y_3 - y_4$ Subject to $y_1 + y_3 \geq 3$, $y_2 + y_3 - y_4 \geq -2$, $y_1, y_2, y_3 \geq 0$
16. Max. $G = -2y_1 + 2y_2 + 6y_5$ Subject to $4y_3 - y_4 - 3y_5 \leq 4$, $3y_3 - 2y_4 - y_5 \leq 1$, $y_3, y_4, y_5 \geq 0$
17. Min. $G = 10y_1 - 15y_2 + 7y_3$ Subject to $y_1 - 4y_2 + y_3 \geq 3$, $y_1 + y_2 + y_3 \geq 4$, $3y_1 + y_2 + y_3 = 7$, $y_1, y_2 \geq 0$, y_3 unrestricted in sigh.
18. From $F_1 \to 10$ units to w_1, 50 units to w_2 and 40 units to w_3, From $F_2 \to 50$ units to w_1 zero units to w_2 and w_3. Total transportation cost = Rs. 490.
19. From brick depot A – Zero to P, 20000 to Q and 10000 to R. From brick depot B – 15000 to P, zero to Q and 5000 to R. Total transportation cost = Rs. 1200.

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Chapter -10

चक्रवृद्धि ब्याज
(Compound Interest)

मान लो एक आदमी बैंक से या किसी दूसरे आदमी से एक निश्चित अवधि के लिए कर्ज लेता है। उस अवधि के बाद जितनी धनराशि वो वापिस करेगा उसका मूल्य कर्ज के मूल्य से अधिक होगा। यह अतिरिक्त राशि वह व्यक्ति उस कर्ज का उपयोग करने के लिए बैंक या दूसरे आदमी को देगा। इस अतिरिक्त राशि को ब्याज कहा जाता है तथा कर्ज को मूलधन कहा जाता है। प्रायः ब्याज प्रतिशत में निकाला जाता है जिसे ब्याज दर कहा जाता है। ब्याज दो तरह के होते हैः–

(1) साधारण ब्याज

(2) चक्रवृद्धि ब्याज

यदि जमाकर्ता को हर तिमाही, छःमाही या साल के बाद ब्याज अदा किया जाता है तों इसे साधारण ब्याज कहते हैं। लेकिन यदि यह ब्याज जमाकर्ता को ना देकर मूलधन में जोड़ दिया जाता है तथा अगली समयावधि के लिए ब्याज इस नई राशि (मूलधन + ब्याज) पर निकाला जाता है तो इसे चक्रवृद्धि ब्याज कहते हैं। चक्रवृद्धि ब्याज में जमाकर्ता को कुल धनराशि उस निश्चित अवधि के पूरा होने पर एक ही बार दे दी जाती है।

Suppose a person takes a loan from a bank or from another person for a specified period of time. After this period, the amount he will return will be higher than the amount of loan taken. This additional amount will be paid by the borrower to the bank or second person for use of all loan given to him. This amount is called interest and the amount borrowed is called principal. Generally interest in expressed is percentage which is called rate of interest.

Interest is of two types :–

1) Simple interest

2) Compound interest

If the lender is paid actual interest after every three months, six months or a year, it is called simple interest. But if this interest instead of being paid to the lender, is added to the principal and interest for next period is calculated on this new amount (principle + interest), it is called compound interest. In compound interest, the lender is paid full amount after completion of the period only once.

Example – Suppose Rs. 1000 is lent at 10% per annum for 2 years. Calculate simple interest and compound interest.

Solution. Simple Interest (S.I.)

$$\text{S.I. for } 1^{st} \text{ year} = \frac{1000 \times 10 \times 1}{100} = \text{Rs. } 100$$

$$\text{S.I. for } 2^{nd} \text{ year} = \frac{1000 \times 10 \times 1}{100} = \text{Rs. } 100$$

S.I. for 2 years = Rs. 100 + Rs. 100 = Rs. 200

Compound Interest (C.I.)

$$\text{C.I. for } 1^{st} \text{ year} = \frac{1000 \times 10 \times 1}{100} = \text{Rs. } 100$$

After 1 year, this interest of Rs. 100 is not given to the lender but added to his principal.

So new principal = Rs. 1000 + Rs. 100 = Rs. 1100

$$\text{Now C.I. for } 2^{nd} \text{ year} = \frac{1100 \times 10 \times 1}{100} = \text{Rs. } 110$$

So C.I. for 2 years = Rs. 100 + Rs. 110 = Rs. 210

Theorem 1. IF P is the principle, r % is the rate of interest per period and n is the number of periods, then

(i) Simple Interest (S.I.) = $\frac{P \cdot r \cdot n}{100}$ and

(ii) Compound interest $= P\left(1+\frac{r}{100}\right)^n - P$

Proof : (i) S.I. for 1st period = $\frac{P \cdot r \cdot 1}{100} = \frac{Pr}{100}$

S.I. for 2nd period $= \frac{P \cdot r \cdot 1}{100} = \frac{Pr}{100}$

Continuing in this way.

S.I. for nth period $= \frac{P \cdot r \cdot 1}{100} = \frac{Pr}{100}$

So total S.I. for n periods = S.I. for 1st period + S.I. for 2nd period +….+ S.I. for nth period

$$= \frac{Pr}{100} + \frac{Pr}{100} + \ldots\ldots \text{ n times}$$

$$= \frac{Pr\,n}{100}$$

and amount $A = P + S.I.$

$$= P + \frac{Pr\,n}{100}$$

$$= P\left(1+\frac{rn}{100}\right)$$

(ii) C.I. for 1st period $= \frac{P \cdot r \cdot 1}{100} = \frac{P \cdot r}{100}$

Amount after 1st period $= P + \frac{Pr}{100} = P\left(1+\frac{r}{100}\right)$

C.I. for 2nd period $= P\left(1+\frac{r}{100}\right)\frac{r}{100}$

so amount after 2nd period $= P\left(1+\frac{r}{100}\right) + P\left(1+\frac{r}{100}\right).\frac{r}{100} = P\left(1+\frac{r}{100}\right)^2\left[1+\frac{r}{100}\right]$

$$= P\left(1+\frac{r}{100}\right)^2$$

C.I. for 3rd period $= P\left(1+\frac{r}{100}\right)^2\frac{r}{100}$

Amount after 3rd period $= P\left(1+\frac{r}{100}\right)^2 + P\left(1+\frac{r}{100}\right)^2\frac{r}{100}$

$$= P\left(1+\frac{r}{100}\right)^2\left[1+\frac{r}{100}\right] = P\left(1+\frac{r}{100}\right)^3$$

$$\text{Amount after n periods} = P\left(1+\frac{r}{100}\right)^n$$

and Compound Interest

$$\text{C.I. after n periods} = P\left(1+\frac{r}{100}\right)^n - P$$

$$= P\left[\left(1+\frac{r}{100}\right)^n - 1\right]$$

Notes :–

1) If n is not a whole number then it is divided into two parts – (i) a whole number part (k) and (ii) a fractional number (p) so n = k + p then

$$A = P\left(1+\frac{r}{100}\right)^k\left(1+\frac{Pr}{100}\right).$$

For example if n = 15 years 3 months, then n = 15 years + $\frac{1}{4}$ year and

$$A = P\left(1+\frac{r}{100}\right)^{15}\left(1+\frac{r}{4.100}\right)$$

2) Generally the unit of time period is in years. So the interest is compounded annually. In this case the above formula holds good. But if the interest is compounded monthly, quarterly or half yearly then calculations are changed as follows :

(i) Interest is compounded monthly

$$A = P\left(1+\frac{r}{12.100}\right)^{12n}$$

(ii) Interest is compound quarterly

$$A = P\left(1+\frac{r}{4.100}\right)^{4n}$$

(iii) Interest is compounded six monthly or half yearly

$$A = P\left(1+\frac{r}{2.100}\right)^{2n}$$

3) If the rate of interest (r%) changes every year i.e. r_1 in 1^{st} year, r_2 in 2^{nd} year,….., r_n in n^{th} year then

$$A = P\left(1+\frac{r_1}{100}\right)\left(1+\frac{r_2}{100}\right)\left(1+\frac{r_3}{100}\right)....\left(1+\frac{r_n}{100}\right)$$

Example 1. Find the compound interest on Rs. 50000 invested at the rate of 10% for 4 years.

Solution. P = Rs. 50000, r = 10%, n = 4 years

$$A = P\left(1+\frac{r}{100}\right)^n$$

$$= 50000\left(1+\frac{10}{100}\right)^4$$

$$= 5 \times \frac{11}{10}\times\frac{11}{10}\times\frac{11}{10}\times\frac{11}{10}$$

$$= 5000 \times 14641$$
$$= \text{Rs. } 73205$$
$$\text{C.I.} = A - P$$
$$= 73205 - 50000$$
$$= \text{Rs. } 23205$$

Example 2. Ram deposits Rs. 31250 in a bank at a rate of 8% per annum for 3 years. How much amount will be get after 3 years. How much his earning will change, if interest is compounded half yearly.

Solution. (i) P = Rs. 31250, r = 8%, n = 3 years

$$A = 31250\left(1+\frac{8}{100}\right)^3$$
$$= 31250 \times \frac{27}{25}\times\frac{27}{25}\times\frac{27}{25}$$
$$= \text{Rs. } 39366$$

(ii) If the rate is compounded half yearly, then

$$A = P\left(1+\frac{r}{2.100}\right)^{2n}$$
$$= 31250\left(1+\frac{8}{2.100}\right)^{2\times3}$$
$$= 31250\left(1+\frac{1}{25}\right)^6$$
$$= \text{Rs. } 39541.22$$

Change in earnings = 39541.22 − 39366 = Rs. 175.22

So if the interest rate is compounded half yearly, he will earn Rs. 175.22 more.

Example 3. Find the compound interest on a sum of Rs. 100000 at the rate 12% per annum for $2\frac{1}{2}$ years when the interest is compounded (i) annually, (ii) half yearly, (iii) quarterly (iv) monthly

Solution. P = Rs. 100000, r = 12 %, n = $2\frac{1}{2}$ years.

(i) Interest compounded annually

$$A = 100000\left(1+\frac{12}{100}\right)^2\left(1+\frac{12}{2.100}\right)$$
$$= 100000 \times \left(\frac{28}{25}\right)^2\left(\frac{53}{50}\right)$$
$$\text{Log } A = \text{Log } [100000 \times \left(\frac{28}{25}\right)^2\left(\frac{53}{50}\right)]$$
$$= \log 100000 + 2[\log 28 - \log 25] + [\log 53 - \log 50]$$
$$= 5 + 2[1.4471 - 1.3979] + [1.7243 - 1.6990]$$
$$= 5 + 0.0984 + .0253$$
$$= 5.1237$$
$$A = AL[5.1237] = \text{Rs. } 132953$$

So $\quad C.I. = 132953 - 100000 = Rs.\ 32953$

(ii) Interest is compounded half yearly

$$A = 100000\left(1+\frac{12}{2.100}\right)^5$$
$$= 100000\left(1+\frac{3}{50}\right)^5$$
$$= 100000\left(\frac{53}{50}\right)^5$$
$$\text{Log } A = \log\left[100000\times\left(\frac{53}{50}\right)^5\right]$$
$$= \log 100000 + 5(\log 53 - \log 50) = 5 + .1265 = 5.1265$$
$$A = AL[5.1265] = Rs.\ 133822$$
$$C.I. = 133822 - 100000 = Rs.\ 33822$$

(iii) Interest compounded quarterly

$$A = 100000\left(1+\frac{12}{4.100}\right)^{\frac{5}{2}\times 4}$$
$$= 100000\left(\frac{103}{100}\right)^{10}$$
$$\text{Log } A = \log\left[100000\times\left(\frac{103}{100}\right)^{10}\right]$$
$$= \log 100000 + 10[\log 103 - \log 100]$$
$$= 5 + 0.1284 = 5.1284$$
$$A = AL[5.1284] = Rs.\ 134400$$
$$C.I. = 134400 - 100000 = Rs.\ 34400$$

(iv) Interest compounded monthly

$$A = 100000\left(1+\frac{12}{12\times 100}\right)^{\frac{5}{2}\times 12}$$
$$= 100000\left(\frac{101}{100}\right)^{30}$$
$$\text{Log } A = \log\left[100000\times\left(\frac{101}{100}\right)^{30}\right]$$
$$= \log 100000 + 30[\log 101 - \log 100]$$
$$= 5 + 30[0.00432]$$

So $\quad A = AL[5.1296]$
$$= Rs.\ 134785$$
$$C.I. = 134785 - 100000 = Rs.\ 34785$$

Example 4. At what rate % will Rs. 32768 yield Rs. 26281 as compound interest in 5 years.

Solution. P = Rs. 32768

$$A = P + C.I.$$

$$= 32768 + 26281 = \text{Rs. } 59049$$
$$n = 5 \text{ years}$$

Now $$A = P\left(1+\frac{r}{100}\right)^n$$

or $$59049 = 32768\left(1+\frac{r}{100}\right)^5$$

or $$\frac{59049}{32768} = \left(1+\frac{r}{100}\right)^5$$

or $$\left(\frac{9}{8}\right)^5 = \left(1+\frac{r}{100}\right)^5$$

$\therefore$ $$1+\frac{r}{100} = \frac{9}{8}$$

$$\frac{r}{100} = \frac{9}{8} - 1 = \frac{1}{8}$$

$$r = \frac{100}{8} = 12.5\ \%$$

Example 5. At what rate % will a principal double itself in 6 years.

Solution. $$A = P\left(1+\frac{r}{100}\right)^n$$

$$2P = P\left(1+\frac{r}{100}\right)^6$$

or $$\left(1+\frac{r}{100}\right)^6 = 2 \qquad \text{Let } 1+\frac{r}{100} = x$$

$$\therefore\ x^6 = 2$$

Taking logarithms of both sides

$$6 \log x = \log 2$$
$$= 0.3010$$

or $$\log x = 0.0502$$

$\therefore$ $$x = AL[0.0502] = 1.1225$$

So now $$1 + \frac{r}{100} = 1.1225$$

$$\frac{r}{100} = 0.1225$$

or $$r = 12.25\ \%$$

Example 6. In how many years will Rs. 30000 becomes Rs. 43923 at 10% rate of interest.

Solution. $$A = P\left(1+\frac{r}{100}\right)^n$$

$$43923 = 30000\left(1+\frac{10}{100}\right)^n$$

$$= 30000\left(\frac{11}{10}\right)^n$$

or $$\frac{43923}{30000} = \left(\frac{11}{10}\right)^n$$

or $$\frac{14641}{10000} = \left(\frac{11}{10}\right)^n$$

or $$\left(\frac{11}{10}\right)^4 = \left(\frac{11}{10}\right)^n$$

$$\therefore\ n = 4$$

So in 4 years Rs. 30000 will become Rs. 43923 at 10% rate of interest.

Example 7. Sita invested equal amounts are at 8% simple interest and the other at 8% compound interest. If the latter earns Rs. 3466.40 more as interest after 5 years, find the total amount invested.

Solution. Let amount invested in each = P

So S.I. on P for 5 years at 8% = $\frac{P \times 8 \times 5}{100} = \frac{2}{5}P$

and C.I. on P for 5 years at 8% = $P\left(1+\frac{8}{100}\right)^5 - P$

$$= P\left[\left(\frac{27}{25}\right)^5 - 1\right]$$

$$\text{Difference} = P\left[\left(\frac{27}{25}\right)^5 - 1\right] - \frac{2}{5}P$$

$$= P\left[\left(\frac{27}{25}\right)^5 - 1 - \frac{2}{5}\right]$$

$$= P\left[\left(\frac{27}{25}\right)^5 - \frac{7}{5}\right] = P\left[\frac{14348907}{9765625} - \frac{7}{5}\right]$$

$$= P\left[\frac{14348907 - 13671875}{9765625}\right]$$

$$= \frac{677032}{9765625}P$$

So now $$\frac{677032}{9765625}P = 3466.40$$

or $$P = \frac{3466.40 \times 9765625}{677032} = \text{Rs. } 50000$$

So total amount invested = 50000 + 50000

= Rs. 100000.

Example 8. A sum of money invested at C.I. becomes Rs. 28231.63 after 4 years and Rs. 33542.00 after 6 years. Find the principal and the rate of interest.

Solution. Let principal be P and rate of interest be r.

So $$28231.63 = P\left(1+\frac{r}{100}\right)^4 \qquad \ldots(i)$$

and $$33542.00 = P\left(1+\frac{r}{100}\right)^6 \qquad \ldots(ii)$$

Dividing (ii) by (i)

$$\frac{33542}{28231.63} = \left(1+\frac{r}{100}\right)^2$$

Put $1 + \frac{r}{100} = x$

$$\therefore \quad \frac{33542}{28231.63} = x^2$$

Taking logarithms of both sides

$$\log 33542 - \log 28231.63 = 2 \log x$$

$$4.52559 - 4.45073 = 2 \log x$$

or $$2\log x = .07486$$

$$\log x = 0.03743$$

or $$x = AL[0.03743]$$

$$= 1.09$$

$$\therefore \quad 1 + \frac{r}{100} = 1.09$$

$$\frac{r}{100} = 1.09 - 1 = 0.09$$

or $$r = 9\%$$

Now substituting this value in equation (i)

$$28231.63 = P\left(1+\frac{9}{100}\right)^4$$

or $$P = 28231.63\left(\frac{100}{109}\right)^4$$

$$= \text{Rs. } 20000$$

Example 9. The difference between S.I. and C.I. on a certain sum of money for 3 years at $8\frac{1}{2}$ % rate of interest is Rs. 3566.26. Find the sum.

Solution. Let principal = Rs. P

$$\text{S.I.} = \frac{x \times 3 \times 17}{2 \times 100} = \frac{51}{200} P$$

$$\text{C.I.} = P\left(1+\frac{17}{2\times 100}\right)^3 - P$$

$$= P\left[\left(\frac{217}{200}\right)^3 - 1\right] = \frac{2218313}{8000000} P$$

$$\therefore \quad \frac{2218313}{8000000} P - \frac{51}{200} P = 3566.26$$

or $$\frac{2218313P - 2040000P}{8000000} = 3566.26$$

or $$\frac{178313}{8000000} P = 3566.26$$

$$P = \frac{3566.26 \times 8000000}{178313}$$

$$= \frac{356626}{100} \times \frac{8000000}{178313}$$

$$= \text{Rs. } 160000$$

Example 10. A person invests a part of Rs. 221000 at 10% C.I. for 5 years and remaining part for three years at the same rate. At time of maturity amount of both the investments is same. Find the sum deposited in each option.

Solution. Let principal in first option = P, r = 10% and n = 5 years

$$\therefore \quad A = P\left(1+\frac{10}{100}\right)^5 = P\left(\frac{11}{10}\right)^5$$

Sum invested in 2nd option = (221000 – P)

$$\therefore \quad A = (221000 - P)\left(1+\frac{10}{100}\right)^3$$

$$= (221000 - P)\left(\frac{11}{10}\right)^3$$

Now $$P\left(\frac{11}{10}\right)^5 = (221000 - P)\left(\frac{11}{10}\right)^3$$

or $$P\left(\frac{11}{10}\right)^2 = 221000 - P$$

or $$121\,P = 22100000 - 100P$$

or $$221\,P = 22100000$$

or $$P = \frac{22100000}{221} = \text{Rs. } 100000$$

So the sum invested in first option is Rs. 100000 and the sum invested in 2nd option is (221000 – 100000) Rs. 121000

Continuos Compounding of Interest

If the interest rate is compounded continuously, such that compounding frequency (λ) is infinitely large then

$$A = \lim_{\lambda\to\infty} P\left[1+\frac{r}{\lambda.100}\right]^{n.\lambda}$$

$$A = \lim_{\lambda\to\infty} P\left[1+\frac{r}{100\lambda}\right]^{\left(\frac{100\lambda}{r}\right)\left(\frac{nr}{100}\right)}$$

$$P = \left[\lim \frac{100\lambda}{r} \to \infty \left(1+\frac{r}{100\lambda}\right)^{\frac{100\lambda}{r}}\right]^{\frac{nr}{100}}$$

$$= P \quad r^{\left(\frac{nr}{100}\right)} \quad \left[\because \lim_{m\to\infty}\left(1+\frac{1}{m}\right)^m = e\right]$$

Here e = 2.71828.

Example 11. Rs 8000 are invested at 6% per annum. Find the amount after 5 years if interest is compounded continuously.

Solution $A = P\ e^{\left(\frac{nr}{100}\right)}$

Now P = Rs. 8000, n = 3 years and r = 6%

$$A = Rs.(2.71825)^{\frac{3\times 6}{100}}$$
$$= 8000 \times 1.197$$
$$= Rs\ 9576.$$

Example 12. At what rate %, a sum will be doubled in 5 years if interest is compounded continuously.

Solution. $A = P\ e^{\left(\frac{nr}{100}\right)}$

So $2P = P.\ e^{\frac{5r}{100}}$

or $2 = e^{\frac{r}{20}}$

Taking logarithms of both sides

$$\log 2 = \frac{r}{20} \log e$$

$$0.3010 = \frac{r}{20} \times 0.4343$$

or $r = \dfrac{20\times .3010}{.4343} = 13.86\%$

Effective Rate of Interest

As we have seen in the example 3 that we get higher yields, if instead of annual compounding, interest is compounded monthly, quarterly or half yearly. So at the same rate of interest, we get higher interest as a result of increased compounding interest. Similarly if we want same interest in a given period, the effective rates will be higher if interest is compounded monthly, quarterly or half yearly instead of annually.

Example 13. A company offers 13% interest rate per annum on its debentures. What are the effective rates if interest is compounded (i) half yearly. (ii) quarterly (iii) monthly and (iv) continuously.

Solution. Let principal = Rs 100

Time = 1 year

$\therefore$ C.I. at 13% = $\dfrac{100\times 13\times 1}{100}$ = Rs.13.

(i) Interest compounded half yearly

$$A = P\left(1+\frac{r}{2\times 100}\right)^{2n}$$

$$= 100\left(1+\frac{13}{2\times 100}\right)^{2\times 1}$$

$$= 100\times\frac{213}{200}\times\frac{213}{200} = 113.42$$

So effective rate of interest = 113.42 − 100 = 13.42%

(ii) Interest is compounded quarterly

$$A = 100\left(1+\frac{13}{4\times100}\right)^4$$

$$= 100\times\left(\frac{413}{400}\right)^4 = 100\times\frac{2.91\times10^{10}}{2.56\times10^{10}} = \text{Rs. } 113.67$$

So effective rate of interest = 113.67 − 100 = 13.67%

(iii) Interest is compounded monthly

$$A = 100\left(1+\frac{13}{12\times100}\right)^{12}$$

$$= 100\times\left(\frac{1213}{1200}\right)^{12} = 100\left(\frac{12.13}{12}\right)^{12}$$

$$= 100\times\frac{10.147\times10^{12}}{8.916\times10^{12}} = \text{Rs. } 113.81$$

So effective rate or interest = 113.81 − 100 = 13.81%

(iv) Interest rate compounded continuously

$$A = P\ e^{\left(\frac{nr}{100}\right)}$$

$$= 100\times e^{\frac{13\times1}{100}}$$

$$= 100\times(2.71828)^{.13}$$

$$= 100\times1.1388$$

$$= 113.88$$

So effective interest rate = 113.88 − 100 = 13.88%

So we can see that as frequency of compounding increases, effective interest rate also goes on increasing.

Exercise 10.1

1. Find the amount after 3 years if Rs. 16000 is invested at a rate of 10% per annum.
2. Find the compound interest earned on Rs. 5000 at a rate of 8% p.a. for 5 years.
3. Find the amount and compound interest on a sum of Rs. 80000 for $2\frac{1}{2}$ years at a rate of 6.5 % p.a.
4. Find the difference in compound interest if interest is compounded (i) annually and (ii) half yearly on a sum of Rs. 20000 for 3 years at a rate of 6% p.a.
5. Find compound interest on Rs. 5000 at 8% p.a. compounded quarterly for nine months.
6. At what rate percent when annum will a sum double itself in 5 years.
7. At what rate percent per annum will Rs 20000 become Rs. 30000 in 3 years if the interest is compounded
   (i) half yearly and (ii) quarterly.
8. A person borrows certain amount of money at 3 % per annum simple interest and invests it at 5% p.a. compound interest. After three years, he makes a profit of Rs 5410. Find the amount borrowed
9. In how much time will a sum be doubled if the rate of interest is 10% per annum.
10. A certain sum of money becomes Rs. 5995.08 after 3 years at 6% p.a. find the principal.

11. The compound interest on a certain sum for 4 years at 8% rate is Rs. 404.89 more than simple interest on the same sum at the same rate and for the same time. Find the principal.
12. A sum of money amounts to Rs. 8988.8 in 2 years and to Rs. 10099.82 in 4 years at compound interest. Find the principal and the rate of interest.
13. Difference between C.I. and C.I. on a certain sum of money for 2 years at 5% p.a. is Rs 10. Find the sum
14. A sum of Rs. 16896 is to be invested in two schemes one for 3 years and the other for 2 years. Rate of interest in both the schemes is 6.25% p.a. If the amount received at the maturity of the two schemes is same, find the sum invested in each scheme.
15. In how many years will a money treble itself at 8% if the interest is compounded continuously?
16. A company offers 12% rate of interest p.a. on its deposits. What is the effective rate of interest if it is compounded (i) six monthly (ii) quarterly (iii) monthly and (iv) continuously.
17. Which is better investment 8% compounded half yearly or 7.5% compounded quarterly.

Answers

1. Rs 21296 2. Rs. 2346.64 3. Rs. 93686.98 and Rs. 13686.98 4. Rs. 60.73
5. Rs.307 6. 14.87 % 7. (i) 14%, (ii) 13.76% 8. Rs. 15912
9. 7.27 years 10. Rs. 5034 11. Rs. 10000 12. Rs.8000 & 6%
13. Rs. 4000 14. Rs. 8192 and Rs. 8704 15. 8.53 years
16. (i) 12.36% (ii) 12.55% (iii) 12.68% and (iv) 12.75% 17. Ist option

Chapter-11

वार्षिकी (Annuity)

समय के समान अन्तराल में समान राशि कें किए गए भुगतान के क्रम को वार्षिकी कहते है ।
Annuity is a series of equal payment made over equal interval of time periods.

उदाहरण के लिए यदि एक व्यक्ति 2 वर्ष तक हर महीनें की प्रथम तिथि को 2000 रुपये जमा करवाता है तो ये एक वार्षिकी है । इस वार्षिकी में 2000 रु. की राशि को किश्त (instalment) कहा जाता है । क्योकि किन्हीं भी दो किश्तो के बीच की समयावधि एक महीना है तो इस वार्षिकी की भुगतान अवधि 1 महीना है । इसके अतिरिक्त, प्रथम तथा अन्तिम किश्त के बीच की समयावधि 2 वर्ष यानि 24 मास है तो इस वार्षिकी की समय सीमा 2 वर्ष है ।

For example, if a person deposits Rs. 2000 on first of every month for 2 years, it is an annuity. In this annuity amount of Rs. 2000 paid every month is called **instalment** of the annuity. Because the time difference between two instalments is one month, so the **payment** period of this annuity is one month. Besides this, since the time period between first and last payments is two years i.e. 24 months, so the **term** of the annuity is 24 months.

वार्षिकी का वर्गीकरण : निर्धारित, आकस्मिक तथा चिरस्थायी वार्षिकी

निर्धारित वार्षिकी (Annuity certain) में किश्तों की राशि तथा संख्या स्थिर होती है तथा किसी भी आकस्मिक कारण से उनमे कोई परिवर्तन नही होता ।

In annuity certain, number and amount of instalments is fixed and there is no change in then be causes of any contingency. For example instalment paid in recurring deposit in a bank, and for purchase of a plot of land are Annuities Certain.

आकस्मिक वार्षिकी (Annuity contingent) में किश्तों की अदायगी तभी तक दी जाती हैं जब तक कि कोई खास घटना ना घट जाएँ

In annuity contingent, instalments are paid till the happening of some specified event. For example, premium on an insurance policy is paid only as long as the policy holder is alive. In case of his/her then death before the maturity of the policy, further instalments are not paid.

चिरस्थायी वार्षिकी ;।ददनपजल चमतचमजनंसद्ध वो वार्षिकी हैं जिनमें किश्तों के भुगतान की कोई समय सीमा नही होती, उनका भुगतान लगातार होता रहता है ।

In annuity perpetual, there is no time limit for payment of instalments, they are paid for ever. For example, the instalments of interest earned by endowment fund is a perpetual annuity as they are received regularly for ever.

Besides this, if the payment of the instalments is made at the beginning of the corresponding period it is called a Annuity Due and if made at the end of the period, it is called Annuity Immediate. Annuity immediate is called ordinary annuity also. The total amount, to be received, after the maturity of the annuity is the sum of the accumulated values (principal + interest) of all the instalments paid.

Case I. When the annuity is annuity immediate

Let a and n be the amount and number of instalments of an annuity immediate. Further let r be the rate of interest per period. Since 1st instalment is paid at the end of first period, so it will earn on interest for (n−1) periods. Similarly 2nd instalment will earn interest for (n−2) periods and so an. Second last instalment will earn interest for 1 period only and last instalment will not earn any interest.

So total amount of the annuity

$$= a\left(1+\frac{r}{100}\right)^{n-1} + a\left(1+\frac{r}{100}\right)^{n-2} + \ldots + a\left(1+\frac{r}{100}\right) + a$$

$$= a\left[\left(1+\frac{r}{100}\right)^{n-1} + \left(1+\frac{r}{100}\right)^{n-2} + \ldots + \left(1+\frac{r}{100}\right) + 1\right]$$

$$= a\,[(1+i)^{n-1} + (1+i)^{n-2} + \ldots + (1+i) + 1] = a\,[1+(1+i) + \ldots + (1+i)^{n-2} + (1+i)^{n-1}]$$

Now this is a geometrical progression and so the sum is given by

$$\text{Amount} = a\left[\frac{1((1+i)^n - 1)}{1+i-1}\right] = 0 \qquad \left\{\because S = \left[a\frac{(r^n-1)}{r-1}\right]\right\}$$

$$= a\left[\frac{1((1+i)^n - 1)}{i}\right].$$

Case 2. When the annuity is annuity due.

In this case, first instalment is paid at the beginning of 1st period, 2nd instalment at the beginning of 2nd period and so on. So 1st instalment will earn interest for n period, 2nd instalment for (n−1) periods and so on. Last instalment will earn interest for one period only.

$$\text{So Amount} = a\left(1+\frac{r}{100}\right)^{n} + a\left(1+\frac{r}{100}\right)^{n-1} + \ldots + a\left(1+\frac{r}{100}\right)^{2} + a\left(1+\frac{r}{100}\right)$$

$$= a\,[(1+i)^n + (1+i)^{n-1} + \ldots + (1+i)^2 + (1+i)]$$

$$= a[((1+i) + (1+i)^2 + \ldots + (1+i)^{n-1} + (1+i)^n]$$

$$= a\left[\frac{(1+i)\{(1+i)^n - 1\}}{1+i-1}\right]$$

$$= \frac{a(1+i)}{i}[(1+i)^n - 1]$$

or $\quad \text{Amount} = a\left[\dfrac{(1+i)^n - 1}{\dfrac{i}{1+i}}\right]$

Example 1. A person deposits Rs. 2000 per month in a bank for 2 years. If bank pays compound interest at the rate of 8% p.a. find the amount he will receive if the annuity is (i) immediate and (ii) due

Solution. a = Rs. 2000, n = 12× 2 = 24 months, r = $\frac{8}{12} = \frac{2}{3}\%$ monthly

(i) Annuity immediate

$$A = a\left[\frac{(1+i)^n - 1}{i}\right], \quad \text{where } i = \frac{2}{3}\times\frac{1}{100} = \frac{2}{300}$$

$$\therefore \qquad A = 2000\frac{\left[\left(1+\frac{2}{300}\right)^{24} - 1\right]}{\frac{2}{300}}$$

$$= \frac{2000\times 300}{2}[1.1729 - 1] = 1000\times 300\times .1729$$

$$= \text{Rs. } 51870$$

(ii) Annuity due

$$A = a\left[\frac{\frac{(1+i)^n - 1}{i}}{1+i}\right]$$

$$= 2000\frac{\left[\left(1+\frac{2}{300}\right)^{24} - 1\right]}{\frac{2/300}{1+2/300}}$$

$$= \frac{2000\times(1.1729-1)}{\frac{2}{300}\times\frac{300}{302}} = 2000\times .1729\times\frac{302}{2}$$

$$= \text{Rs. } 52215.80$$

Example 2. Find the future value of an ordinary annuity of Rs. 4000 per year for 3 years at 10% compound interest rate per annum.

Solution. Here a = 4000, n = 3 and $i = \frac{10}{100} = 0.1$

So $\quad A = 4000\left[\frac{(1+.1)^3 - 1}{0.1}\right]$

$$= \frac{4000}{0.1}(1.331 - 1) = 40000\times .331$$

$$= \text{Rs. } 13240$$

Example 3. Find the future value of an annuity due of Rs. 5000 per year for 10 years at rate of 12 % p.a. the interest being compounded half yearly.

Solution. Here a = 5000, n = 10×2 = 20 half years, $r = \frac{12}{2} = 6\%$ half yearly

So $\quad i = \frac{r}{100} = \frac{6}{100} = .06$

Now Amount $= 5000\left[\frac{\frac{(1+.06)^{20} - 1}{.06}}{1+.06}\right]$

$$= 5000\,[(1.06)^{20} - 1] \times \frac{1.06}{.06}$$

Let $x = (1.06)^{20}$

$\therefore$ $\log x = 20 \log 1.06$

$= 20 \times 0.0253$

$= 0.5061$

$\therefore$ $x = AL\,[0.5061]$

$= 3.2071$

Now $\text{Amount} = 5000\,(3.2071 - 1) \times \frac{1.06}{.06}$

$$= 5000 \times 2.2071 \times \frac{1.06}{.06}$$

= Rs. 194960.5

Example 4. Find the future amount of Rs. 50000 payable at the end of each quarter for 5 years at 10 % p.a. compounded quarterly.

Solution a = 40000, n 5×4 = 20 quarters, $r = \frac{10}{4} = 2.5\%$ quarterly

So $i = \frac{2.5}{100} = .025$

Now Amount $= a\left[\frac{(1+i)^n - 1}{i}\right]$

$$= 50000\left[\frac{(1+.025)^{20} - 1}{.025}\right]$$

Let $(1.025)^{20} = x$

$\therefore$ $\log x = 20 \log (1.025)$

$= 20 \times 0.0107$

$= 0.2145$

So $x = AL\,[0.2145] = 1.6386$

So amount $= 50000 \times \left(\frac{1.6386 - 1}{.025}\right)$

$$= 50000 \times \frac{.6386}{.025}$$

= Rs. 1277200

To find the instalment of given annuity when amount is given

Example 5. What instalment has a person to pay at the end of each year if he wants to get Rs. 5,00,000 after 10 years at 5% compound rate of interest per annum.

Solution. We know that

$$\text{Amount (A)} = a\left[\frac{(1+i)^n - 1}{i}\right]$$

Given A = Rs. 500000, n = 10, r = 5 so i = 0.0.5

Now $500000 = a\left[\frac{(1+.05)^{10}-1}{.05}\right]$

$= a\left[\frac{(1.05)^{10}-1}{.05}\right]$

$= a\left[\frac{1.6289-1}{.05}\right]$

$= \frac{.6289}{.05}.a$

$\therefore$ $a = \frac{500000\times.05}{.6289}$

=Rs. 39751.95

Example 6. A company creates a sinking fund to provide for paying Rs. 1000000 debt maturing in 5 years. Find the amount of annual deposits at the end of each year if rate of interest is 18% compounded annually.

Solution. A = 1000000, n = 5, r = 18% so $i = \frac{18}{100} = 0.18$

$A = a\left[\frac{(1+i)^n-1}{i}\right]$

$1000000 = a\left[\frac{(1+.18)^5-1}{.18}\right]$

$= a\left[\frac{(1.18)^5-1}{.18}\right]$

Let $x = (1.18)^5$

So $\log x = 5 \log 1.18 = 5\times .0719$

$= 0.3595$

$x = AL\ [0.3595]$

$= 2.2877$

$1000000 = a\left[\frac{2.2877-1}{.18}\right]$

or $a = \frac{1000000\times.18}{1.2877}$

= Rs. 151553.42

Example 7. A machine costs Rs. 1,50,000 and has a life of 10 years.

If the scrap value of the machine is Rs. 5000, how much amount should be accumulated at the end of each year so that after 12 years a new machine could be purchased after 10 years at the same price. Annual compound rate of interest is 8 %.

Solution. Amount required after 10 years = 150000 – 5000 = 145000

we are given A = 145000, n = 12, r = 8 %, i = .08

Now $A = a\left[\frac{(1+i)^n - 1}{i}\right]$

$$145000 = a\left[\frac{(1+.08)^{10} - 1}{.08}\right]$$

$$= a\left[\frac{2.1589 - 1}{.08}\right]$$

$$\therefore \quad a = \frac{145000 \times .08}{1.1589}$$

$$= \text{Rs. } 10009.49$$

Example 8. Find the minimum number of years for which an annuity of Rs. 2000 must sum in order to have at least total amount of Rs. 32000 at 5% compound rate of interest

Solution. A = 32000, a = 2000, r = 5% , i = .05

So $32000 = 2000\left[\frac{(1+.05)^n - 1}{.05}\right]$

or $\frac{32000 \times .05}{2000} = (1.05)^n - 1$

or $(1.05)^n = 1.8$

taking logarithms of both sides

n. log (1.05) = log 1.8

$n \times 0.0212 = 0.2553$

$$n = \frac{0.2553}{0.0212} = 12.04$$

∴ The amount of annuity will take 13 years to exceed Rs. 32000 as total amount.

Example 9. What will be the instalment of an annuity having a total amount of Rs. 75000 for 12 years at 8% p.a., rate of interest compounded half yearly.

Solution. We are given that

$$A = 75000,\ n = 12 \times 2 = 24,\ r = \frac{8}{2} = 4\% \text{ and } i = .04$$

So $75000 = a\left[\frac{(1+.04)^{24} - 1}{.04}\right]$

or $75000 \times .04 = a\,[(1.04)^{24} - 1]$

Let $x = (1.04)^{24}$

$\log x = 24 \log 1.04$

$= 24 \times 0.01703$

$= 0.4088$

$x = AL\,[0.4088]$

$= 2.5633$

So $75000 \times .04 = a\,(2.5633 - 1)$

$$A = \frac{3000}{1.5633}$$

$$= \text{Rs. } 1919.02$$

Amount of an annuity when the interest is compounded continuously

In this case, the amount of the annuity is calculated by using the formula

$$A = a \int_0^n e^{it}\, dt \text{ where } i = \frac{r}{100}$$

Example 10. In an annuity, Rs. 5000 are deposited each year for 8 years. Find the amount if interest rate of 10% is compounded continuously.

Solution. We are given

$$a = 5000,\ n = 8,\ r = 10\ \%\ \text{and}\ i = \frac{10}{100} = 0.10$$

So $A = 5000 \int_0^8 e^{0.1t}\, dt$

$= 5000 \left[\frac{e^{0.1t}}{0.1}\right]_0^8$

$= \frac{5000}{0.1}[e^{0.8} - e^0]$

$= 50000\ (2.71828\text{-}1)^{0.8}$ $\qquad [\because e^0 = 1]$

Let $x = (2.71828)^{0.8}$

$\log x = 0.8 \log 2.712828$

$= 0.8 \times 0.4343$

$= 0.3474$

$x = AL\ [0.3474]$

$= 2.2255$

So $A = 50000 \times (2.2255 - 1) = 50000 \times 1.2255$

$= \text{Rs. } 61275$

Example 11. A person wants to have Rs. 20000 in his recurring account at the end of 6 years. How much amount he should deposit each year if the rate of interest is 8 % p.a. compound continuously.

Solution. Here $A = 20000,\ n = 6,\ r = 8$ and $i = \frac{8}{100} = .08$

Now $A = a \int_0^n e^{it}\, dt$

or $20000 = a \int_0^6 e^{.08t}\, dt$

$= a \left[\frac{e^{.08t}}{.08}\right]_0^6$

$= \frac{a}{.08}[e^{0.48} - e^0]$

Let $x = e^{0.48}$

$\log x = 0.48 \log e$

$= 0.48 \times 0.4343$

$= 0.20846$

$x = AL\ [0.20846]$

$= 1.6161$

So $20000 \times .08 = a\,[1.6161 - 1]$

$$a = \frac{1600}{0.6161} =$$

$$= \text{Rs. } 2597.$$

Exercise 11.1

1. A person deposits Rs. 10000 at the end of each year for 5 years. Find the amount, he will receive after 5 years if rate of compound interest is 10% p.a.
2. Calculate the future value of an ordinary annuity of Rs. 8000 per annum for 12 years at 15% p.a. compounded annually.
3. A company has set up a sinking fund account to replace an old machine after 8 years. If deposits in this account Rs. 3000 at the end of each year and rate of compound interest is 5% p.a. find the cost of the machine.
4. To meet the expenses of her daughter a woman deposits Rs. 3000 every six months at rate of 10% per annum. Find the amount she will receive after 18 years.
5. A sinking fund is created by a company for redemption of debentures of Rs. 1000000 at the end of 25 years. How much funds should be provided at the end of each year if rate of interest is 4% compounded annually.
6. The parents of a child have decided to deposit same amount at every six months so that they receive an amount of Rs. 100000 after 10 year. The rate of interest is 5% p.a. compounded half yearly.
7. Which is a better investment – An annuity of Rs. 2000 each year for 10 year at a rate of 12 % compounded annually or an annuity of Rs. 2000 each year for 10 years at a rate of 11.75 % compounded half yearly.

वार्षिकी का वर्तमान मूल्य (Present value of an annuity)

एक वार्षिकी का वर्तमान मूल्य, भविष्य में मिलने वाली सभी राशियों का, वार्षिकी शूरु होने के समय कुल मूल्य है। मूल्य सभी किश्तो के वर्तमान मूल्य का योग होता है।

Present value of an annuity is equal to the total worth, at the time of beginning of the annuity, of all the future payments that are to be received. This value is equal to the sum of present values of all the instalments.

Let a be the amount of each instalment, n be the term (time periods) of the annuity and r % be the rate of interest per period. Further let $V_1, V_2 \ldots V_n$ be the present values of instalments paid in periods 1,2,…n respectively.

From our previous discussion, we know that the future value of (FV) of an annuity is given by

$$FV = a\left(1+\frac{r}{100}\right)^{n}$$

So if an instalment is paid at the end of period 1.

$$\text{Then } FV = a\left(1+\frac{r}{100}\right)^{n-1}$$

If we want to calculate present value (PV) of this instalment then it is calculated as

$$PV = \frac{a}{\left(1+\frac{r}{100}\right)}$$

So present value of an instalment is the amount of money today which is equivalent to the amount of that instalment, to be received after a specific period. In general, if an instalment, a, is paid in nth period and rate of interest is r % then

$$PV_n = \frac{a}{\left(1+\frac{r}{100}\right)^n} \text{ or } \frac{a}{(1+i)^n}$$

Now we will find present value of both ordinary annuity and annuity due.

Ordinary Annuity or Annuity immediate

Present value of the annuity

$$V = V_1 + V_2 + V_3 \ldots + U_n$$

$$= \frac{a}{1+i} + \frac{a}{(1+i)^2} + \frac{a}{(1+i)^3} + \ldots + \frac{a}{(1+i)^n}$$

$$= a\left[\frac{1}{1+i} + \frac{1}{(1+i)^2} + \frac{1}{(1+i)^3} + \ldots + \frac{1}{(1+i)^n}\right]$$

$$= a\left[\left(\frac{1}{1+i}\right)\left\{\frac{1-\left(\frac{1}{1+i}\right)^n}{1-\frac{1}{1+i}}\right\}\right]$$

$$= a\left[\frac{1}{(1+i)}\left(\frac{(1+i)^n-1}{(1+i)^n}\right)\times\left(\frac{1+i}{1+i-1}\right)\right]$$

$$= a\left[\frac{(1+i)^n-1}{(1+i)^n}\times\frac{1}{i}\right] \text{ or } a\left[\frac{1-(1+i)^{-n}}{i}\right]$$

(ii) Annuity Due

In this type of annuity, each instalment is paid at the beginning of every period. So first instalment is paid at time zero, 2nd instalment at time 1 and so on last instalment is paid at period (n−1) Hence PV of 1st instalment is equal to a, of 2nd instalment is $\frac{a}{1+\frac{r}{100}}$, of 3rd instalment is $\frac{a}{\left(1+\frac{r}{100}\right)^2}$ and PV of last instalment is $\frac{a}{\left(1+\frac{r}{100}\right)^{n-1}}$

So $\quad V = V_1 + V_2 + V_3 + \ldots + V_n$

$$= a + \frac{a}{1+\frac{r}{100}} + \frac{a}{\left(1+\frac{r}{100}\right)^2} + \ldots + \frac{a}{\left(1+\frac{r}{100}\right)^{n-1}}$$

$$= a\left[1+\frac{1}{1+i}+\frac{1}{(1+i)^2}+\ldots+\frac{1}{(1+i)^{n-1}}\right]$$

$$= a\left[\left(\frac{1-\left(\frac{1}{1+i}\right)^n}{1-\frac{1}{1+i}}\right)\right]$$

$$= a\left[\frac{1-(1+i)^{-n}}{\frac{1+i-1}{1+i}}\right]$$

$$= a(1+i)\left[\frac{1-(1+i)^{-n}}{i}\right]$$

or $$= a\left[\frac{1-(1+i)^{-n}}{\frac{i}{1+i}}\right]$$

Example 11. Find the present value of an ordinary annuity of Rs. 1500 per year for 5 years at 8% rate of interest.

Solution. $V = a\left[\frac{1-(1+i)^{-n}}{i}\right]$

Here $a = 1500$, $n = 5$, $r = 8$ and thus $i = \frac{8}{100} = .08$

So $V = 1500\left[\frac{1-(1+.08)^{-5}}{.08}\right]$

$= \frac{1500}{.08}\left[1-(1.08)^{-5}\right]$

Let $x = (1.08)^{-5}$

$\log x = -5 \log 1.08$

$= -5 \times 0.0334$

$= (-.1670 + 1) - 1$ [∵ Mantissa can never be negative so making the value positive, we add and subtract 1 to the negative value]

$= \bar{1}.8330$

$x = AL(\bar{1}.8330)$

$= 0.6806$

So $V = \frac{1500}{.08}\ [1-0.6806]$

$= \frac{1500\times 100}{8}\times 0.3194$

$=$ Rs. 5988.75

Example 12. Find the present value of an annuity due of Rs. 800 per year for 10 years at a rate of 4 % p.a.

Solution. We are given

$$a = 800,\ n = 10,\ r = 4\ \%\ \text{ and so } i = \frac{4}{100} = .04$$

Now $$V = a\left[\frac{1-(1+i)^{-n}}{(1+i)}\right] = a\left[1-(1+i)^{-n}\right]\left(\frac{1+i}{i}\right)$$

$$= 800\ [\ 1-(1+.04)^{-10}]\left[\frac{1.04}{0.04}\right]$$

Let $x = (1.04)^{-10}$

$\log x = -10 \log (1.04)$

$= -10 \times 0.01703$

$= -0.1703 + 1 - 1$

$= \bar{1}.8297$

$x = AL\ [\bar{1}.8297]$

$= 0.6756$

So now $$V = 800\ (1-0.6756)\left(\frac{1.04}{0.04}\right)$$

$$= \frac{800 \times .3244 \times 1.04}{.04}$$

$= \text{Rs. } 6747.52$

Example 13. A dealer sells a scooter to a customer on the condition that he will pay Rs 10000 in cash and balance to be paid in 36 month end instalment of Rs 400.

If rate of interest is 12 % p.a. find the cash price of the scooter.

Solution. We are given

$$a = 400,\ n = 36,\ r = \frac{12}{12} = 1\ \%\ \text{per month and so } i = \frac{1}{100} = 0.01$$

Now $$V = a\left[\frac{1-(1+i)^{-n}}{i}\right]$$

$$= 400\left[\frac{1-(1+.01)^{-36}}{0.01}\right]$$

$$= \frac{400}{0.01}[1-(1.01)^{-36}]$$

Let $x = (1.01)^{-36}$

$\log x = -36 \log 1.01$

$= -36 \times 0.00432$

$= -0.15557 + 1 - 1$

$= \bar{1}.84443$

$x = AL\ [\bar{1}.84443]$

$= 0.6989$

So $V = 40000\,[1-.6989]$

$= 40000 \times 0.3011$

$= 12044$

PV of 36 instalments = Rs. 12044

Cash paid = Rs. 10000

So cash price of the scooter = 12044 + 10000

= 22044

Example 14. A person takes a loan from a finance company for construction of a house, to be repayable in 120 monthly instalments of Rs. 1020 each. Find the present value of the instalments if the company charges interest @ 9 % p.a.

Solution. We are given

$$a = 1020,\ n = 120,\ r = \frac{9}{12} = 0.75\ \%\ \text{per month}$$

and $$i = \frac{0.75}{100} = .0075$$

$$V = a\left[\frac{1-(1+i)^{-n}}{i}\right]$$

$$= 1020\left[\frac{1-(1+.0075)^{-120}}{0.0075}\right]$$

$$= \frac{1020}{0.0075}\,[1-(1.0075)^{-120}]$$

Let $x = (1.0075)^{-120}$

so $\log x = -120 \log (1.0075)$

$= -120 \times 0.003245$

$= -0.3894 + 1 - 1$

$= \bar{1}.6106$

$x = AL\,[\bar{1}.6106]$

$= 0.4079$

Hence $$V = \frac{1020}{0.0075}\,[1-0.4079]$$

$$= \frac{1020}{0.0075} \times .5921$$

$= \text{Rs. } 80525.64$

Type 2. To find amount of instalment when present value is given

Example 15. Find the amount of instalment on a loan of Rs 40000 to be payable in 10, at the end of year, equal instalments at a rate of 10 % interest per annum.

Solution. We are given

$V = 40000$

$n = 10$

$$r = 10\ \%\ \text{or}\ i = \frac{10}{100} = 0.1$$

Now $$V = a\left[\frac{1-(1+i)^{-n}}{i}\right]$$

$$40000 = a\left[\frac{1-(1+0.1)^{-10}}{0.1}\right]$$

or $$a = \frac{40000\times 0.1}{1-(1.1)^{-10}}$$

Let $x = (1.1)^{-10}$

So $\log x = -10 \log 1.1$

$= -10 \times 0.04139$

$= -0.4139 + 1 - 1$

$= \bar{1}.5861$

$x = AL[\bar{1}.5861]$

$= 0.3856$

So $$a = \frac{4000}{1-0.3856}$$

$$= \frac{4000}{0.6144}$$

= Rs. 6510.42

Example 16. A person takes a loan of Rs. 600000 to be repaid in 60 equal end of month instalments at a rate of 8 % per annum. Find the amount of each instalment.

Solution. We are given

$$V = 600000,\ n = 60,\ r = \frac{8}{12} = \frac{2}{3}\% \text{ or } i = \frac{2}{300} = \frac{1}{150}$$

Now $$V = a\left[\frac{1-(1+i)^{-n}}{i}\right]$$

$$600000 = a\left[\frac{1-\left(1+\frac{1}{150}\right)^{-60}}{\frac{1}{150}}\right]$$

or $$600000\times\frac{1}{150} = a\left[1-\left(\frac{151}{150}\right)^{-60}\right]$$

Let $$x = \left(\frac{151}{150}\right)^{-60}$$

So $\log x = -60\,[\log 151 - \log 150]$

$= -60\,[2.1790 - 2.17611]$

$= -60 \times 0.00289$

$= -0.17314 + 1 - 1$

$= \bar{1}.82686$

$x = AL[\bar{1}.82686]$

$= 0.6712$

So now $4000 = a\,[1 - 0.6712]$

$= a \times 00.3388$

$$a = \frac{4000}{0.3388}$$

$= \text{Rs. } 11806.37$

So monthly instalment is Rs 11806.37

Type 3. Interest is compounded continuously

Present value in this case is given by

$$V = a\int_0^n e^{-it}\,dt. \text{ where } i = \frac{r}{100}$$

Example 17. Find the present value of an annuity of Rs. 12000 per year for 4 years at a rate of 8 %. The interest is compounded continuously.

Solution. We are given

$$a = 12000,\ n = 4,\ r = 8\ \% \qquad \text{or} \quad i = \frac{8}{100} = 0.08$$

Now $V = a\int_0^n e^{-it}\ dt$

$$= a\left[\frac{e^{-it}}{-i}\right]_0^n$$

$$= -\frac{a}{i}\,[e^{-in} - e^0]$$

$$= -\frac{a}{i}\,[e^{-in} - 1]$$

So $V = \dfrac{12000}{0.08}\,[(2.71828)^{-4\times.08} - 1]$

Let $x = (2.71828)^{-0.32}$

log $x = -0.32 \log (2.71828]$

$= -032 \times 0.43429$

$= -0.1390 + 1 - 1$

$= \bar{1}.8610$

$x = AL\,[\ \bar{1}.8610]$

$= 0.7261$

So $V = -150000 \times (0.7261 - 1)$

$= -150000 \times (-0.2739)$

$= \text{Rs. } 41085$

स्थगित वार्षिकि (Deferred Annuity)

स्थगित वार्षिकि वह वार्षिकि है जिसमें प्रथम किस्त का भुगतान एक निर्दिष्ट भुगतान अवधि के गुजरने के बाद किया जाता है । इस अवधि को स्थगन अवधि कहा जाता है।

Deferred annuity is an annuity in which payment of first instalment is made after lapse of some specified number of payment periods. This period is called deferment period. For example, payment of first instalment in case of educational loans and housing loans is paid after a deferment period of one to four years.

स्थगित वार्षिकि की राशि (Amount of deferred annuity)

Let a be the instalment, n be the time periods, r % be the rate of interest and m the deferment period of a deferred annuity. Amount of this annuity is same as in case of other annuities. This amount is not affected by deferment period.

1. Annuity immediate

$$A = a\left[\frac{(1+i)^n - 1}{i}\right]$$

2. Annuity due

$$A = a\left[\frac{(1+i)^n - 1}{\frac{i}{1+i}}\right]$$

स्थगित वार्षिकि का वर्तमान मूल्य (Present value of a deferred annuity)

Let $V_1, V_2 \ldots V_n$ be the present values of the 1st, 2nd , nth instalments respectively.

Case 1. Annuity immediate

Since m is the deferment period, so 1st instalment will be paid after (m+1) periods, 2nd after (m+2) periods and the last instalment is paid after (m+n) periods, so

$$V_1 = \frac{a}{\left(1+\frac{r}{100}\right)^{m+1}}, V_2 = \frac{a}{\left(1+\frac{r}{100}\right)^{m+2}}, \ldots \text{ and } V_n = \frac{a}{\left(1+\frac{r}{100}\right)^{m+n}}$$

$$= \frac{a}{(1+i)^{m+1}}, V_2 = \frac{a}{(1+i)^{m+2}} \ldots V_n$$

Now the present value of the annuity

$$V = V_1 + V_2 + V_3 + \ldots + V_n$$

$$= \frac{a}{(1+i)^{m+1}} + \frac{a}{(1+i)^{m+2}} + \ldots + \frac{a}{(1+i)^{m+n}}$$

$$= \frac{a}{(1+i)^m(1+i)} + \frac{a}{(1+i)^m(1+i)^2} + \ldots + \frac{a}{(1+i)^m(1+i)^n}$$

$$= \frac{a}{(1+i)^m}\left[\frac{1}{1+i} + \frac{1}{(1+i)^2} + \ldots + \frac{1}{(1+i)^n}\right]$$

$$= \frac{a}{(1+i)^m}\left[\frac{\frac{1}{1+i}\left(1-\left(\frac{1}{1+i}\right)^n\right)}{1-\frac{1}{1+i}}\right]$$

$$= \frac{a}{(1+i)^m}\left[\frac{1-\left(\frac{1}{1+i}\right)^n}{i}\right]$$

$$= a(1+i)^{-m}\left[\frac{1-(1+i)^{-n}}{i}\right]$$

Case II. Annuity Due

In this case, 1st instalment is paid after m periods, 2nd after (m+1) periods and so on. Last instalment wil be paid after (m+n−1) periods.

So $V_1 = \frac{a}{\left(1+\frac{r}{100}\right)^m}$, $V_2 = \frac{a}{\left(1+\frac{r}{100}\right)^{m+1}}$,… $V_n = \frac{a}{\left(1+\frac{r}{100}\right)^{m+n-1}}$

∴ $V_1 = \frac{a}{(1+i)^m} = \frac{a}{(1+i)^{m+1}}\ V_n = \frac{a}{(1+i)^{m+n-1}}$

Now $V = V_1 + V_2 + \ldots + V_n$

$$= \frac{a}{(1+i)^m} + \frac{a}{(1+i)^{m+1}} + \ldots + \frac{a}{(1+i)^{m+n-1}}$$

$$= \frac{a}{(1+i)^m} + \frac{a}{(1+i)^m(1+i)} + \ldots + \frac{a}{(1+i)^m(1+i)^{n-1}}$$

$$= \frac{a}{(1+i)^m}\left[1+\frac{1}{(1+i)}+\frac{1}{(1+i)^2}+\ldots+\frac{1}{(1+i)^{n-1}}\right]$$

$$= \frac{a}{(1+i)^m}\left[\frac{1-\left(\frac{1}{1+i}\right)^n}{1-\frac{1}{1+i}}\right]$$

$$= a(1+i)^{-m}\left[\frac{1-(1+i)^{-n}}{\frac{i}{1+i}}\right]$$

Example 18. Find the present value of a deferred annuity of Rs. 8000 per year for 8 years at 10 % p.a. rate, the first instalment to be paid at the end of 4 years.

Solution. We are given

$a = 8000,\ n = 8,\ m = 4,\ r = 10\ \%$ so $i = \frac{10}{100} = 0.1$

Now $V = a(1+i)^{-m}\left[\frac{1-(1+i)^{-n}}{i}\right]$

$$= 8000\ (1+0.1)^{-4}\left[\frac{1-(1+0.1)^{-8}}{0.1}\right] = 800\ (1.1)^{-4}\left[\frac{1-(1.1)^{-8}}{0.1}\right]$$

Let $x = (1.1)^{-4}$ and $y = (1.1)^{-8}$

$\log x = -4 \log 1.1$
$= -4\times0.04139$
$= -0.1656+1-1$
$= \bar{1}.8344$
$x = AL[\bar{1}.8344]$
$= 0.6830$

$\log y = -8 \log 1.1$
$= -8\times0.04139$
$= -0.3312+1-1$
$= \bar{1}.6688$
$y = AL[\bar{1}.6688]$
$= 0.4664$

So $V = 8000\times0.6830 \left[\frac{1-0.4664}{0.1}\right]$

$= 80000\times0.6830\times0.5336$

$= \text{Rs. } 29155.90$

Example 19. A car is sold for Rs 75000 down and 30 half yearly instalments of Rs. 6000 each, the first to be paid after 4 years. Find the cash price of the car, if rate of interest is 12 % p.a. compounded half yearly.

Solution. We are given

$$a = 6000,\ n = 30,\ m = 3.5\times2 = 7,\ r = \frac{12}{2} = 6\ \%\ \text{or}\ i = \frac{6}{100} = 0.06$$

Now $V = a(1+i)^{-m}\left[\frac{1-(1+i)^{-n}}{i}\right]$

$= 6000(1+0.06)^{-7}\left[\frac{1-(1+0.06)^{-30}}{0.06}\right]$

$= \frac{6000}{0.06}(1.06)^{-7}\,[1-(1.06)^{-30}]$

Let $x = (1.06)^{-7}$ and $y = (1.06)^{-30}$

$\log x = -7 \log 1.06$
$= -7\times0.0253$
$= -0.1771+1-1$
$= \bar{1}.8229$
$x = AL[\bar{1}.8229]$
$= 0.6651$

$\log y = -30 \log 1.06$
$= -30\times0.0253$
$= -0.7592+1-1$
$= \bar{1}.2408$
$y = AL[\bar{1}.2408]$
$= 0.1741$

So $V = 100000\times0.6651\times(1-0.174)$

$= 100000\times0.6651\times0.8259$

$= \text{Rs. } 54930.$

Cash price = Cash payment + present value of future instalments

= 75000+54930

= Rs. 129930

Exercise 11.2

1. Find the present value of an annuity due of Rs 4000 per annum for 10 years at a rate of 8 % per annum.
2. Find the present value of an ordinary annuity of Rs. 5625 per year for 6 year at rate of 9 % per annum.
3. Find the present values of an ordinary annuity of Rs. 5000 per six months for 12 years at rate of 4 % p.a. if the interest is compounded half yearly.

4. John buys a plot for Rs 3,00,000 for which he agrees to equal payments at the end of each year for 10 years . If the rate of interest is 10 % p.a. find the amount of each instalment.
5. Lalita buys a house by paying Rs 1,00,000 in cash immediately and promises to pay the balance amount in 15 equal annual instalments of Rs. 8000 each at 15 % compound interest rate. Find the cash price of the house.
6. Find the amount of instalment on a loan of Rs. 250000 to be paid in 20 equal annual instalments at a rate of 8 % per annum.
7. A persons buys a car for Rs. 2,50,000. He pays Rs. 1,00,000 in cash and promises to pay the balance amount in 10 annual equal instalments. If the rate of interest is 12 % per annum, find the instalment.
8. Find the present value of an annuity of Rs 11000 per year for 6 years at a rate of 11 % if the interest is compounded continuously.
9. Find the present value of a deferred ordinary annuity of 12000 per year for 10 years at a rate of 6 % p.a., the first instalment being paid after 3 years.

Answers

Exercise 11.1

1. Rs. 61050 2. Rs. 232008 3. Rs. 28647 4. Rs. 143754.48
5. Rs. 24081.9 6. Rs.3924.64
7. Amount 1st = Rs. 35098 and for 2nd = Rs. 34674. So first investment is better.

Exercise 11.2

1. Rs. 28987.2 2. Rs. 25233.75 3. Rs. 94570 4. Rs. 1127.90
5. Rs. 146779 6. Rs. 25463.43 7. Rs. 26548.67 8. Rs. 53141
9. Rs. 74154.35